文章编号: 1000-5641(2019)04-0011-08

二阶广义 Emden-Fowler 型微分方程的振荡性

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摘要: 研究了一类二阶广义 Emden-Fowler 型非线性变时滞中立型泛函微分方程的振荡性. 利用广义 Riccati 变换技术及一些分析技巧, 在条件 $\int_{t_0}^{+\infty} a^{-1/\beta}(t)\mathrm{d}t < +\infty$ 下建立了该类方程振荡的两个新的判别准则. 所举例子说明, 这些准则不仅推广和改进了一些已有的结果, 而且具有较好的实用性和可操作性.

关键词: 振荡性; Emden-Fowler 型微分方程; Riccati 变换

中图分类号: O175.7 **文献标志码**: A **DOI:** 10.3969/j.issn.1000-5641.2019.04.002

Oscillation of second-order generalized Emden-Fowler-type differential equations

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Abstract: The oscillatory behavior of a class of second-order generalized Emden-Fowler-type nonlinear variable delay neutral functional differential equations is studied in this article. By using the generalized Riccati transformation and some analytic techniques, we establish two new oscillation criteria for the equations under the condition $\int_{t_0}^{+\infty} a^{-1/\beta}(t)\mathrm{d}t < +\infty$. Illustrative examples are provided to show that our results extend and improve those previously reported in the literature, and the results are both practical and implementable.

Keywords: oscillation; Emden-Fowler-type differential equation; Riccati transformation

0 引 言

考虑如下一类二阶广义 Emden-Fowler 型非线性的中立型变时滞微分方程

$$[a(t)\phi_1(z'(t))]' + q(t)f(\phi_2(x(\delta(t)))) = 0, \quad t \geqslant t_0 \geqslant 0, \tag{1}$$

的振荡性, 其中$z(t) = x(t) + p(t)x(\tau(t))$, $\phi_1(u) = |u|^{\beta-1}u$, $\phi_2(u) = |u|^{\lambda-1}u$, 这里 β、λ 均为正的实常数, 并考虑如下假设条件:

收稿日期: 2018-05-05

基金项目: 湖南省自然科学基金(12JJ3008); 湖南省教育厅教学改革研究项目(2016jg671); 邵阳市科技计划项目(2016GX04)

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(C_1) 函数$a \in C^1([t_0,+\infty),(0,+\infty))$, 而$p \in C([t_0,+\infty),[0,+\infty))$, 且$0 \leqslant p(t) \leqslant p_0 < +\infty$ (这里p_0为常数), $q \in C([t_0,+\infty),(0,+\infty))$.

(C_2) 时滞函数$\tau, \delta : [t_0,+\infty) \to (0,+\infty)$满足$\tau(t) \leqslant t$, $\lim\limits_{t\to+\infty} \tau(t) = +\infty$, $\delta(t) \leqslant t$, $\lim\limits_{t\to+\infty} \delta(t) = +\infty$, 并且$\tau \circ \delta = \delta \circ \tau$, $\tau'(t) \geqslant \tau_0$ (这里 τ_0 是正常数).

(C_3) 函数$f \in C(\mathbf{R},\mathbf{R})$, 且当$u \neq 0$时有$uf(u) > 0$, $\frac{f(u)}{u} \geqslant L$ (这里L是正常数).

关于方程(1)的解及其振荡性的定义, 可参见文献 [1-2] 及其参考文献. 近年来, 二阶 Emden-Fowler 型微分动力方程的振荡性问题引起了国内外学者的极大兴趣和广泛关注[1-10]. 文献 [2] 研究了方程(1)的特殊情形 (即方程(1)中 $f(u) = u$ 时的情形) 的振荡性, 并得到如下结论.

定理 A[2] 设$\beta \geqslant \lambda$, $a'(t) \geqslant 0$, $\delta'(t) > 0$, $0 \leqslant p(t) < 1$, 且

$$\int_{t_0}^{+\infty} a^{-1/\beta}(t)\mathrm{d}t = +\infty. \tag{2}$$

如果存在函数 $\varphi \in C^1([t_0,+\infty),(0,+\infty))$, 使得

$$\int_{t_0}^{+\infty} \left[\varphi(s)q(s)[1-p(\delta(s))]^\lambda - \frac{a(\delta(s))(\varphi'(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}(k\varphi(s)\delta'(s))^\lambda}\right]\mathrm{d}s = +\infty,$$

其中$k > 0$是某常数, 则方程

$$\{a(t)|[x(t)+p(t)x(\tau(t))]'|^{\beta-1}[x(t)+p(t)x(\tau(t))]'\}' + q(t)|x(\delta(t))|^{\lambda-1}x(\delta(t)) = 0 \tag{3}$$

是振荡的.

在此基础上, 文献 [3] 研究了方程(1)的振荡性, 取消了限制条件"$\beta \geqslant \lambda$, $a'(t) \geqslant 0$", 并将条件"$0 \leqslant p(t) < 1$"放宽到了"$0 \leqslant p(t) \leqslant p_0 < +\infty$ (p_0为常数)", 得到了方程(1)振荡的一些新准则. 而文献[4]则进一步改进了文献 [3] 中的结论, 得到了如下关于方程(1)振荡的一个较为精细的结果.

定理 B[4] 设条件(C_1)—(C_3)及式(2)成立, 如果存在函数 $\varphi \in C^1([t_0,+\infty),(0,+\infty))$, 使得当 $\beta \leqslant \lambda$时

$$\limsup_{t\to+\infty} \int_T^t \varphi(s)\left[L_0 Q(s)\Psi^\lambda(s,t_0) - \frac{(\beta/\lambda)^\beta}{\eta^{\lambda-\beta}(\beta+1)^{\beta+1}}\left(a(s) + \frac{p_0^\lambda a(\tau(s))}{\tau_0^{\beta+1}}\right)\left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\beta+1}\right]\mathrm{d}s$$
$$= +\infty; \tag{4}$$

当 $\beta > \lambda$ 时

$$\limsup_{t\to+\infty} \int_T^t \varphi(s)\left[L_0 Q(s)\Psi^\lambda(s,t_0) - \frac{\eta^{(\beta-\lambda)/\beta}}{(\lambda+1)^{\lambda+1}}\left(a^{\lambda/\beta}(s) + \frac{p_0^\lambda a^{\lambda/\beta}(\tau(s))}{\tau_0^{\lambda+1}}\right)\left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\lambda+1}\right]\mathrm{d}s$$
$$= +\infty. \tag{5}$$

其中: 常数 $T \geqslant t_0$ 足够大; 常数 $L_0 = \begin{cases} L, & 0 < \lambda \leqslant 1, \\ 2^{1-\lambda}L, & \lambda > 1; \end{cases}$ η为正常数; 函数$Q(t) = \min\{q(t), q(\tau(t))\}$; $\Psi(t,t_0) = \left(\int_{t_0}^{\delta(t)} a^{-1/\beta}(s)\mathrm{d}s\right)\left(\int_{t_0}^{t} a^{-1/\beta}(s)\mathrm{d}s\right)^{-1}$; $\varphi_+(t) = \max\{\varphi(t), 0\}$. 则方程(1)是振荡的.

显然, 上述结果是在条件(2)成立下取得的. 如果条件(2)不满足, 即该积分收敛时, 文献 [3-4, 6-10] 等没有得到方程的振荡性结果, 而文献 [2] 得到如下结论.

定理 C[2] 设$\beta \geqslant \lambda$, $a'(t) \geqslant 0$, $\tau'(t) > 0$, $0 \leqslant p(t) < 1$且$p'(t) \geqslant 0$. 如果

$$\int_{t_0}^{+\infty} a^{-1/\beta}(t)\mathrm{d}t < +\infty, \tag{6}$$

并且有

$$\int_{t_0}^{+\infty} q(t)[1-p(\delta(t))]^{\lambda}\mathrm{d}t = +\infty \text{ 和} \int_{t_0}^{+\infty}\left(\frac{1}{a(t)}\int_{t_0}^{t} q(s)\mathrm{d}s\right)^{1/\beta}\mathrm{d}t = +\infty$$

成立, 则方程(3)的每一个解 $x(t)$ 或者振荡, 或者 $\lim\limits_{t\to+\infty} x(t) = 0$.

定理 C 就是文献 [2] 中的定理 2.4. 显然, 定理 C 中有较为严格的条件"$\beta \geqslant \lambda$, $a'(t) \geqslant 0$且$p'(t) \geqslant 0$", 而且其结论是不确定的. 在这种情形下, 其他文献中得到的结论(如文献 [5] 中的结论)也是如此. 这就使得在实际应用时不方便, 因为我们不知道方程 (3) 在什么条件下其解$x(t)$是振荡的, 而在什么条件下其解$x(t)$是渐近于0的. 因而对下列欧拉方程

$$(t^2x'(t))' + mx(t) = 0, \quad t \geqslant 1, \tag{E$_1$}$$

也就不能用定理C来确定其是否振荡了. 此外, 对于方程

$$\{t|[x(t)+(9+\sin 2t)x(t/2)]'|^{-1/2}[x(t)+(9+\sin 2t)x(t/2)]'\}' + t^4x^3(t/3) = 0, \quad t \geqslant 1, \tag{E$_2$}$$

由于不满足条件"$\beta \geqslant \lambda$, $0 \leqslant p(t) < 1$", 因此定理C也不能用于它的振荡性判别.

本文将在条件(6)下研究方程(1)的振荡性, 利用广义 Riccati 变换技术及不等式技巧, 得到方程(1)振荡的一系列新的判别准则, 改进且推广一系列已有的结果.

1 主要结果及证明

引理 1[3] 设 $A > 0$, $B > 0$ 和 $\alpha > 0$ 均为常数, 则当 $x > 0$ 时,

$$Ax - Bx^{\frac{\alpha+1}{\alpha}} \leqslant \frac{\alpha^{\alpha}A^{\alpha+1}}{(\alpha+1)^{\alpha+1}B^{\alpha}}.$$

定理 1 设条件(C_1)—(C_3)及式(6)成立, 并且当$\beta \leqslant \lambda$时式(4)成立; 当$\beta > \lambda$时式(5)成立. 如果存在实常数$\omega > 0$, 使得

$$\limsup_{t\to+\infty}\int_{T}^{t}(t-s)^{\omega}\left[L_0Q(s)R(s) - \left(1+\frac{p_0^{\lambda}}{\tau_0}\right)\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta+1}}\right]\mathrm{d}s > 0. \tag{7}$$

其中: 常数$T \geqslant t_0$足够大; $L_0 = \begin{cases} L, & 0 < \lambda \leqslant 1, \\ 2^{1-\lambda}L, & \lambda > 1; \end{cases}$ 函数$Q(t) = \min\{q(t), q(\tau(t))\}$, $R(t) = \begin{cases} k, & \beta > \lambda, \\ 1, & \beta = \lambda, \\ kA^{\lambda-\beta}(t), & \beta < \lambda \end{cases}$ ($k > 0$为常数); $A(t) = \int_{t}^{+\infty} a^{-1/\beta}(s)\mathrm{d}s$. 则方程(1)是振荡的.

证 明 使用反证法. 设方程(1)存在一个非振荡解$x(t)$, 不妨设$x(t)$为最终正解(当$x(t)$为最终负解时类似可证), 则有$x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ $(t \geqslant t_1 \geqslant t_0)$. 由文献 [4] 中定理 1 的证明知, $a(t)\phi_1(z'(t)) = a(t)|z'(t)|^{\beta-1}z'(t)$是严格单调减少且最终定号

的, 从而$z'(t)$最终为正或最终为负. 故只需考虑下列两种情形: (a) 当$t \geqslant t_1$时$z'(t) > 0$; (b) 当$t \geqslant t_1$时$z'(t) < 0$.

情形 a $z'(t) > 0$ $(t \geqslant t_1)$. 此时完全与文献 [4] 中定理 1 的证明相同. 当 $\beta \leqslant \lambda$ 时得到与式(4)矛盾; 当 $\beta > \lambda$ 时得到与式(5)矛盾.

情形 b $z'(t) < 0$ $(t \geqslant t_1)$. 首先, 定义广义的 Riccati 变换 $v(t)$ 为

$$v(t) = \frac{a(\tau(t))\phi_1(z'(\tau(t)))}{\phi_1(z(t))} = \frac{a(\tau(t))(-z'(\tau(t)))^{\beta-1}z'(\tau(t))}{z^{\beta}(t)}, \quad t \geqslant t_1, \tag{8}$$

则 $v(t) < 0$ $(t \geqslant t_1)$. 由于 $a(t)[-z'(t)]^{\beta-1}z'(t)$ 是单调减少的, 所以当 $t \geqslant t_1$ 时, 有$a(\tau(t))[-z'(\tau(t))]^{\beta-1}z'(\tau(t)) \geqslant a(t)[-z'(t)]^{\beta-1}z'(t)$, 即$a(\tau(t))[-z'(\tau(t))]^{\beta} \leqslant a(t)[-z'(t)]^{\beta}$. 进一步可得$z'(t) \leqslant \left(\frac{a(\tau(t))}{a(t)}\right)^{1/\beta} z'(\tau(t))$. 于是由式(8) (并注意到$z'(t) < 0$) 可得

$$\begin{aligned} v'(t) &= \frac{[a(\tau(t))\phi_1(z'(\tau(t)))]'}{z^{\beta}(t)} - \frac{a(\tau(t))(-z'(\tau(t)))^{\beta-1}z'(\tau(t))\beta z^{\beta-1}(t)z'(t)}{z^{2\beta}(t)} \\ &\leqslant \frac{[a(\tau(t))\phi_1(z'(\tau(t)))]'}{z^{\beta}(t)} - \frac{\beta a(\tau(t))(-z'(\tau(t)))^{\beta-1}z'(\tau(t))}{z^{\beta+1}(t)} \left(\frac{a(\tau(t))}{a(t)}\right)^{1/\beta} z'(\tau(t)) \\ &= \frac{[a(\tau(t))\phi_1(z'(\tau(t)))]'}{z^{\beta}(t)} - \frac{\beta(-v(t))^{(\beta+1)/\beta}}{a^{1/\beta}(t)}. \end{aligned} \tag{9}$$

再定义广义的 Riccati 变换$w(t)$为

$$w(t) = \frac{a(t)\phi_1(z'(t))}{\phi_1(z(t))} = \frac{a(t)(-z'(t))^{\beta-1}z'(t)}{z^{\beta}(t)}, \quad t \geqslant t_1,$$

则 $w(t) < 0$ $(t \geqslant t_1)$. 按与上面类似的方法可得

$$w'(t) \leqslant \frac{[a(t)\phi_1(z'(t))]'}{z^{\beta}(t)} - \frac{\beta(-w(t))^{(\beta+1)/\beta}}{a^{1/\beta}(t)}. \tag{10}$$

由文献 [4] 中定理 1 的证明已得下式:

$$[a(t)\phi_1(z'(t))]' + \frac{p_0^{\lambda}}{\tau_0}[a(\tau(t))\phi_1(z'(\tau(t)))]' \leqslant -L_0 Q(t) z^{\lambda}(\delta(t)) \leqslant 0. \tag{11}$$

于是, 利用式(9)—(11)并注意到 $z(\delta(t)) \geqslant z(t)$, 容易得到

$$\begin{aligned} w'(t) + \frac{p_0^{\lambda}}{\tau_0}v'(t) \leqslant& \frac{1}{z^{\beta}(t)} \left\{[a(t)\phi_1(z'(t))]' + \frac{p_0^{\lambda}}{\tau_0}[a(\tau(t))\phi_1(z'(\tau(t)))]'\right\} \\ &- \frac{\beta}{a^{1/\beta}(t)}\left[(-w(t))^{(\beta+1)/\beta} + \frac{p_0^{\lambda}}{\tau_0}(-v(t))^{(\beta+1)/\beta}\right] \\ \leqslant& -\frac{L_0 Q(t) z^{\lambda}(\delta(t))}{z^{\beta}(t)} - \frac{\beta}{a^{1/\beta}(t)}\left[(-w(t))^{(\beta+1)/\beta} + \frac{p_0^{\lambda}}{\tau_0}(-v(t))^{(\beta+1)/\beta}\right] \\ \leqslant& -L_0 Q(t) z^{\lambda-\beta}(t) - \frac{\beta}{a^{1/\beta}(t)}\left[(-w(t))^{(\beta+1)/\beta} + \frac{p_0^{\lambda}}{\tau_0}(-v(t))^{(\beta+1)/\beta}\right]. \end{aligned} \tag{12}$$

若 $\beta > \lambda$, 则根据 $z(t) > 0$, $z'(t) < 0$ $(t \geqslant t_1)$ 得, $z(t) \leqslant z(t_1)$, 即 $z^{\lambda-\beta}(t) \geqslant z^{\lambda-\beta}(t_1) = k$.

若 $\beta = \lambda$, 则 $z^{\lambda-\beta}(t) = 1$.

若 $\beta<\lambda$, 则利用 $a(t)(-z'(t))^{\beta-1}z'(t)$ $(t\geqslant t_1)$ 的单调减少性, 可得

$$a(s)(-z'(s))^{\beta-1}z'(s)\leqslant a(t_1)(-z'(t_1))^{\beta-1}z'(t_1)=-M,\quad s\geqslant t_1,$$

其中常数 $M=-a(t_1)(-z'(t_1))^{\beta-1}z'(t_1)>0$. 于是 $a(s)(-z'(s))^{\beta}\geqslant M$, 即 $z'(s)\leqslant -M^{1/\beta}a^{-1/\beta}(s)$. 进一步就有 $z(T)-z(t)\leqslant -M^{1/\beta}\int_t^T a^{-1/\beta}(s)\mathrm{d}s$, 即

$$z(t)\geqslant z(T)+M^{1/\beta}\int_t^T a^{-1/\beta}(s)\mathrm{d}s\geqslant M^{1/\beta}\int_t^T a^{-1/\beta}(s)\mathrm{d}s.$$

在上式中令$T\to+\infty$, 得$z(t)\geqslant M^{1/\beta}\int_t^{+\infty}a^{-1/\beta}(s)\mathrm{d}s=M^{1/\beta}A(t)$, 即$z^{\lambda-\beta}(t)\geqslant kA^{\lambda-\beta}(t)$, 这里$k=M^{(\beta-\lambda)/\lambda}>0$是常数.

根据函数$R(t)$的定义, 有

$$z^{\lambda-\beta}(t)\geqslant R(t). \tag{13}$$

于是, 将式(13)代入式(12), 得

$$L_0Q(t)R(t)\leqslant -w'(t)-\frac{p_0^{\lambda}}{\tau_0}v'(t)-\frac{\beta}{a^{1/\beta}(t)}\left[(-w(t))^{(\beta+1)/\beta}+\frac{p_0^{\lambda}}{\tau_0}(-v(t))^{(\beta+1)/\beta}\right]. \tag{14}$$

将式(14)两边同乘以$(t-s)^{\omega}$, 并从t_1到t $(t\geqslant t_1)$ 积分, 再利用引理1中的不等式, 则可推得

$$\begin{aligned}
&\int_{t_1}^{t}L_0Q(s)R(s)(t-s)^{\omega}\mathrm{d}s\\
\leqslant&-\int_{t_1}^{t}(t-s)^{\omega}w'(s)\mathrm{d}s-\frac{p_0^{\lambda}}{\tau_0}\int_{t_1}^{t}(t-s)^{\omega}v'(s)\mathrm{d}s\\
&-\int_{t_1}^{t}\frac{\beta(t-s)^{\omega}}{a^{1/\beta}(s)}\left[(-w(s))^{(\beta+1)/\beta}+\frac{p_0^{\lambda}}{\tau_0}(-v(s))^{(\beta+1)/\beta}\right]\mathrm{d}s\\
=&(t-t_1)^{\omega}w(t_1)+\int_{t_1}^{t}\left[\omega(t-s)^{\omega-1}(-w(s))-\frac{\beta(t-s)^{\omega}}{a^{1/\beta}(s)}(-w(s))^{\frac{\beta+1}{\beta}}\right]\mathrm{d}s\\
&+\frac{p_0^{\lambda}}{\tau_0}\left\{(t-t_1)^{\omega}v(t_1)+\int_{t_1}^{t}\left[\omega(t-s)^{\omega-1}(-v(s))-\frac{\beta(t-s)^{\omega}}{a^{1/\beta}(s)}(-v(s))^{\frac{\beta+1}{\beta}}\right]\mathrm{d}s\right\}\\
\leqslant&(t-t_1)^{\omega}w(t_1)+\int_{t_1}^{t}\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta-\omega+1}}\mathrm{d}s\\
&+\frac{p_0^{\lambda}}{\tau_0}\left\{(t-t_1)^{\omega}v(t_1)+\int_{t_1}^{t}\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta-\omega+1}}\mathrm{d}s\right\}.
\end{aligned}$$

由上式进一步得

$$\begin{aligned}
&\int_{t_1}^{t}\left[L_0Q(s)R(s)(t-s)^{\omega}-\left(1+\frac{p_0^{\lambda}}{\tau_0}\right)\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta-\omega+1}}\right]\mathrm{d}s\\
&\leqslant(t-t_1)^{\omega}w(t_1)+\frac{p_0^{\lambda}}{\tau_0}(t-t_1)^{\omega}v(t_1),
\end{aligned}$$

这与条件(7)矛盾！定理证毕.

考虑集合$D=\{(t,s)|t\geqslant s\geqslant t_0\}$和集合$D_0=\{(t,s)|t>s\geqslant t_0\}$, 称函数$K(t,s)\in\Omega$, 如果$K(t,s)\in C(D,\mathbf{R})$, 并且当$t\geqslant t_0$时$K(t,t)=0$; 当$(t,s)\in D_0$时$K(t,s)>0$且$K(t,s)$对变量$s$有连续非正的偏导数.

下面给出判别方程(1)振荡的一般性定理.

定理 2 设条件(C_1)—(C_3)及式(6)成立, 并且当$\beta \leqslant \lambda$时(4)式成立; 当 $\beta > \lambda$时(5)式成立. 如果存在函数$K \in \Omega$使得

$$\limsup_{t\to+\infty}\int_T^t \left[L_0Q(s)R(s)K(t,s)-\left(1+\frac{p_0^\lambda}{\tau_0}\right)\frac{a(s)}{(\beta+1)^{\beta+1}K^\beta(t,s)}\left|\frac{\partial K(t,s)}{\partial s}\right|^{\beta+1}\right]\mathrm{d}s>0. \quad (15)$$

其中: 常数 $T \geqslant t_0$ 足够大; 常数 L_0 及函数 $Q(t)$, $R(t)$ 和 $A(t)$ 如定理 1. 则方程(1)是振荡的.

证 明 证明的前面部分同定理1, 函数$v(t)$和$w(t)$的定义也同定理 1, 则可得式(14), 即当$s \geqslant t_1$时, 有

$$L_0Q(s)R(s)\leqslant -w'(s)-\frac{p_0^\lambda}{\tau_0}v'(s)-\frac{\beta}{a^{1/\beta}(s)}\left[(-w(s))^{(\beta+1)/\beta}+\frac{p_0^\lambda}{\tau_0}(-v(s))^{(\beta+1)/\beta}\right].$$

将上式两边同乘以 $K(t,s)$, 并从 t_1 到 t $(t\geqslant t_1)$ 积分, 再利用引理1中的不等式, 则可推得

$$\begin{aligned}
&\int_{t_1}^t L_0Q(s)R(s)K(t,s)\mathrm{d}s\\
\leqslant&-\int_{t_1}^t K(t,s)w'(s)\mathrm{d}s-\frac{p_0^\lambda}{\tau_0}\int_{t_1}^t K(t,s)v'(s)\mathrm{d}s\\
&-\int_{t_1}^t\frac{\beta K(t,s)}{a^{1/\beta}(s)}\left[(-w(s))^{(\beta+1)/\beta}+\frac{p_0^\lambda}{\tau_0}(-v(s))^{(\beta+1)/\beta}\right]\mathrm{d}s\\
=&K(t,t_1)w(t_1)+\int_{t_1}^t\left[\left(-\frac{\partial K(t,s)}{\partial s}\right)(-w(s))-\frac{\beta K(t,s)}{a^{1/\beta}(s)}(-w(s))^{\frac{\beta+1}{\beta}}\right]\mathrm{d}s\\
&+\frac{p_0^\lambda}{\tau_0}\left\{K(t,t_1)v(t_1)+\int_{t_1}^t\left[\left(-\frac{\partial K(t,s)}{\partial s}\right)(-v(s))-\frac{\beta K(t,s)}{a^{1/\beta}(s)}(-v(s))^{\frac{\beta+1}{\beta}}\right]\mathrm{d}s\right\}\\
\leqslant&K(t,t_1)w(t_1)+\int_{t_1}^t\frac{a(s)}{(\beta+1)^{\beta+1}K^\beta(t,s)}\left|\frac{\partial K(t,s)}{\partial s}\right|^{\beta+1}\mathrm{d}s\\
&+\frac{p_0^\lambda}{\tau_0}\left\{K(t,t_1)v(t_1)+\int_{t_1}^t\frac{a(s)}{(\beta+1)^{\beta+1}K^\beta(t,s)}\left|\frac{\partial K(t,s)}{\partial s}\right|^{\beta+1}\mathrm{d}s\right\}.
\end{aligned}$$

由上式进一步得

$$\begin{aligned}
&\int_{t_1}^t\left[L_0Q(s)R(s)K(t,s)-\left(1+\frac{p_0^\lambda}{\tau_0}\right)\frac{a(s)}{(\beta+1)^{\beta+1}K^\beta(t,s)}\left|\frac{\partial K(t,s)}{\partial s}\right|^{\beta+1}\right]\mathrm{d}s\\
\leqslant&K(t,t_1)w(t_1)+\frac{p_0^\lambda}{\tau_0}K(t,t_1)v(t_1),
\end{aligned}$$

这与条件(15)矛盾! 定理证毕.

2 例子分析

例 1 考虑欧拉方程(E_1), 即下列方程

$$(t^2x'(t))'+mx(t)=0,\quad t\geqslant 1,$$

其中常数 $m>0$. 这相当于在方程(1)取中 $a(t)=t^2$, $p(t)\equiv 0$, $q(t)=m$, $\tau(t)=\delta(t)=t$, $f(u)=u$, $\lambda=\beta=1$, $t_0=1$. 容易验证条件(C_1)—(C_3)及式(6)都满足. 在定理1 中取$\varphi(t)=t$, $\omega=1$, $T=2$, 注意到$L=1$, $R(t)=1$. 则当$m>\frac{1}{4}$时, 有

$$\begin{aligned}&\limsup_{t\to+\infty}\int_T^t\varphi(s)\left[L_0Q(s)\Psi^\lambda(s,t_0)-\frac{(\beta/\lambda)^\beta}{\eta^{\lambda-\beta}(\beta+1)^{\beta+1}}\left(a(s)+\frac{p_0^\lambda a(\tau(s))}{\tau_0^{\beta+1}}\right)\left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\beta+1}\right]\mathrm{d}s\\&=\limsup_{t\to+\infty}\int_2^t s\left[m-\frac{1}{4}\right]\mathrm{d}s=+\infty,\end{aligned}$$

且

$$\begin{aligned}&\limsup_{t\to+\infty}\int_T^t(t-s)^\omega\left[L_0Q(s)R(s)-\left(1+\frac{p_0^\lambda}{\tau_0}\right)\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta+1}}\right]\mathrm{d}s\\&=\limsup_{t\to+\infty}\int_2^t(t-s)\left[m-\frac{s^2}{4(t-s)^2}\right]\mathrm{d}s>0,\end{aligned}$$

于是由定理 1 知, 当 $m>\frac{1}{4}$ 时方程(E_1)是振荡的. 显然这是一个确定的结果, 而且与我们所熟知的结果是一致的.

例 2 考虑方程(E_2), 即下列方程

$$\{t|[x(t)+(9+\sin 2t)x(t/2)]'|^{-1/2}[x(t)+(9+\sin 2t)x(t/2)]'\}'+t^4x^3(t/3)=0,\quad t\geqslant 1.$$

这相当于在方程(1)中取$a(t)=t$, $p(t)=9+\sin 2t$, $q(t)=t^4$, $\tau(t)=\frac{t}{2}$, $\delta(t)=\frac{t}{3}$, $f(u)=u$, $\beta=\frac{1}{2}$, $\lambda=3$, $t_0=1$. 容易验证条件(C_1)—(C_3)及式(6)都是满足的. 在定理1 中取$\varphi(t)=1$, $\omega=1$, $T=3$, 注意到$\beta<\lambda$, $L=\frac{1}{4}$, 且

$$\begin{aligned}&A(t)=\int_t^{+\infty}a^{-1/\beta}(s)\mathrm{d}s=\int_t^{+\infty}s^{-2}\mathrm{d}s=\frac{1}{t},\quad Q(t)=\min\{q(t),q(\tau(t))\}=\frac{t^4}{16},\\&R(t)=kA^{\lambda-\beta}(t)=\frac{k}{t^{5/2}},\quad \Psi(t,t_0)=\left(\int_{t_0}^{\delta(t)}a^{-1/\beta}(s)\mathrm{d}s\right)\left(\int_{t_0}^t a^{-1/\beta}(s)\mathrm{d}s\right)^{-1}=\frac{t-3}{t-1}.\end{aligned}$$

则有

$$\begin{aligned}&\limsup_{t\to+\infty}\int_T^t\varphi(s)\left[L_0Q(s)\Psi^\lambda(s,t_0)-\frac{(\beta/\lambda)^\beta}{\eta^{\lambda-\beta}(\beta+1)^{\beta+1}}\left(a(s)+\frac{p_0^\lambda a(\tau(s))}{\tau_0^{\beta+1}}\right)\left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\beta+1}\right]\mathrm{d}s\\&=\limsup_{t\to+\infty}\int_3^t\left[\frac{1}{4}\cdot\frac{s^4}{16}\left(\frac{s-3}{s-1}\right)^3-0\right]\mathrm{d}s=+\infty,\end{aligned}$$

且

$$\begin{aligned}&\limsup_{t\to+\infty}\int_T^t(t-s)^\omega\left[L_0Q(s)R(s)-\left(1+\frac{p_0^\lambda}{\tau_0}\right)\frac{\omega^{\beta+1}a(s)}{(\beta+1)^{\beta+1}(t-s)^{\beta+1}}\right]\mathrm{d}s\\&=\limsup_{t\to+\infty}\int_3^t(t-s)\left[\frac{1}{4}\cdot\frac{s^4}{16}\cdot\frac{k}{s^{5/2}}-\left(1+\frac{10^3}{1/2}\right)\frac{s}{(3/2)^{3/2}(t-s)^{3/2}}\right]\mathrm{d}s>0.\end{aligned}$$

于是由定理1知, 方程(E_2)是振荡的.

注 1 非常明显, 文献 [1-10] 中的定理都不能判别方程(E_1)和(E_2)是否振荡.

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