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线性矩阵方程的斜 Hermit$\{P,k+1\}$ Hamilton 解

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摘要: 给定矩阵 $P \in \mathbf{C}^{n\times n}$ 且 $P^* = -P = P^{k+1}$. 考虑了矩阵方程 $AX = B$ 存在斜 Hermite $\{P, k+1\}$ (斜) Hamilton 解的充要条件, 并给出了解的表达式. 进一步, 对于任意给定的矩阵 $\tilde{A} \in \mathbf{C}^{n\times n}$, 给出了使得 Frobenius 范数 $||\tilde{A} - \bar{A}||$ 取得最小值的最佳逼近解 $\bar{A} \in \mathbf{C}^{n\times n}$. 当矩阵方程 $AX = B$ 不相容时, 给出了斜 Hermite$\{P, k+1\}$ (斜) Hamilton 最小二乘解, 在此条件下, 给出了对于任意给定矩阵的最佳逼近解. 最后给出一些数值实例.

关键词: 斜 Hermite 矩阵; Hamilton 矩阵; 最小二乘解; 斜 Hermite$\{P, k+1\}$ Hamilton 矩阵

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The skew-Hermitian $\{P,k+1\}$ Hamiltonian solutions of a linear matrix equation

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Abstract: Given $P\in\mathbf{C}^{n\times n}$ and $P^*=-P=P^{k+1}$, we consider the necessary and sufficient conditions such that the matrix equation $AX = B$ is consistent with the skew-Hermitian $\{P, k+1\}$ (skew-) Hamiltonian structural constraint. Then, the corresponding expressions of the constraint solutions are also obtained. For any given matrix $\tilde{A} \in \mathbf{C}^{n\times n}$, we present the optimal approximate solution $\bar{A} \in \mathbf{C}^{n\times n}$ such that $||\tilde{A} - \bar{A}||$ is minimized in the Frobenius norm sense. If the matrix equation $AX = B$ is not consistent, its least-squares skew-Hermitian $\{P, k+1\}$ (skew-) Hamiltonian solutions are given. Under the least-square sense, we consider the best approximate solutions to any given matrix. Finally, some illustrative experiments are also presented.

Keywords: skew-Hermitian matrix; Hamiltonian matrix; least-squares solution;

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skew-Hermitian $\{P,k+1\}$ Hamiltonian matrix

0 引　言

令 $\mathbf{R}^{m\times n}(\mathbf{C}^{m\times n})$ 和 $\mathbf{U}^{n\times n}$ 表示所有 $m\times n$ 阶实 (复) 矩阵和 $n\times n$ 阶酉矩阵的集合; $\mathbf{C}_H^n$ 和 $\mathbf{C}_{SH}^n$ 表示所有 $n\times n$ 阶 Hermite 和斜 Hermite 矩阵的集合; A^* 表示矩阵 A 的共轭转置, I_n 表示 $n\times n$ 阶单位矩阵. $A\circ B$ 和 $\langle A,B\rangle=\mathrm{tr}(B^{\mathrm{H}}A)$ 分别表示 Hadamard 乘积和内积; $||A||=\sqrt{\langle A,A\rangle}$ 表示矩阵 A 的 Frobenius 范数; $A^{\dagger}$ 表示矩阵 A 的 Moore-Penrose 广义逆.

首先, 给出 $n\times n$ 阶斜 Hermite$\{P,k+1\}$Hamilton 矩阵和斜 Hermite$\{P,k+1\}$ 斜 Hamilton 矩阵的定义.

定义 0.1　设 $P\in\mathbf{C}^{n\times n}$ 满足 $P^*=-P=P^{k+1}$.

(1) 如果 $A\in\mathbf{C}^{n\times n}$ 满足 $A=-A^*=PAP$, 则称 A 是斜 Hermite$\{P,k+1\}$ Hamilton 矩阵, 所有斜 Hermite$\{P,k+1\}$Hamilton 矩阵构成的集合记为 SHH.

(2) 如果 $A\in\mathbf{C}^{n\times n}$ 满足 $A=-A^*=-PAP$, 则称 A 是斜 Hermite$\{P,k+1\}$ 斜 Hamilton 矩阵, 所有斜 Hermite$\{P,k+1\}$ 斜 Hamilton 矩阵构成的集合记为 $SHSH$.

当 P 是斜对称正交矩阵时, 有 $P^{\mathrm{T}}=-P=P^{-1}$, 白正简研究了 Hermite 和广义斜 Hamilton 解的逆特征值问题[1]. 我们考虑线性矩阵方程的斜 Hermite$\{P,k+1\}$ (斜) Hamilton 解. 在 Hopfield 神经网络中经常需要考虑如下微分方程

$$\frac{\mathrm{d}u}{\mathrm{d}t}=T(-u+\Omega g(u)).$$

该方程可以离散后得到如下关于 A 的线性矩阵方程:

$$AX=B,\quad B,X\in\mathbf{C}^{n\times m}.\tag{0.1}$$

很多学者研究了方程 (0.1) 的具有特殊结构的解, 如自反和反自反解等[2-8], $\{P,k+1\}$ 自反和反自反解[9]、(P,Q) 广义自反和反自反解[10]. 我们知道 Hamilton 和斜 Hamilton 矩阵的应用很广泛, 如最优控制[11]、H_∞ 最优[12]等.

本文主要考虑关于方程 (0.1) 的 4 个问题, 首先考虑方程 (0.1) 的斜 Hermite$\{P,k+1\}$ (斜) Hamilton 矩阵解存在的条件和表达式; 其次, 研究方程 (0.1) 最佳近似问题; 进一步, 当方程 (0.1) 不相容时, 考虑最小二乘解并给出表示形式; 最后给出部分数值实例.

问题 1　方程 (0.1) 的斜 Hermite$\{P,k+1\}$ (斜) Hamilton 矩阵解存在的条件和表达式.

问题 2　设 K 是问题 1 的解集, 任意给定矩阵 $\widetilde{A}\in\mathbf{C}^{n\times n}$, 求 $\overline{A}\in K$ 使得

$$||\widetilde{A}-\overline{A}||=\min_{A\in K}||\widetilde{A}-A||.$$

矩阵方程的最佳逼近问题被广泛用于实验设计[13-14], 通过实验可得到矩阵 $\widetilde{A}$, 但由于误差等的影响所得矩阵 $\widetilde{A}$ 不满足所需要的结构, 此时需要寻求在 Frobenius 范数下满足所需结构的最佳逼近矩阵 $\overline{A}$.

由于测量误差的影响, 得到的质量矩阵和刚度矩阵不满足方程 (0.1), 即方程不相容时需要考虑满足特殊条件的最小二乘解.

问题 3 给定矩阵 $\widetilde{A}\in\mathbf{C}^{n\times n}$, B, $X\in\mathbf{C}^{n\times m}$. 设 $\Psi_1=\{A|A\in\mathbf{C}^n_{SHH},||AX-B||=\min\}$. 求矩阵 $\overline{A}\in\Psi_1$, 使得

$$||\widetilde{A}-\overline{A}||=\min_{A\in\Psi_1}||\widetilde{A}-A||.$$

问题 4 给定矩阵 $\widetilde{A}\in\mathbf{C}^{n\times n}$, B, $X\in\mathbf{C}^{n\times m}$. 设 $\Psi_2=\{A|A\in\mathbf{C}^n_{SHSH},||AX-B||=\min\}$. 求矩阵 $\overline{A}\in\Psi_2$, 使得

$$||\widetilde{A}-\overline{A}||=\min_{A\in\Psi_2}||\widetilde{A}-A||.$$

1 预备知识

先引进一些有关斜 Hermite$\{P,k+1\}$ (斜) Hamilton 矩阵基本性质的引理.

引理 1.1 设 $P\in\mathbf{C}^{n\times n}$. 则 $P^*=-P=P^{k+1}$ 当且仅当存在矩阵 $U\in\mathbf{U}^{n\times n}$, 使得

$$P=U\begin{pmatrix}\mathrm{iI}_s & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathrm{iI}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^*, \tag{1.1}$$

其中 s, r 分别为矩阵 P 的特征值 i, $-$i 的重数, $k\in\{4m+2|m\in\mathbf{N}\}$.

证 明 如果 $-P=P^{k+1}$, 则 $x^{k+1}+x$ 是其零化多项式. 由于 $P=-P^*$, 则矩阵 P 的特征值是 0 或者纯虚数, 因此, 当 $k=4m+2$, $m\in\mathbf{N}$ 时, P 的特征值为 0 或 $\pm$i, 设 s, r 分别为 i, $-$i 的重数. 由谱分解理论知, 存在 $U\in\mathbf{U}^{n\times n}$ 使得式 (1.1) 成立.

引理 1.2 设 $P\in\mathbf{C}^{n\times n}$ 且 $P^*=-P=P^{k+1}$, 则 $A\in\mathbf{C}^{n\times n}$ 是斜 Hermite$\{P,k+1\}$ Hamilton 矩阵的充要条件是 A 可以表示为

$$A=U\begin{pmatrix}\mathbf{0} & A_{12} & \mathbf{0}\\ -A_{12}^* & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^*,\quad A_{12}\in\mathbf{C}^{s\times r}.$$

证 明 由定义 0.1 和引理 1.1, 有下列等式

$$U\begin{pmatrix}\mathrm{iI}_s & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathrm{iI}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^*AU\begin{pmatrix}\mathrm{iI}_s & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathrm{iI}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^*=-A^*.$$

令 $U^*AU=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ -A_{12}^* & A_{22} & A_{23}\\ -A_{13}^* & -A_{23}^* & A_{33}\end{pmatrix}$, 将其代入上式比较两边可得: $A_{11}=-A_{11}$, $A_{22}=-A_{22}$, $A_{13}=\mathbf{0}$, $A_{23}=\mathbf{0}$, $A_{33}=\mathbf{0}$. 定理得证.

类似于引理 1.2 的证明, 我们得如下引理.

引理 1.3 设 $P\in\mathbf{C}^{n\times n}$ 且 $P^*=-P=P^{k+1}$, 则 $A\in\mathbf{C}^{n\times n}$ 是斜 Hermite$\{P,k+1\}$ 斜 Hamilton 矩阵的充要条件是 A 可以表示为

$$A=U\begin{pmatrix}A_{11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & A_{22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^*,\quad A_{11}\in\mathbf{C}^s_{SH},\quad A_{22}\in\mathbf{C}^r_{SH}. \tag{1.2}$$

引理 1.4[15] 设矩阵 $B, X \in \mathbf{C}^{n\times m}$. 则方程 (0.1) 有斜 Hermite 解的充要条件是 $BX^{\dagger}X = B$ 且 $X^*B = -B^*X$, 其一般解可表示为

$$A = BX^{\dagger} - (X^{\dagger})^*B^* + \frac{1}{2}(X^{\dagger})^*(B^*X - X^*B)X^{\dagger} + (\mathrm{I}_n - XX^{\dagger})W(\mathrm{I}_n - XX^{\dagger}),$$

其中 $W \in \mathbf{C}^n_{SH}$.

引理 1.5[16] 设矩阵 $A \in \mathbf{C}^{m\times n}$, $C \in \mathbf{C}^{p\times q}$, $B \in \mathbf{C}^{m\times p}$ 和 $D \in \mathbf{C}^{n\times q}$. 则矩阵方程组

$$AX = B, \quad XC = D. \tag{1.3}$$

存在解 $X \in \mathbf{C}^{n\times p}$ 的充要条件为 $AA^{\dagger}B = B$, $DC^{\dagger}C = D$, $AD = BC$.

在该条件下的一般解可表示为

$$X = A^{\dagger}B + DC^{\dagger} - A^{\dagger}ADC^{\dagger} + (\mathrm{I}_n - A^{\dagger}A)V(\mathrm{I}_p - CC^{\dagger}),$$

其中 $V \in \mathbf{C}^{n\times p}_H$.

引理 1.6[17] 设矩阵 $B, A \in \mathbf{C}^{n\times n}$, 存在唯一矩阵解 $\overline{Y} = \frac{A+B}{2} \in \mathbf{C}^{n\times n}$ 使得

$$||\overline{Y} - A||^2 + ||\overline{Y} - B||^2 = \min_{Y\in\mathbf{C}^{n\times n}}\{||Y - A||^2 + ||Y - B||^2\}. \tag{1.4}$$

引理 1.7[18] 设 $L \in \mathbf{C}^{r\times p}$, $M \in \mathbf{C}^{l\times q}$, $H \in \mathbf{C}^{p\times q}$, $T \in \mathbf{C}^{r\times l}$, $L^*L = \mathrm{I}_p$, $M^*M = \mathrm{I}_q$. 则 $||LHM^* - T||^2 = ||H - L^*TM||^2 + ||T - LL^*TMM^*||^2$.

2 问题 1 的解的存在性

现在, 我们来考虑问题 1 的解的存在条件, 并给出方程 (0.1) 的斜 Hermite$\{P,k+1\}$ (斜) Hamilton 解.

定理 2.1 设矩阵 $B, X \in \mathbf{C}^{n\times m}$. 对于式 (1.1) 中的 U, 令

$$U^*X = \begin{pmatrix} X_{11} \\ X_{21} \\ X_{31} \end{pmatrix}, \quad X_{11} \in \mathbf{C}^{s\times m}, \quad X_{21} \in \mathbf{C}^{r\times m}, \quad X_{31} \in \mathbf{C}^{(n-s-r)\times m}. \tag{2.1}$$

$$U^*B = \begin{pmatrix} B_{11} \\ B_{21} \\ B_{31} \end{pmatrix}, \quad B_{11} \in \mathbf{C}^{s\times m}, \quad B_{21} \in \mathbf{C}^{r\times m}, \quad B_{31} \in \mathbf{C}^{(n-s-r)\times m}. \tag{2.2}$$

则方程 (0.1) 有斜 Hermite$\{P,k+1\}$ Hamilton 解 A 的充要条件是

$$A_{12}X_{21} = B_{11}, \quad -X_{11}^*A_{12} = B_{21}^*, \quad B_{31} = \mathbf{0}. \tag{2.3}$$

进一步

$$B_{21}X_{11}^{\dagger}X_{11} = B_{21}, \quad B_{11}X_{21}^{\dagger}X_{21} = B_{11}, \quad X_{11}^*B_{11} = -B_{21}^*X_{21}, \quad B_{31} = \mathbf{0}. \tag{2.4}$$

其解可表示为

$$A = U\begin{pmatrix} \mathbf{0} & A_{12} & \mathbf{0} \\ -A_{12}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}U^*,$$

其中

$$\begin{aligned}A_{12}=&-(X_{11}^{\dagger})^{*}B_{21}^{*}+B_{11}X_{21}^{\dagger}-X_{11}X_{11}^{\dagger}B_{11}X_{21}^{\dagger}\\&+(\mathrm{I}_s-X_{11}X_{11}^{\dagger})W(\mathrm{I}_r-X_{21}X_{21}^{\dagger}),\quad W\in\mathbf{C}^{s\times r}.\end{aligned}\tag{2.5}$$

证　明　假设矩阵 A 是问题 1 的斜 Hermite$\{P,k+1\}$ Hamilton 解, 由引理 1.5, 问题 1 有斜 Hermite$\{P,k+1\}$ Hamilton 解的充要条件是存在矩阵 $A_{12}\in\mathbf{C}^{s\times r}$ 使得

$$A=U\begin{pmatrix}\mathbf{0}&A_{12}&\mathbf{0}\\-A_{12}^{*}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^{*}.$$

代入方程 (0.1), 由式 (2.1) 和式 (2.2) 可得

$$A_{12}X_{21}=B_{11},\quad -X_{11}^{*}A_{12}=B_{21}^{*},\quad B_{31}=\mathbf{0}.\tag{2.6}$$

根据引理 1.5 可知, 方程组 (2.6) 存在解 $A_{12}\in\mathbf{C}^{s\times r}$ 的充要条件为

$$B_{21}X_{11}^{\dagger}X_{11}=B_{21},\quad B_{11}X_{21}^{\dagger}X_{21}=B_{11},\quad X_{11}^{*}B_{11}=-B_{21}^{*}X_{21},\quad B_{31}=\mathbf{0}.$$

此时 A_{12} 可用式 (2.5) 来表示, 即定理得证.

定理 2.2　给定矩阵 $B,X\in\mathbf{C}^{n\times m}$. 令 $U^{*}X$ 和 $U^{*}B$ 分别用 (2.1) 和 (2.2) 来表示. 则方程 (0.1) 有斜 Hermite$\{P,k+1\}$ 斜 Hamilton 解 A 的充要条件是

$$B_{11}X_{11}^{\dagger}X_{11}=B_{11},X_{11}^{*}B_{11}=-B_{11}^{*}X_{11},B_{21}X_{21}^{\dagger}X_{21}=B_{21},X_{21}^{*}B_{21}=-B_{21}^{*}X_{21},B_{31}=\mathbf{0}.\tag{2.7}$$

其解可表示为

$$A=U\begin{pmatrix}A_{11}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&A_{22}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^{*},$$

$$\begin{aligned}A_{11}=&B_{11}X_{11}^{\dagger}-(X_{11}^{\dagger})^{*}B_{11}^{*}+\frac{1}{2}(X_{11}^{\dagger})^{*}(B_{11}^{*}X_{11}-X_{11}^{*}B_{11})X_{11}^{\dagger}\\&+(\mathrm{I}_s-X_{11}X_{11}^{\dagger})W(\mathrm{I}_s-X_{11}X_{11}^{\dagger}),\end{aligned}\tag{2.8}$$

$$\begin{aligned}A_{22}=&B_{21}X_{21}^{\dagger}-(X_{21}^{\dagger})^{*}B_{21}^{*}+\frac{1}{2}(X_{21}^{\dagger})^{*}(B_{21}^{*}X_{21}-X_{21}^{*}B_{21})X_{21}^{\dagger}\\&+(\mathrm{I}_r-X_{21}X_{21}^{\dagger})V(\mathrm{I}_r-X_{21}X_{21}^{\dagger}),\end{aligned}\tag{2.9}$$

其中 $W\in\mathbf{C}_{SH}^{s}$, $V\in\mathbf{C}_{SH}^{r}$ 为任意矩阵.

证　明　假设矩阵 A 是问题 1 的斜 Hermite$\{P,k+1\}$ 斜 Hamilton 解, 由引理 1.3, 问题 1 有斜 Hermite$\{P,k+1\}$ 斜 Hamilton 解的充要条件是存在矩阵 $A_{11}\in\mathbf{C}_{SH}^{s}$, $A_{22}\in\mathbf{C}_{SH}^{r}$, 满足

$$A=U\begin{pmatrix}A_{11}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&A_{22}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^{*},\quad AX=B,\ \text{即}\ \begin{pmatrix}A_{11}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&A_{22}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^{*}X=U^{*}B.$$

由 (2.1) 式和 (2.2) 式得

$$A_{11}X_{11}=B_{11}, A_{22}X_{21}=B_{21}, B_{31}=\mathbf{0}. \tag{2.10}$$

根据引理 1.4, 方程组 (2.10) 有解的充要条件是

$$B_{11}X_{11}^{\dagger}X_{11}=B_{11},\quad X_{11}^{*}B_{11}=-B_{11}^{*}X_{11},\quad B_{21}X_{21}^{\dagger}X_{21}=B_{21},\quad X_{21}^{*}B_{21}=-B_{21}^{*}X_{21},\quad B_{31}=\mathbf{0}.$$

此时 A_{11}, A_{22} 可分别表示为式 (2.8) 和式 (2.9).

3　最佳逼近解

下面考虑问题 2 的解, 即研究问题 2 的斜 Hermite$\{P,k+1\}$ (斜) Hamilton 解. 对于给定矩阵 $\widetilde{A}\in\mathbf{C}^{n\times n}$, 令

$$U^{*}\left(\frac{\widetilde{A}-\widetilde{A}^{*}}{2}\right)U=\begin{pmatrix}\widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13}\\ -\widetilde{A}_{12}^{*} & \widetilde{A}_{22} & \widetilde{A}_{23}\\ -\widetilde{A}_{13}^{*} & -\widetilde{A}_{23}^{*} & \widetilde{A}_{33}\end{pmatrix}, \tag{3.1}$$

其中 $\widetilde{A}_{11},\widetilde{A}_{22}\in\mathbf{C}_{SH}^{s}$ 和 $\widetilde{A}_{12}\in\mathbf{C}^{s\times r}$, $\widetilde{A}_{13}\in\mathbf{C}^{s\times(n-s-r)}$, $\widetilde{A}_{33}\in\mathbf{C}_{SH}^{n-s-r}$.

对式 (2.1) 和式 (2.2) 中的 X_{11} 和 X_{21} 进行奇异值分解得

$$X_{11}=U_{1}\begin{pmatrix}\Sigma_{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}V_{1}^{*}, \tag{3.2}$$

$$X_{21}=U_{2}\begin{pmatrix}\Sigma_{2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}V_{2}^{*}, \tag{3.3}$$

其中 $\Sigma_1=\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{r_1})>0$, $r_1=\mathrm{rank}(X_{11})$, $\Sigma_2=\mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_{r_2})>0$, $r_2=\mathrm{rank}(X_{21})$. 令

$$U_1=(U_{11}\quad U_{12}),\quad V_1=(V_{11}\quad V_{12}),\quad U_2=(U_{21}\quad U_{22}),\quad V_2=(V_{21}\quad V_{22}). \tag{3.4}$$

定理 3.1　任意给定 $\widetilde{A}\in\mathbf{C}^{n\times n}$, 在定理 2.1 的条件下, 令 $U^{*}(\frac{\widetilde{A}-\widetilde{A}^{*}}{2})U$ 表示为式 (3.1) 的形式, 则问题 2 有唯一斜 Hermite$\{P,k+1\}$ Hamilton 解

$$\overline{A}=U\begin{pmatrix}\mathbf{0} & A_{12} & \mathbf{0}\\ -A_{12}^{*} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}U^{*}, \tag{3.5}$$

其中 $A_{12}=A_{12}^{0}+(\mathrm{I}_s-X_{11}X_{11}^{\dagger})(\widetilde{A}_{12}-A_{12}^{0})(\mathrm{I}_r-X_{21}X_{21}^{\dagger})$, 这里 $A_{12}^{0}=-(B_{21}X_{11}^{\dagger})^{*}+B_{11}X_{21}^{\dagger}-X_{11}X_{11}^{\dagger}B_{11}X_{21}^{\dagger}$.

证　明　设 K 是问题 1 的解集, 则问题 2 等价于

$$||\widetilde{A}-A||=\min,\forall A\in K. \tag{3.6}$$

令

$$U^{*}\widetilde{A}U=\begin{pmatrix}Y_{11} & Y_{12} & Y_{13}\\ Y_{21} & Y_{22} & Y_{23}\\ Y_{31} & Y_{32} & Y_{33}\end{pmatrix}. \tag{3.7}$$

由引理 1.2 有

$$||\widetilde{A}-A||^2=||U^*\widetilde{A}U-U^*AU||^2=\|Y_{12}-A_{12}\|^2+\|Y_{21}+A_{12}^*\|^2+\|Y_{11}\|^2+\|Y_{13}\|^2 \\ +\|Y_{22}\|^2+\|Y_{23}\|^2+\|Y_{31}\|^2+\|Y_{32}\|^2+\|Y_{33}\|^2.$$

由定理 2.1 知 $A_{12}=A_{12}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)W(\mathrm{I}_r-X_{21}X_{21}^\dagger)$, 则 (3.6) 等价于

$$\left\|Y_{12}-(A_{12}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)W(\mathrm{I}_r-X_{21}X_{21}^\dagger))\right\|^2 \\ +\left\|-Y_{21}^*-(A_{12}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)W(\mathrm{I}_r-X_{21}X_{21}^\dagger))\right\|^2=\min,\quad W\in\mathbf{C}^{s\times s}. \tag{3.8}$$

由式 (3.2)、(3.3) 和 (3.4), 有 $X_{11}X_{11}^\dagger=U_1\begin{pmatrix}\mathrm{I}_{r_1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}U_1^*=U_{11}U_{11}^*$, $X_{21}X_{21}^\dagger=U_2\begin{pmatrix}\mathrm{I}_{r_2}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}U_2^*=U_{21}U_{21}^*$, 从而 $\mathrm{I}_s-X_{11}X_{11}^\dagger=U_{12}U_{12}^*$, $\mathrm{I}_r-X_{21}X_{21}^\dagger=U_{22}U_{22}^*$, $A_{12}=A_{12}^0+U_{12}U_{12}^*WU_{22}U_{22}^*$. 令 $W_1=U_{12}^*WU_{22}$, 根据式 (3.8) 和引理 1.7, 有

$$\begin{aligned}&\left\|A_{12}^0+U_{12}W_1U_{22}^*-Y_{12}\right\|^2+\left\|A_{12}^0+U_{12}W_1U_{22}^*+Y_{21}^*\right\|^2\\=&\left\|W_1-U_{12}^*(Y_{12}-A_{12}^0)U_{22}\right\|^2+\left\|(Y_{12}-A_{12}^0)-U_{12}U_{12}^*(Y_{12}-A_{12}^0)U_{22}U_{22}^*\right\|^2\\&+\left\|W_1-U_{12}^*(-A_{12}^0-Y_{21}^*)U_{22}\right\|^2+\left\|(-Y_{21}^*-A_{12}^0)-U_{12}U_{12}^*(-Y_{21}^*-A_{12}^0)U_{22}U_{22}^*\right\|^2.\end{aligned}$$

再由引理 1.6, 我们有

$$\begin{aligned}W_1&=U_{12}^*\frac{Y_{12}-Y_{21}^*}{2}U_{22}-U_{12}^*A_{12}^0U_{22},\\A_{12}&=A_{12}^0+U_{12}U_{12}^*\left(\frac{Y_{12}-Y_{21}^*}{2}-A_{12}^0\right)U_{22}U_{22}^*\\&=A_{12}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)(\widetilde{A}_{12}-A_{12}^0)(\mathrm{I}_r-X_{21}X_{21}^\dagger).\end{aligned}$$

因此式 (3.6) 取到最小值当且仅当

$$A_{12}=A_{12}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)(\widetilde{A}_{12}-A_{12}^0)(\mathrm{I}_r-X_{21}X_{21}^\dagger).$$

此时最佳逼近解为式 (3.5), 定理得证.

当问题 2 具有斜 Hermite$\{P,k+1\}$ 斜 Hamilton 解时, 类似可得下列结论.

定理 3.2 任意给定 $\widetilde{A}\in\mathbf{C}^{n\times n}$, 在定理 2.2 的条件下, 令 $U^*\left(\frac{\widetilde{A}-\widetilde{A}^*}{2}\right)U$ 表示为式 (3.1) 的形式, 则问题 2 有唯一斜 Hermite$\{P,k+1\}$ 斜 Hamilton 解

$$\overline{A}=U\begin{pmatrix}A_{11}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&A_{22}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^*, \tag{3.9}$$

其中 $A_{11}=A_{11}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)(\widetilde{A}_{11}-A_{11}^0)(\mathrm{I}_s-X_{11}X_{11}^\dagger)$, $A_{22}=A_{22}^0+(\mathrm{I}_r-X_{21}X_{21}^\dagger)(\widetilde{A}_{22}-A_{22}^0)(\mathrm{I}_r-X_{21}X_{21}^\dagger)$. 这里

$$\begin{aligned}A_{11}^0&=B_{11}X_{11}-(X_{11}^\dagger)^*B_{11}^*+\frac{1}{2}(X_{11}^\dagger)^*(B_{11}^*X_{11}-X_{11}^*B_{11})X_{11}^\dagger,\\A_{22}^0&=B_{21}X_{21}-(X_{21}^\dagger)^*B_{21}^*+\frac{1}{2}(X_{21}^\dagger)^*(B_{21}^*X_{21}-X_{21}^*B_{21})X_{21}^\dagger.\end{aligned}$$

证 明 设 K 是问题 1 的解集, 则问题 2 等价于式 (3.6), 由式 (3.7) 和引理 1.3, 我们有

$$\begin{aligned}||\widetilde{A}-A||^2 =& ||U^*\widetilde{A}U-U^*AU||^2 \\ =& \|Y_{11}-A_{11}\|^2+\|Y_{22}-A_{22}\|^2+\|Y_{12}\|^2+\|Y_{13}\|^2 \\ &+\|Y_{21}\|^2+\|Y_{23}\|^2+\|Y_{31}\|^2+\|Y_{32}\|^2+\|Y_{33}\|^2.\end{aligned}$$

由定理 2.2 知式 (3.6) 等价于

$$||Y_{11}-A_{11}||^2+||Y_{22}-A_{22}||^2=\min, \tag{3.10}$$

其中 $A_{11}=A_{11}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)W(\mathrm{I}_s-X_{11}X_{11}^\dagger)$, $A_{22}=A_{22}^0+(\mathrm{I}_r-X_{21}X_{21}^\dagger)W(\mathrm{I}_r-X_{21}X_{21}^\dagger)$. 而 $||Y_{11}-A_{11}||^2=\left\|\frac{Y_{11}+Y_{11}^*}{2}\right\|^2+\left\|\frac{Y_{11}-Y_{11}^*}{2}-A_{11}\right\|^2$, $||Y_{22}-A_{22}||^2=\left\|\frac{Y_{22}+Y_{22}^*}{2}\right\|^2+\left\|\frac{Y_{22}-Y_{22}^*}{2}-A_{22}\right\|$. 根据式 (3.2)、(3.3) 和 (3.4) 得 $A_{11}=A_{11}^0+U_{12}U_{12}^*WU_{12}U_{12}^*$, $A_{22}=A_{22}^0+U_{22}U_{22}^*WU_{22}U_{22}^*$. 令 $W_1=U_{12}^*WU_{12}$, $V_1=U_{22}^*VU_{22}$. 则式 (3.6) 取最小值当且仅当 $\left\|\frac{Y_{11}-Y_{11}^*}{2}-(A_{11}^0+U_{12}U_{12}^*WU_{12}U_{12}^*)\right\|^2=\min$ 和 $\left\|\frac{Y_{11}-Y_{11}^*}{2}-(A_{11}^0+U_{12}U_{12}^*WU_{12}U_{12}^*)\right\|^2=\min$, 对于任意的 W, $V\in\mathbf{C}^{s\times s}$ 同时成立.

根据引理 1.7, 我们有

$$\begin{aligned}&\left\|\frac{Y_{11}-Y_{11}^*}{2}-(A_{11}^0+U_{12}U_{12}^*WU_{12}U_{12}^*)\right\|^2 \\ =&\left\|W_1-U_{12}^*\Big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\Big)U_{12}\right\|^2+\left\|\Big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\Big)-U_{12}U_{12}^*\Big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\Big)U_{12}U_{12}^*\right\|^2.\end{aligned}$$

因此有 $W_1=U_{12}^*\big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\big)U_{12}$, 从而得

$$\begin{aligned}A_{11}&=A_{11}^0+U_{12}U_{12}^*\Big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\Big)U_{12}U_{12}^* \\ &=A_{11}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)\Big(\frac{Y_{11}-Y_{11}^*}{2}-A_{11}^0\Big)(\mathrm{I}_s-X_{11}X_{11}^\dagger).\end{aligned}$$

同理可得 $A_{22}=A_{22}^0+(\mathrm{I}_r-X_{21}X_{21}^\dagger)\big(\frac{Y_{22}-Y_{22}^*}{2}-A_{22}^0\big)(\mathrm{I}_r-X_{21}X_{21}^\dagger)$. 由式 (3.1) 可得

$$A_{11}=A_{11}^0+(\mathrm{I}_s-X_{11}X_{11}^\dagger)(\widetilde{A}_{11}-A_{11}^0)(\mathrm{I}_s-X_{11}X_{11}^\dagger),$$

$$A_{22}=A_{22}^0+(\mathrm{I}_r-X_{21}X_{21}^\dagger)(\widetilde{A}_{22}-A_{22}^0)(\mathrm{I}_r-X_{21}X_{21}^\dagger).$$

所以根据引理 1.3 和定理 2.2 可知式 (3.6) 的唯一解可用式 (3.9) 来表示, 定理得证.

4 最小二乘解及其最佳逼近解

由实验或测量所得矩阵 X, B 导致方程不相容时, 需考虑最小二乘解. 这一节将研究矩阵方程的斜 Hermite $\{P, k+1\}$ (斜) Hamilton 最小二乘解. 先给出一个引理.

引理 4.1 设 m, $n\in\mathbf{R}$, $b\in\mathbf{C}$. 则 $\min\limits_{x\in\mathbf{C}}(|mx-a|^2+|nx-b|^2)$ 在 $x=\frac{ma+nb}{m^2+n^2}$ 处取得最小值.

根据引理 1.2 和式 (2.1)、(2.2) 的分割形式, 经计算可得

$$\|AX-B\|^2=||A_{12}X_{21}-B_{11}||^2+||A_{12}^*X_{11}+B_{21}||^2+||B_{13}||^2. \tag{4.1}$$

因此, $||AX-B||^2=\min, A\in C_{SHH}^n$ 当且仅当

$$||A_{12}X_{21}-B_{11}||^2+||A_{12}^*X_{11}+B_{21}||^2=\min, A_{12}\in \mathbf{C}^{s\times r}, \tag{4.2}$$

有最小二乘解.

设 X_{11} 和 X_{21} 的奇异值分解由式 (3.2)、(3.3) 和 (3.4) 给出, 令

$$U_1^*A_{12}U_2=\begin{pmatrix} D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}. \tag{4.3}$$

则有

$$\begin{aligned}
&\|A_{12}X_{21}-B_{11}\|^2+\|A_{12}^*X_{11}+B_{21}\|^2\\
&\quad=\|U_1^*A_{12}U_2U_2^*X_{21}-U_1^*B_{11}\|^2+\|U_2^*A_{12}^*U_1U_1^*X_{11}+U_2^*B_{21}\|^2\\
&\quad=\left\|\begin{pmatrix} D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}\begin{pmatrix} U_{21}^*X_{21}\\ \mathbf{0}\end{pmatrix}-\begin{pmatrix} U_{11}^*B_{11}\\ U_{12}^*B_{11}\end{pmatrix}\right\|^2\\
&\qquad+\left\|\begin{pmatrix} D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}^*\begin{pmatrix} U_{11}^*X_{11}\\ \mathbf{0}\end{pmatrix}+\begin{pmatrix} U_{21}^*B_{21}\\ U_{22}^*B_{21}\end{pmatrix}\right\|^2\\
&\quad=\|D_{11}U_{21}^*X_{21}-U_{11}^*B_{11}\|^2+\|D_{21}U_{21}^*X_{21}-U_{12}^*B_{11}\|^2\\
&\qquad+\|D_{11}^*U_{11}^*X_{11}+U_{21}^*B_{21}\|^2+\|D_{12}^*U_{11}^*X_{11}+U_{22}^*B_{21}\|^2.
\end{aligned}$$

于是, $\|A_{12}X_{21}-B_{11}\|^2+\|A_{12}^*X_{11}+B_{21}\|^2$ 取到最小值当且仅当

$$\begin{cases}\|D_{21}U_{21}^*X_{21}-U_{12}^*B_{11}\|^2=\min,\\ \|D_{12}^*U_{11}^*X_{11}+U_{22}^*B_{21}\|^2=\min,\\ \|D_{11}U_{21}^*X_{21}-U_{11}^*B_{11}\|^2+\|D_{11}^*U_{11}^*X_{11}+U_{21}^*B_{21}\|^2=\min.\end{cases} \tag{4.4}$$

首先, 我们有

$$\begin{aligned}
\|D_{21}U_{21}^*X_{21}-U_{12}^*B_{11}\|^2&=\left\|D_{21}U_{21}^*U_2\begin{pmatrix}\Sigma_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}V_2^*-U_{12}^*B_{11}\right\|^2\\
&=\left\|D_{21}U_{21}^*U_2\begin{pmatrix}\Sigma_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}-U_{12}^*B_{11}V_2\right\|^2\\
&=\|D_{21}\Sigma_2-U_{12}^*B_{11}V_{21}\|+\|U_{12}^*B_{11}V_{22}\|^2.
\end{aligned} \tag{4.5}$$

当 $\|D_{21}\Sigma_2-U_{12}^*B_{11}V_{21}\|=\mathbf{0}$ 时, 式 (4.5) 取得最小值, 此时有 $D_{21}=U_{12}^*B_{11}V_{21}\Sigma_2^{-1}$.

同理有

$$\|D_{12}^*U_{11}^*X_{11}+U_{22}^*B_{21}\|^2=\|D_{12}^*\Sigma_1+U_{22}^*B_{21}V_{11}\|^2+\|U_{22}^*B_{21}V_{12}\|^2. \tag{4.6}$$

当 $D_{12}=-\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}$ 时式 (4.6) 取得最小值.

我们再来考虑下式

$$\begin{aligned}&\|D_{11}U_{21}^*X_{21}-U_{11}^*B_{11}\|^2+\|D_{11}^*U_{11}^*X_{11}+U_{21}^*B_{21}\|^2\\=&\|D_{11}\Sigma_2-U_{11}^*B_{11}V_{21}\|^2+\|U_{11}^*B_{11}V_{22}\|^2+\|D_{11}^*\Sigma_1+U_{21}^*B_{21}V_{11}\|^2+\|U_{21}^*B_{21}V_{12}\|^2,\end{aligned}\tag{4.7}$$

则式 (4.7) 取得最小值只需 $\|D_{11}\Sigma_2-U_{11}^*B_{11}V_{21}\|^2+\|D_{11}^*\Sigma_1+U_{21}^*B_{21}V_{11}\|^2$ 取得最小值.

令 $D_{11}=(a_{ij})$, $U_{11}^*B_{11}V_{21}=(b_{ij})$, $U_{21}^*B_{21}V_{11}=(c_{ij})$. 由引理 4.1 有

$$\|D_{11}\Sigma_2-U_{11}^*B_{11}V_{21}\|^2+\|D_{11}^*\Sigma_1+U_{21}^*B_{21}V_{11}\|^2=\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}(|a_{ij}\beta_j-b_{ij}|^2+|a_{ij}\alpha_i+\overline{c}_{ji}|^2).\tag{4.8}$$

当 $a_{ij}=\frac{\beta_jb_{ij}-\overline{c}_{ji}\alpha_i}{\alpha_i^2+\beta_j^2}$, $i=1,2,\cdots,r_1$, $j=1,2,\cdots,r_2$, 时, 式 (4.8) 取得最小值. 即 $D_{11}=\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21})$, 其中 $\Omega=(\omega_{ij})\in\mathbf{C}^{r_1\times r_2}$, $\omega_{ij}=\frac{1}{\alpha_i^2+\beta_j^2}$, $i=1,2,\cdots,r_1$, $j=1,2,\cdots,r_2$.

将 D_{11}, D_{12}, D_{21} 的值代入式 (4.3) 可得

$$A_{12}=U_1\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21}) & -\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\ U_{12}^*B_{11}V_{21}\Sigma_2^{-1} & \widehat{D}_{22}\end{pmatrix}U_2^*,\tag{4.9}$$

其中 $\widehat{D}_{22}\in\mathbf{C}^{(s-r_1)\times(s-r_2)}$ 是任意矩阵.

对于问题 3, 给定矩阵 X, $B\in\mathbf{C}^{n\times m}$ 和 $\widetilde{A}\in\mathbf{C}^{n\times n}$. 求矩阵 $\overline{A}\in\Psi_1$, 使得

$$||\overline{A}-\widetilde{A}||=\min_{A\in\Psi_1}||A-\widetilde{A}||.\tag{4.10}$$

由式 (3.7) 和引理 1.2 有

$$\begin{aligned}\left\|\widetilde{A}-A\right\|^2&=\left\|U^*\widetilde{A}U-U^*AU\right\|^2\\&=||Y_{12}-A_{12}||^2+||Y_{21}+A_{12}^*||^2+||Y_{11}||^2+||Y_{13}||^2\\&\quad+||Y_{22}||^2+||Y_{23}||^2+||Y_{31}||^2+||Y_{32}||^2+||Y_{33}||^2.\end{aligned}$$

再根据式 (4.9) 有

$$\begin{aligned}&\|Y_{12}-A_{12}\|^2+\|Y_{21}+A_{12}^*\|^2\\=&\left\|Y_{12}-U_1\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21}) & -\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\ U_{12}^*B_{11}V_{21}\Sigma_2^{-1} & \widehat{D}_{22}\end{pmatrix}U_2^*\right\|^2\\&+\left\|-Y_{21}^*-U_1\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21}) & -\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\ U_{12}^*B_{11}V_{21}\Sigma_2^{-1} & \widehat{D}_{22}\end{pmatrix}U_2^*\right\|^2\\=&\left\|U_1^*Y_{12}U_2-\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21}) & -\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\ U_{12}^*B_{11}V_{21}\Sigma_2^{-1} & \widehat{D}_{22}\end{pmatrix}\right\|^2\\&+\left\|-U_1^*Y_{21}^*U_2-\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21}) & -\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\ U_{12}^*B_{11}V_{21}\Sigma_2^{-1} & \widehat{D}_{22}\end{pmatrix}\right\|^2.\end{aligned}\tag{4.11}$$

根据式 (3.2)、(3.3) 和 (3.4) 知 (4.10) 等价于

$$\min_{\widehat{D}_{22}\in\mathbf{C}^{(s-r_1)\times(s-r_2)}}(||U_{12}^*Y_{12}U_{22}-\widehat{D}_{22}||^2+||-U_{12}^*Y_{21}^*U_{22}-\widehat{D}_{22}||^2). \tag{4.12}$$

由引理 1.6 可知式 (4.12) 取得最小值当且仅当

$$\widehat{D}_{22}=\frac{U_{12}^*(Y_{12}-Y_{21}^*)U_{22}}{2}=U_{12}^*\widetilde{A}_{12}U_{22}. \tag{4.13}$$

根据以上的讨论, 我们可得下列结果.

定理 4.2 给定 $X,B\in\mathbf{C}^{m\times n}$. 设 U^*X、U^*B 分别用式 (2.1)、(2.2) 表示, X_{11}、X_{21} 的奇异值分解分别用式 (3.2)、(3.3) 表示, $\widetilde{A}_{12}$ 由式 (3.1) 给出. 则问题 3 在最小二乘意义下的斜 Hermite$\{P,k+1\}$ Hamilton 最佳逼近解为

$$\overline{A}=U\begin{pmatrix}\mathbf{0}&A_{12}&\mathbf{0}\\-A_{12}^*&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^*,$$

其中 $A_{12}=U_1\begin{pmatrix}\Omega\circ(U_{11}^*B_{11}V_{21}\Sigma_2-\Sigma_1V_{11}^*B_{21}^*U_{21})&-\Sigma_1^{-1}V_{11}^*B_{21}^*U_{22}\\U_{12}^*B_{11}V_{21}\Sigma_2^{-1}&\widehat{D}_{22}\end{pmatrix}U_2^*$, $\widehat{D}_{22}=U_{12}^*\widetilde{A}_{12}U_{22}$, $\Omega=(\omega_{ij})\in\mathbf{C}^{r_1\times r_2}$, $\omega_{ij}=\frac{1}{\alpha_i^2+\beta_j^2}$, $i=1,2,\cdots,r_1$, $j=1,2,\cdots,r_2$.

对于问题 4, 我们可得如下结论.

定理 4.3 给定 $B,\ X\in\mathbf{C}^{m\times n}$. 设 U^*X、U^*B 分别用式 (2.1)、(2.2) 表示, X_{11}、X_{21} 的奇异值分解分别用式 (3.2)、(3.3) 表示, $\widetilde{A}_{11}$、$\widetilde{A}_{22}$ 由式 (3.1) 给出. 则问题 4 在最小二乘意义下的斜 Hermite $\{P,k+1\}$ 斜 Hamilton 最佳逼近解为

$$\overline{A}=U\begin{pmatrix}A_{11}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&A_{22}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}U^*,$$

其中

$$\begin{aligned}
A_{11}=&U_1\begin{pmatrix}\Phi_1\circ(U_{11}^*B_{11}V_{11}\Sigma_1-\Sigma_1V_{11}^*B_{11}^*U_{11})&-\Sigma_1^{-1}V_{11}^*B_{11}^*U_{12}\\U_{12}^*B_{11}V_{11}\Sigma_1^{-1}&\widetilde{D}_{22}\end{pmatrix}U_1^*,\\
\widetilde{D}_{22}=&U_{12}^*\widetilde{A}_{11}U_{12},\quad \Phi_1=(\varphi_{ij})\in\mathbf{C}^{r_1\times r_1},\quad \varphi_{ij}=\frac{1}{\alpha_i^2+\alpha_j^2},1\leqslant i,j\leqslant r_1,\\
A_{22}=&U_2\begin{pmatrix}\Phi_2\circ(U_{21}^*B_{21}V_{21}\Sigma_2-\Sigma_2V_{21}^*B_{21}^*U_{21})&-\Sigma_2^{-1}V_{21}^*B_{21}^*U_{22}\\U_{22}^*B_{21}V_{21}\Sigma_2^{-1}&\widetilde{C}_{22}\end{pmatrix}U_2^*,\\
\widetilde{C}_{22}=&U_{12}^*\widetilde{A}_{22}U_{12},\quad \Phi_2=(\psi_{ij})\in\mathbf{C}^{r_1\times r_1},\quad \psi_{ij}=\frac{1}{\beta_i^2+\beta_j^2},1\leqslant i,j\leqslant r_2.
\end{aligned}$$

证 明 由式 (1.3)、(2.1) 和 (2.2) 可得

$$\|AX-B\|^2=||A_{11}X_{11}-B_{11}||^2+||A_{22}X_{21}-B_{21}||^2+||B_{13}||^2. \tag{4.14}$$

$||AX-B||^2=\min$ 当且仅当 $||A_{11}X_{11}-B_{11}||^2=\min$ 和 $||A_{22}X_{21}-B_{21}||^2=\min$ 同时成立.

由式 (1.3), 设 X_{11} 和 X_{21} 的奇异值分解由式 (3.2)、(3.3) 和 (3.4) 给出, 令 $U_1^*A_{11}U_1=\begin{pmatrix} D_{11} & -\mathrm{D}_{21}^* \\ D_{21} & D_{22}\end{pmatrix}$, 其中 $D_{11}\in \mathbf{C}_{SH}^{r_1}$, $D_{22}\in \mathbf{C}_{SH}^{s-r_1}$. 则有

$$\begin{aligned}||A_{11}X_{11}-B_{11}||^2&=||U_1^*A_{11}U_1U_1^*X_{11}-U_1^*B_{11}||^2\\&=||D_{11}U_{11}^*X_{11}-U_{11}^*B_{11}||^2+||D_{21}U_{11}^*X_{11}-U_{12}^*B_{11}||^2.\end{aligned}$$

当 $D_{21}=U_{12}^*B_{11}V_{11}\Sigma_1^{-1}$ 时 $||D_{21}U_{11}^*X_{11}-U_{12}^*B_{11}||^2$ 取得最小值. 又 $||D_{11}U_{11}^*X_{11}-U_{11}^*B_{11}||^2=||D_{11}\Sigma_1-U_{11}^*B_{11}V_{11}||^2+||U_{11}^*B_{11}V_{12}||^2$. 设 $D_{11}=(a_{ij})$, $U_{11}^*B_{11}V_{11}=(b_{ij})$, 由引理 (4.1) 有

$$||D_{11}\Sigma_1-U_{11}^*B_{11}V_{11}||^2=\sum_{i<j,i=1}^{r_1}\sum_{j=1}^{r_2}(|a_{ij}\alpha_j-b_{ij}|^2+|a_{ij}\alpha_j+\overline{b}_{ij}|^2)+\sum_{i=1}^{r_1}|a_{ii}\alpha_i-b_{ii}|^2,$$

当 $a_{ij}=\frac{\alpha_jb_{ij}-\alpha_i\overline{b}_{ij}}{\alpha_i^2+\alpha_j^2}$ 时, $a_{ij}=-\overline{a}_{ji}$, $i,j=1,2,\cdots,r_1$. 所以有 $D_{11}=\Phi_1\circ(U_{11}^*B_{11}V_{11}\Sigma_1-\Sigma_1V_{11}^*B_{11}^*U_{11})$, 其中 $\Phi_1=(\varphi_{ij})\in\mathbf{C}^{r_1\times r_1}$, $\varphi_{ij}=\frac{1}{\alpha_i^2+\alpha_j^2}$, $1\leqslant i,j\leqslant r_1$. 因此可得

$$A_{11}=U_1\begin{pmatrix}\Phi_1\circ(U_{11}^*B_{11}V_{11}\Sigma_1-\Sigma_1V_{11}^*B_{11}^*U_{11}) & -\Sigma_1^{-1}V_{11}^*B_{11}^*U_{12}\\ U_{12}^*B_{11}V_{11}\Sigma_1^{-1} & \widetilde{D}_{22}\end{pmatrix}U_1^*, \tag{4.15}$$

其中 $\widetilde{D}\in\mathbf{C}_{SH}^{s-r_1}$.

同理可得

$$A_{22}=U_2\begin{pmatrix}\Phi_2\circ(U_{21}^*B_{21}V_{21}\Sigma_2-\Sigma_2V_{21}^*B_{21}^*U_{21}) & -\Sigma_2^{-1}V_{21}^*B_{21}^*U_{22}\\ U_{22}^*B_{21}V_{21}\Sigma_2^{-1} & \widetilde{C}_{22}\end{pmatrix}U_2^*, \tag{4.16}$$

$\widetilde{C}_{22}\in\mathbf{C}_{SH}^{s-r_2}$,　$\Phi_2=(\psi_{ij})\in\mathbf{C}^{r_1\times r_1}$,　$\psi_{ij}=\frac{1}{\beta_i^2+\beta_j^2}$,　$1\leqslant i,j\leqslant r_2$.

对于问题 4, 给定矩阵 $X,B\in\mathbf{C}^{n\times m}$ 和 $\widetilde{A}\in\mathbf{C}^{n\times n}$. 求矩阵 $\overline{A}\in\Psi_2$, 使得

$$||\overline{A}-\widetilde{A}||=\min_{A\in\Psi_2}||A-\widetilde{A}||. \tag{4.17}$$

由式 (4.7) 和引理 1.3 有

$$\begin{aligned}\left\|\widetilde{A}-A\right\|^2&=\left\|U^*\widetilde{A}U-U^*AU\right\|^2\\&=||Y_{11}-A_{11}||^2+||Y_{22}-A_{22}||^2+||Y_{12}||^2+||Y_{13}||^2\\&\quad+||Y_{21}||^2+||Y_{23}||^2+||Y_{31}||^2+||Y_{32}||^2+||Y_{33}||^2.\end{aligned}$$

我们知道 (4.17) 取得最小值当且仅当 $||Y_{11}-A_{11}||^2$ 和 $||Y_{22}-A_{22}||^2$ 同时取得最小值, 其中 A_{11}、A_{22} 分别由式 (4.15)、(4.16) 表示. 根据式 (4.15) 有

$$\begin{aligned}||Y_{11}-A_{11}||^2&=\left\|\frac{Y_{11}-Y_{11}^*}{2}-A_{11}\right\|^2+\left\|\frac{Y_{11}+Y_{11}^*}{2}\right\|^2\\&=\left\|\frac{Y_{11}-Y_{11}^*}{2}-U_1\begin{pmatrix}\Phi_1\circ(U_{11}^*B_{11}V_{11}\Sigma_1-\Sigma_1V_{11}^*B_{11}^*U_{11}) & -\Sigma_1^{-1}V_{11}^*B_{11}^*U_{12}\\ U_{12}^*B_{11}V_{11}\Sigma_1^{-1} & \widetilde{D}_{22}\end{pmatrix}U_1^*\right\|^2\end{aligned}$$

$$+\left\|\frac{Y_{11}+Y_{11}^*}{2}\right\|^2$$
$$=\left\|U_1^*\frac{Y_{11}-Y_{11}^*}{2}U_1-\begin{pmatrix}\Phi_1\circ(U_{11}^*B_{11}V_{11}\Sigma_1-\Sigma_1V_{11}^*B_{11}^*U_{11}) & -\Sigma_1^{-1}V_{11}^*B_{11}^*U_{12}\\ U_{12}^*B_{11}V_{11}\Sigma_1^{-1} & \widetilde{D}_{22}\end{pmatrix}\right\|^2$$
$$+\left\|\frac{Y_{11}+Y_{11}^*}{2}\right\|^2.$$

由式 (3.4), $||Y_{11}-A_{11}||^2$ 取得最小值当且仅当 $\left\|U_{12}^*\frac{Y_{11}-Y_{11}^*}{2}U_{12}-\widetilde{D}_{22}\right\|$ 取得最小值. 根据式 (3.1) 可得 $\widetilde{D}_{22}=U_{12}^*\frac{Y_{11}-Y_{11}^*}{2}U_{12}=U_{12}^*\widetilde{A}_{11}U_{12}$. 同理可得 $\widetilde{C}_{22}=U_{12}^*\widetilde{A}_{22}U_{12}$, 定理得证.

5 算法和数值例子

根据定理 3.1、定理 3.2、定理 4.1、定理 4.2 建立求解问题 2、问题 3、问题 4 的算法并用部分数值实例进行说明.

算法 1

输入: X, B, $\widetilde{A}$, P;

输出: $\overline{A}$;

步 1: 根据式 (1.1) 计算矩阵 U;

步 2: 根据式 (2.1)、(2.1) 形成矩阵 X_{11}, X_{21}, B_{11}, B_{21}, B_{31};

步 3: 计算 $X_{11}^\dagger$ 和 $X_{21}^\dagger$;

步 4: 如果 $B_{21}X_{11}^\dagger X_{11}=B_{21}$, $B_{11}X_{21}^\dagger X_{21}=B_{11}$, $X_{11}^*B_{11}=-B_{21}^*X_{21}$, $B_{31}=\mathbf{0}$ (或 $B_{11}X_{11}^\dagger X_{11}=B_{11}, X_{11}^*B_{11}=-B_{11}^*X_{11}, B_{21}X_{21}^\dagger X_{21}=B_{21}, X_{21}^*B_{21}=-B_{21}^*X_{21}, B_{31}=\mathbf{0}$), 则执行步 5——步 7, 否则转到步 8;

步 5: 根据式 (2.1) 时计算 $\widetilde{A}_{12}$ (或 $\widetilde{A}_{11}$ 和 $\widetilde{A}_{22}$);

步 6: 根据式 (3.5) 计算 A_{12} (或根据式 (3.9) 计算 A_{11} 和 A_{22});

步 7: 根据定理 3.1 (或定理 3.2) 计算矩阵 $\overline{A}$;

步 8: 输出矩阵 $\overline{A}$ 或者解不存在, 结束.

算法 2

输入: X, B, $\widetilde{A}$, P;

输出: $\overline{A}$;

步 1: 根据式 (1.1) 计算矩阵 U;

步 2: 根据式 (2.1)、(2.1) 形成矩阵 X_{11}, X_{21}, B_{11}, B_{21}, B_{31};

步 3: 计算 $X_{11}^\dagger$ 和 $X_{21}^\dagger$;

步 4: 根据式 (2.1) 时计算 $\widetilde{A}_{12}$ (或 $\widetilde{A}_{11}$ 和 $\widetilde{A}_{22}$);

步 5: 根据式 (4.9) 和式 (4.13) 计算 A_{12} (或根据式 (4.15)、(4.16)、(4.18)、(4.19) 计算 A_{11} 和 A_{22});

步 6: 根据定理 4.1 (或定理 4.2) 计算矩阵 $\overline{A}$;

步 7: 输出矩阵 $\overline{A}$.

用 MATLAB2016a 在精度为 10^{-16} 的计算机上对算法 1 和算法 2 进行验证. 在实验中

取 $k=2$, $P=(P_1\quad P_2)$, 其中

$$P_1=\begin{pmatrix} 0.4224\mathrm{i} & 0.7007\mathrm{i} & 0.2997\mathrm{i} & 0.3623\mathrm{i}\\ 0.7007\mathrm{i} & 0.0356\mathrm{i} & 0.2182\mathrm{i} & 0.1256\mathrm{i}\\ 0.2997\mathrm{i} & 0.2182\mathrm{i} & 0.4980\mathrm{i} & 0.3940\mathrm{i}\\ 0.3623\mathrm{i} & 0.1256\mathrm{i} & 0.3940\mathrm{i} & 0.3440\mathrm{i}\\ 0.1617\mathrm{i} & 0.2731\mathrm{i} & 0.3775\mathrm{i} & 0.6543\mathrm{i}\\ 0.1965\mathrm{i} & 0.3662\mathrm{i} & 0.3061\mathrm{i} & 0.0706\mathrm{i}\\ 0.2046\mathrm{i} & 0.3800\mathrm{i} & 0.0223\mathrm{i} & 0.1978\mathrm{i}\\ 0.0535\mathrm{i} & 0.3020\mathrm{i} & 0.4718\mathrm{i} & 0.3275\mathrm{i} \end{pmatrix},\quad P_2=\begin{pmatrix} 0.1617\mathrm{i} & 0.1965\mathrm{i} & 0.2046\mathrm{i} & 0.0535\mathrm{i}\\ 0.2731\mathrm{i} & 0.3662\mathrm{i} & 0.3800\mathrm{i} & 0.3020\mathrm{i}\\ 0.3775\mathrm{i} & 0.3061\mathrm{i} & 0.0223\mathrm{i} & 0.4718\mathrm{i}\\ 0.6543\mathrm{i} & 0.0706\mathrm{i} & 0.1978\mathrm{i} & 0.3275\mathrm{i}\\ 0.2965\mathrm{i} & 0.0045\mathrm{i} & 0.0999\mathrm{i} & 0.4805\mathrm{i}\\ 0.0045\mathrm{i} & 0.3844\mathrm{i} & 0.7120\mathrm{i} & 0.2716\mathrm{i}\\ 0.0999\mathrm{i} & 0.7120\mathrm{i} & 0.0697\mathrm{i} & 0.5023\mathrm{i}\\ 0.4805\mathrm{i} & 0.2716\mathrm{i} & 0.5023\mathrm{i} & 0.1386\mathrm{i} \end{pmatrix}.$$

例 5.1　给定下列矩阵 X, B, $\widetilde{A}$,

$$X=\begin{pmatrix} 0.2215 & 0.8575 & 0.5218\\ 0.3058 & 0.1649 & 0.9157\\ 0.1270 & 0.1576 & 0.7922\\ 0.3134 & 0.9706 & 0.9595\\ 0.2324 & 0.9572 & 0.6557\\ 0.0975 & 0.4854 & 0.0357\\ 0.2685 & 0.7003 & 0.8441\\ 0.6469 & 0.1419 & 0.8340 \end{pmatrix},$$

$$B=\begin{pmatrix} -0.7353-0.6689\mathrm{i} & -1.3115-0.9388\mathrm{i} & -1.0931+1.0883\mathrm{i}\\ -0.8599+1.0682\mathrm{i} & -1.1158+0.1436\mathrm{i} & -1.3119+1.5323\mathrm{i}\\ -1.1438-0.3290\mathrm{i} & -0.8991-0.4993\mathrm{i} & 0.0267-1.5611\mathrm{i}\\ 1.8149-1.5199\mathrm{i} & 2.4002-1.9371\mathrm{i} & 3.0453-1.2616\mathrm{i}\\ -2.2359-0.5442\mathrm{i} & -2.1511-0.8277\mathrm{i} & -4.0046-0.3031\mathrm{i}\\ -0.1540-1.1167\mathrm{i} & 0.1235-1.4078\mathrm{i} & -0.0162-1.0713\mathrm{i}\\ 1.9494+1.2438\mathrm{i} & 2.3164+2.3343\mathrm{i} & 0.7536+1.4603\mathrm{i}\\ 1.3742+2.0407\mathrm{i} & 2.0414+2.8587\mathrm{i} & 0.7557+0.9288\mathrm{i} \end{pmatrix},$$

$\widetilde{A}=(\widetilde{A}_1\quad \widetilde{A}_2)$, 其中

$$\widetilde{A}_1=\begin{pmatrix} -0.0000-2.0156\mathrm{i} & -1.2081-0.3912\mathrm{i} & 0.7193+0.5425\mathrm{i} & 0.4353+0.3190\mathrm{i}\\ 0.2081-0.3912\mathrm{i} & 0.0000+2.3238\mathrm{i} & 0.1572+0.1448\mathrm{i} & -2.6389-1.0025\mathrm{i}\\ -0.7193+0.5425\mathrm{i} & -0.1572+0.1448\mathrm{i} & 0.0000+0.0378\mathrm{i} & -1.3388-0.0083\mathrm{i}\\ -1.4353+0.3190\mathrm{i} & 2.6389-1.0025\mathrm{i} & 1.3388-0.0083\mathrm{i} & 0.0000-0.7932\mathrm{i}\\ 0.7302+0.2498\mathrm{i} & -0.8389-0.1965\mathrm{i} & -0.1374-0.1948\mathrm{i} & -0.7816-0.6729\mathrm{i}\\ -0.1371+0.6929\mathrm{i} & 0.7844-1.0027\mathrm{i} & -0.0956+0.2645\mathrm{i} & -0.9978-0.5970\mathrm{i}\\ 1.1888+0.2824\mathrm{i} & 0.2502-0.2792\mathrm{i} & -1.2063-1.2228\mathrm{i} & 0.2359+0.4700\mathrm{i}\\ 0.4824+1.2125\mathrm{i} & -0.2959+0.8760\mathrm{i} & -0.6861-0.8437\mathrm{i} & 0.1865+0.3778\mathrm{i} \end{pmatrix},$$

$$\widetilde{A}_2=\begin{pmatrix} -0.7302+0.2498\mathrm{i} & 0.1371+0.6929\mathrm{i} & -1.1888+0.2824\mathrm{i} & -0.4824+1.2125\mathrm{i}\\ 0.8389-0.1965\mathrm{i} & -0.7844-1.0027\mathrm{i} & -0.2502-0.2792\mathrm{i} & 0.2959+0.8760\mathrm{i}\\ 0.1374-0.1948\mathrm{i} & 0.0956+0.2645\mathrm{i} & 1.2063-1.2228\mathrm{i} & 0.6861-0.8437\mathrm{i}\\ 0.7816-0.6729\mathrm{i} & 0.9978-0.5970\mathrm{i} & -0.2359+0.4700\mathrm{i} & -0.1865+0.3778\mathrm{i}\\ 0.0000-0.4075\mathrm{i} & -0.1916-0.6059\mathrm{i} & -1.1328+0.5363\mathrm{i} & -1.8381+0.4336\mathrm{i}\\ 0.1916-0.6059\mathrm{i} & 0.0000-0.3161\mathrm{i} & 0.3916+0.1756\mathrm{i} & -0.1089+0.1899\mathrm{i}\\ 1.1328+0.5363\mathrm{i} & -0.3916+0.1756\mathrm{i} & -0.0000+1.7940\mathrm{i} & 0.0251+0.2498\mathrm{i}\\ 1.8381+0.4336\mathrm{i} & 0.1089+0.1899\mathrm{i} & -0.0251+0.2498\mathrm{i} & 0.0000-0.6233\mathrm{i} \end{pmatrix}.$$

求满足问题 3 的斜 Hermite$\{P, k+1\}$ 斜 Hamilton 最佳逼近解.

解根据算法 2, 我们可得 $\overline{A} = (\overline{A}_1 \quad \overline{A}_2)$, 其中

$$\overline{A}_1 = \begin{pmatrix} -0.0000+2.0892\mathrm{i} & 7.2727+1.1030\mathrm{i} & -2.7844+1.4772\mathrm{i} & 0.5579-0.6959\mathrm{i} \\ -7.2727+1.1030\mathrm{i} & 0.0000-0.7947\mathrm{i} & -2.3966+3.5395\mathrm{i} & -1.2961-1.2633\mathrm{i} \\ 2.7844+1.4772\mathrm{i} & 2.3966+3.5395\mathrm{i} & 0.0000-1.0521\mathrm{i} & -0.0209+1.0838\mathrm{i} \\ -0.5579-0.6959\mathrm{i} & 1.2961-1.2633\mathrm{i} & 0.0209+1.0838\mathrm{i} & 0.0000-2.5265\mathrm{i} \\ 1.1946-1.5030\mathrm{i} & -2.3575-1.5054\mathrm{i} & 1.6609-1.5822\mathrm{i} & -2.5934-0.9273\mathrm{i} \\ -0.5394+0.6757\mathrm{i} & -1.8498+0.3794\mathrm{i} & 0.3345+0.4120\mathrm{i} & -0.0214+0.3229\mathrm{i} \\ 1.9921-0.9812\mathrm{i} & -5.3370+0.9213\mathrm{i} & 2.3942-3.3906\mathrm{i} & 0.2002+2.9394\mathrm{i} \\ 4.2776+0.2020\mathrm{i} & 0.6171+0.3630\mathrm{i} & 0.8009-2.2529\mathrm{i} & -0.6632-0.0878\mathrm{i} \end{pmatrix},$$

$$\overline{A}_2 = \begin{pmatrix} -1.1946-1.5030\mathrm{i} & 0.5394+0.6757\mathrm{i} & -1.9921-0.9812\mathrm{i} & -4.2776+0.2020\mathrm{i} \\ 2.3575-1.5054\mathrm{i} & 1.8498+0.3794\mathrm{i} & 5.3370+0.9213\mathrm{i} & -0.6171+0.3630\mathrm{i} \\ -1.6609-1.5822\mathrm{i} & -0.3345+0.4120\mathrm{i} & -2.3942-3.3906\mathrm{i} & -0.8009-2.2529\mathrm{i} \\ 2.5934-0.9273\mathrm{i} & 0.0214+0.3229\mathrm{i} & -0.2002+2.9394\mathrm{i} & 0.6632-0.0878\mathrm{i} \\ 0.0000+1.4618\mathrm{i} & -0.1690-1.6796\mathrm{i} & -0.4819+1.4829\mathrm{i} & -1.4702+1.9906\mathrm{i} \\ 0.1690-1.6796\mathrm{i} & 0.0000+0.1864\mathrm{i} & 0.6520-0.8403\mathrm{i} & 1.1144-0.9458\mathrm{i} \\ 0.4819+1.4829\mathrm{i} & -0.6520-0.8403\mathrm{i} & -0.0000-0.3824\mathrm{i} & 3.1299+0.7627\mathrm{i} \\ 1.4702+1.9906\mathrm{i} & -1.1144-0.9458\mathrm{i} & -3.1299+0.7627\mathrm{i} & -0.0000+1.0183\mathrm{i} \end{pmatrix}.$$

通过计算有 $||P\overline{A}P+\overline{A}|| = 2.0910\times10^{-14}$, $||\overline{A}X-B|| = 3.4875$, 且 $\overline{A}$ 与 $-P\overline{A}P$ 几乎相等, 误差接近计算机的精度. 因此, $\overline{A}$ 是矩阵方程 $AX=B$ 的斜 Hermite$\{P, k+1\}$ 斜 Hamilton 最佳逼近解.

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