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一类具有潜伏感染细胞的时滞 HIV-1 传染病模型

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摘要: 提出了一类具有潜伏感染细胞的时滞 HIV-1 传染病模型, 定义了基本再生数R_0, 给出了无病平衡点$P_0(x_0, 0, 0, 0)$和慢性感染平衡点$P^*(x^*, \omega^*, y^*, v^*)$的存在条件. 首先利用线性化方法, 得到了无病平衡点和慢性感染平衡点的局部渐近稳定性. 进一步通过构造相应的Lyapunov 函数, 并结合 LaSalle 不变集原理, 证明了当$R_0 \leqslant 1$时, 无病平衡点$P_0(x_0, 0, 0, 0)$是全局渐近稳定的; 当$R_0 > 1$时, 慢性感染平衡点$P^*(x^*, \omega^*, y^*, v^*)$是全局渐近稳定的, 但无病平衡点$P_0(x_0, 0, 0, 0)$是不稳定的. 结果表明, 模型中的潜伏感染时滞和感染时滞并不影响模型的全局稳定性, 并通过数值模拟验证了所得结论.

关键词: HIV-1 传染病模型; 潜伏感染细胞; 时滞; Lyapunov 函数

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A class of delayed HIV-1 infection models with latently infected cells

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Abstract: A class of delayed HIV-1 infection models with latently infected cells was proposed. The basic reproductive number R_0 was defined, and the existence conditions of disease-free equilibrium $P_0(x_0, 0, 0, 0)$ and chronic-infection equilibrium $P^*(x^*, \omega^*, y^*, v^*)$ were given. First, the local asymptotic stability of infection-free equilibrium and chronic-infection equilibrium was obtained by the linearization method. Further, by constructing the corresponding Lyapunov functions and using LaSalle's invariant principle, it was proved that when the basic reproductive number R_0 was less than or equal to unity, the infection-free equilibrium $P_0(x_0, 0, 0, 0)$ was globally asymptotically stable; moreover, when the basic reproductive number R_0 was greater than unity, the chronic-infective equilibrium $P^*(x^*, \omega^*, y^*, v^*)$ was globally asymptotically stable, but the infection-free equilibrium $P_0(x_0, 0, 0, 0)$ was unstable. The results showed that a latently infected delay and an intracellular delay did not affect the global stability of the model, and numerical simulations were carried out to illustrate the theoretical results.

Keywords: HIV-1 infection model; latently infected cells; delay; Lyapunov function

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0 引　言

艾滋病(AIDS)是一类严重威胁人类健康和生命的传染病, 目前已成为全球重要的公共健康问题. 艾滋病病毒(Human Immunodeficiency Virus, 简称 HIV)主要感染人体免疫系统细胞 CD4+T, 可引起细胞 CD4+T 计数的大幅度下降, 导致人体免疫缺陷, 严重影响患者防御机会性感染的能力[1]. HIV 分为两种类型: HIV-1型和HIV-2型, 其中 HIV-1 是引发艾滋病的主要病原体. 国内外已有很多医学和数学等各方面的工作者投入到艾滋病的防治研究中, 其中借助数学模型来分析艾滋病病毒感染的动力学行为已成为一个热点研究问题[2]. 对于 HIV-1 感染的研究, Perelson、Anderson 等提出了最初的模型[3-6]:

$$\begin{cases} \dot{x}(t) = \lambda - dx(t) - \beta x(t)v(t), \\ \dot{y}(t) = \beta x(t)v(t) - ay(t), \\ \dot{v}(t) = ky(t) - uv(t). \end{cases} \tag{1}$$

其中: $x(t), y(t), v(t)$分别表示t时刻 CD4+T 未感染细胞数、CD4+T 感染细胞数、病毒数; 参数λ表示未感染细胞固有生成率; β表示病毒感染率; d, a, u分别表示未感染细胞、被感染细胞、游离病毒的死亡率; k表示被感染细胞释放病毒的比率. $\lambda, \beta, d, a, u, k$均为正数.

在模型(1)中, 发生率被假设为: t时刻未感染细胞 CD4+T 个数$x(t)$和病毒数$v(t)$之间是双线性的, 然而实际发生率可能不是严格线性的[7-11]. Song 和 Xu 等[7,10-11]提出具有饱和发生率$\frac{\beta xv}{1+\alpha v}$ $(\alpha > 0)$ 的传染病模型. 然而, 以上提到的模型均忽略了一个事实, 即在细胞中并不是所有的病毒都能启动活性病毒的产生. 一部分 CD4+T 细胞在被病毒激活感染后, 进入染病阶段, 但还有一部分 CD4+T 细胞在被激活之后长时间保持静止, 仍然保留在潜伏期[12], 在文献[13]中, 这种细胞被定义为潜伏感染细胞. HIV-1持续潜伏在 CD4+T 细胞内的这种特性目前被认为是细胞从感染中恢复的障碍. 但到目前为止, 关注潜伏感染细胞对 HIV-1 感染过程影响的模型并不多见[14-15]. 其中 Wang[15]讨论了一类具有潜伏感染细胞和饱和发生率的 HIV-1 传染病模型:

$$\begin{cases} \dot{x}(t) = \lambda - dx(t) - \dfrac{\beta x(t)v(t)}{1+\alpha v(t)}, \\ \dot{\omega}(t) = \dfrac{(1-q)\beta x(t)v(t)}{1+\alpha v(t)} - e\omega(t) - \delta\omega(t), \\ \dot{y}(t) = \dfrac{q\beta x(t)v(t)}{1+\alpha v(t)} - ay(t) + \delta\omega(t), \\ \dot{v}(t) = ky(t) - uv(t). \end{cases} \tag{2}$$

其中: $\omega(t)$ 表示 t 时刻的潜伏感染细胞数量; e 表示潜伏感染细胞的死亡率; δ表示潜伏感染细胞转化为感染细胞的速率; e、δ 均为正数. 在该模型中, 假设未感染细胞被病毒激活后, 以速率$\frac{q\beta x(t)v(t)}{1+\alpha v(t)}$产生感染细胞, 而以速率$\frac{(1-q)\beta x(t)v(t)}{1+\alpha v(t)}$保持潜伏感染, 其中$0 < q < 1$, $\alpha > 0$.

本文考虑了具有潜伏感染细胞和饱和发生率的时滞HIV-1传染病模型:

$$\begin{cases} \dot{x}(t) = \lambda - dx(t) - \dfrac{\beta x(t)v(t)}{1+\alpha v(t)}, \\ \dot{\omega}(t) = \dfrac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)} - e\omega(t) - \delta\omega(t), \\ \dot{y}(t) = \dfrac{q\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)} - ay(t) + \delta\omega(t), \\ \dot{v}(t) = ky(t) - uv(t). \end{cases} \tag{3}$$

其中: 时滞τ_1表示 CD4+T 细胞与病毒接触使其成为潜伏感染细胞所需要的时间; 时滞τ_2表示 CD4+T 细胞与病毒接触使其被感染所需要的时间; $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$; 其他参数的生物学意义同上.

系统(3)满足初始条件:

$$\begin{cases} x(\theta) = \phi_1(\theta), \quad \omega(\theta) = \phi_2(\theta), \quad y(\theta) = \phi_3(\theta), \quad v(\theta) = \phi_4(\theta), \\ \phi_i(\theta) \geqslant 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0 \ (i = 1,2,3,4). \end{cases} \tag{4}$$

其中: $\tau = \max\{\tau_1, \tau_2\}$; $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)) \in C([-\tau, 0], \mathbf{R}^4_{+0})$, 表示从区间$[-\tau, 0]$到$\mathbf{R}^4_{+0}$且具有上确界范数的 Banach 空间的连续泛函; $\mathbf{R}^4_{+0} = \{(x_1, x_2, x_3, x_4) : x_i \geqslant 0, i = 1,2,3,4\}$. 由泛函微分方程的基本理论知识[16] 可知, 系统(3)存在满足初始条件(4)的唯一解$(x(t), \omega(t), y(t), v(t))$, 且对任意$t \geqslant 0$, 都有$x(t) > 0, \omega(t) > 0, y(t) > 0, v(t) > 0$.

1　平衡点的存在性

显然, 系统(3)总有一个无病平衡点$P_0(x_0, 0, 0, 0)$, 其中$x_0 = \frac{\lambda}{d}$. 定义基本再生数

$$R_0 = \frac{k\lambda\beta(eq+\delta)}{adu(e+\delta)}.$$

当 $R_0 > 1$ 时, 系统(3)有唯一的慢性感染平衡点$P^*(x^*, \omega^*, y^*, v^*)$, 其中

$$x^* = \frac{\lambda(1+\alpha v^*)}{d(1+\alpha v^*) + \beta v^*}, \quad \omega^* = \frac{\lambda\beta(1-q)v^*}{(e+\delta)[d(1+\alpha v^*) + \beta v^*]}, \quad y^* = \frac{uv^*}{k}, \quad v^* = \frac{d}{\beta + \alpha d}(R_0 - 1).$$

2　平衡点的局部稳定性

关于平衡点的局部稳定性, 我们讨论以下三种情况, 即(i) $\tau_1 = 0, \tau_2 \neq 0$; (ii) $\tau_1 \neq 0, \tau_2 = 0$; (iii) $\tau_1 = \tau_2 \neq 0$.

定理 1　(I) 当$R_0 < 1$, 且满足(i)、(ii)、(iii)中任一种情况时, 系统(3)的无病平衡点$P_0(x_0, 0, 0, 0)$均局部渐近稳定; (II) 当$R_0 > 1$时, $P_0(x_0, 0, 0, 0)$是不稳定的.

证　明　(I) 系统(3)在$P_0(x_0, 0, 0, 0)$处的特征方程为

$$(s+d)\left[s^3 + p_2 s^2 + p_1 s + p_0 + r_0 \mathrm{e}^{-s\tau_1} + (q_1 s + q_0)\mathrm{e}^{-s\tau_2}\right] = 0. \tag{5}$$

其中:

$$p_0 = au(e+\delta); \quad p_1 = (e+\delta)(a+u) + au; \quad p_2 = e + \delta + a + u;$$
$$r_0 = -\delta(1-q)k\beta x_0; \quad q_0 = -(e+\delta)qk\beta x_0; \quad q_1 = -qk\beta x_0.$$

显然, 方程(5)总有负实根$s_1=-d$. 因此方程(5)其余的根取决于方程

$$f(s)=s^3+p_2s^2+p_1s+p_0+r_0\mathrm{e}^{-s\tau_1}+(q_1s+q_0)\mathrm{e}^{-s\tau_2}=0. \tag{6}$$

下面分三种情形来讨论.

(i) $\tau_1=0,\tau_2\neq 0$

此时, 方程(6)变为

$$f(s)=s^3+p_2s^2+p_1s+p_0+r_0+(q_1s+q_0)\mathrm{e}^{-s\tau_2}=0, \tag{7}$$

令$s=\mathrm{i}\omega$ $(\omega>0)$ 是方程(7)的一个纯虚根, 将其实部与虚部分离得

$$\begin{cases}\omega^3-p_1\omega=q_1\omega\cos(\omega\tau_2)-q_0\sin(\omega\tau_2),\\ p_2\omega^2-p_0-r_0=q_1\omega\sin(\omega\tau_2)+q_0\cos(\omega\tau_2).\end{cases} \tag{8}$$

再将方程组(8)的两个方程分别平方后相加得

$$\omega^6+(p_2^2-2p_1)\omega^4+[p_1^2-2(p_0+r_0)p_2-q_1^2]\omega^2+(p_0+r_0)^2-q_0^2=0. \tag{9}$$

令$\omega^2=z$, 则方程(9) 变为

$$z^3+(p_2^2-2p_1)z^2+[p_1^2-2(p_0+r_0)p_2-q_1^2]z+(p_0+r_0)^2-q_0^2=0. \tag{10}$$

由于

$$\begin{aligned}p_2^2-2p_1&=(e+\delta+a+u)^2-2[(e+\delta)(a+u)+au]=(e+\delta)^2+a^2+u^2>0,\\ p_1^2-2(p_0+r_0)p_2-q_1^2&=(e+\delta)^2(a^2+u^2)+2(1-q)\delta(e+\delta+a+u)k\beta x_0\\ &\quad+(au)^2-(qk\beta x_0)^2\\ &>(e+\delta)^2(a^2+u^2)+(au)^2(1-R_0^2)>0,\\ p_0^2-q_0^2&=[au(e+\delta)]^2-[(e+\delta)qk\beta x_0]^2>[au(e+\delta)]^2(1-R_0^2)>0,\end{aligned}$$

$$\begin{aligned}&(p_2^2-2p_1)(p_1^2-2p_0p_2-q_1^2)-(p_0^2-q_0^2)\\ =&\left[(e+\delta)^2+a^2+u^2\right]\cdot\left[(e+\delta)^2(a^2+u^2)+2(1-q)\delta(e+\delta+a+u)k\beta x_0\right]\\ &+(a^2+u^2)\left[(au)^2-(qk\beta x_0)^2\right]>0.\end{aligned}$$

由 Routh-Hurwitz 准则得, 方程(10)的所有特征值均具有负实部, 这与$z=\omega^2>0$相矛盾. 因此方程(7)的任意根均具有负实部. 所以, 当$\tau_1=0,\tau_2\neq 0$时, 无病平衡点$P_0(x_0,0,0,0)$是局部渐近稳定的.

(ii) $\tau_1\neq 0,\tau_2=0$

此时, 方程(6)变为

$$f(s)=s^3+p_2s^2+(p_1+q_1)s+p_0+q_0+r_0\mathrm{e}^{-s\tau_1}=0. \tag{11}$$

令$s=\mathrm{i}\omega$ $(\omega>0)$ 是方程(11)的一个纯虚根, 将其实部与虚部分离得

$$\begin{cases}\omega^3-(p_1+q_1)\omega=r_0\sin(\omega\tau_1),\\ -p_2\omega^2+p_0+q_0=r_0\cos(\omega\tau_1).\end{cases} \tag{12}$$

再将方程组(12)的两个方程分别平方后相加得

$$\omega^6+\left[p_2^2-2(p_1+q_1)\right]\omega^4+\left[(p_1+q_1)^2-2(p_0+q_0)p_2\right]\omega^2+(p_0+q_0)^2-r_0^2=0. \tag{13}$$

令$\omega^2=z$, 则方程(13)变为

$$z^3+\left[p_2^2-2(p_1+q_1)\right]z^2+\left[(p_1+q_1)^2-2(p_0+q_0)p_2\right]z+(p_0+q_0)^2-r_0^2=0. \tag{14}$$

由于

$$p_2^2-2(p_1+q_1)=(e+\delta)^2+a^2+u^2+2qk\beta x_0>0,$$

$$\begin{aligned}(p_1+q_1)^2-2(p_0+q_0)p_2&=(e+\delta)^2(a^2+u^2)+(au)^2+(qk\beta x_0)^2\\&\quad-2auqk\beta x_0+2(e+\delta)^2qk\beta x_0\\&\geqslant(e+\delta)^2(a^2+u^2)+2(e+\delta)^2qk\beta x_0>0,\\(p_0+q_0)^2-r_0^2&=(p_0+q_0+r_0)(p_0+q_0-r_0)>\left[au(e+\delta)(1-R_0)\right]^2,\\\left[p_2^2-2(p_1+q_1)\right]&\cdot\left[(p_1+q_1)^2-2(p_0+q_0)p_2\right]-\left[(p_0+q_0)^2-r_0^2\right]>0.\end{aligned}$$

因此, 当$\tau_1\neq0,\tau_2=0$时, 无病平衡点$P_0(x_0,0,0,0)$是局部渐近稳定的.

(iii) $\tau_1=\tau_2\neq0$

令 $\tau_1=\tau_2=\overline{\tau}$ ($\overline{\tau}\neq0$), 此时, 方程(6)变为

$$f(s)=s^3+p_2s^2+p_1s+p_0+(q_1s+q_0+r_0)\mathrm{e}^{-s\bar{\tau}}=0, \tag{15}$$

令$s=\mathrm{i}\omega$ ($\omega>0$)是方程(15)的一个纯虚根, 将其实部与虚部分离得

$$\begin{cases}\omega^3-p_1\omega=q_1\omega\cos(\omega\bar{\tau})-(q_0+r_0)\sin(\omega\bar{\tau}),\\p_2\omega^2-p_0=q_1\omega\sin(\omega\bar{\tau})+(q_0+r_0)\cos(\omega\bar{\tau}).\end{cases} \tag{16}$$

再将方程组(16)的两个方程分别平方后相加得

$$\omega^6+(p_2^2-2p_1)\omega^4+(p_1^2-2p_0p_2-q_1^2)\omega^2+p_0^2-(q_0+r_0)^2=0. \tag{17}$$

令$\omega^2=z$, 则方程(17) 变为

$$z^3+(p_2^2-2p_1)z^2+(p_1^2-2p_0p_2-q_1^2)z+p_0^2-(q_0+r_0)^2=0. \tag{18}$$

由于

$$\begin{aligned}&p_2^2-2p_1=(e+\delta)^2+a^2+u^2>0,\\&p_1^2-2p_0p_2-q_1^2=(e+\delta)^2(a^2+u^2)+(au)^2-(qk\beta x_0)^2>(e+\delta)^2(a^2+u^2)\\&\qquad\qquad+(au)^2(1-R_0^2)>0,\\&p_0^2-(q_0+r_0)^2=a^2u^2(e+\delta)^2(1-R_0^2)>0,\\&(p_2^2-2p_1)(p_1^2-2p_0p_2-q_1^2)-\left[p_0^2-(q_0+r_0)^2\right]>0.\end{aligned}$$

因此, 当$\tau_1=\tau_2\neq 0$时, 无病平衡点$P_0(x_0,0,0,0)$均局部渐近稳定.

综上可知, 当$R_0<1$, 且满足(i)、(ii)、(iii)中任一种情况时, 系统(3)的无病平衡点$P_0(x_0,\,0,\,0,\,0)$是局部渐近稳定的.

(II) 当$R_0>1$时, 对实数s,

$$f(0)=au(e+\delta)-k\beta x_0(eq+\delta)=au(e+\delta)(1-R_0)<0,$$
$$\lim_{s\to+\infty}f(s)=+\infty.$$

因此, 方程$f(s)=0$至少有一个正实根. 所以, 当$R_0>1$时, 无病平衡点$P_0(x_0,0,0,0)$是不稳定的.

定理 2 当$R_0>1$, 且满足(i)、(ii)、(iii)中任一种情况时, 系统(3)的慢性感染平衡点$P^*(x^*,\,\omega^*,\,y^*,\,v^*)$是局部渐近稳定的.

证 明 系统(3)在$P^*(x^*,\,\omega^*,\,y^*,\,v^*)$处的特征方程为

$$s^4+a_3s^3+a_2s^2+a_1s+a_0+(b_1s+b_0)\mathrm{e}^{-s\tau_1}+(c_2s^2+c_1s+c_0)\mathrm{e}^{-s\tau_2}=0. \tag{19}$$

其中:

$$a_0=au(e+\delta)\left(d+\frac{\beta v^*}{1+\alpha v^*}\right);\quad a_1=au(e+\delta)+[(e+\delta)(a+u)+au]\left(d+\frac{\beta v^*}{1+\alpha v^*}\right);$$
$$a_2=(e+\delta)(a+u)+au+(e+\delta+a+u)\left(d+\frac{\beta v^*}{1+\alpha v^*}\right);$$
$$a_3=e+\delta+a+u+d+\frac{\beta v^*}{1+\alpha v^*};$$
$$b_0=\frac{-d\delta(1-q)k\beta x^*}{(1+\alpha v^*)^2};\quad b_1=\frac{-\delta(1-q)k\beta x^*}{(1+\alpha v^*)^2};$$
$$c_0=\frac{-d(e+\delta)qk\beta x^*}{(1+\alpha v^*)^2};\quad c_1=\frac{-(d+e+\delta)qk\beta x^*}{(1+\alpha v^*)^2};\quad c_2=\frac{-d(e+\delta)qk\beta x^*}{(1+\alpha v^*)^2}.$$

下面分三种情形来讨论.

(i) $\tau_1=0,\tau_2\neq 0$

此时, 方程(19)变为

$$s^4+a_3s^3+a_2s^2+(a_1+b_1)s+a_0+b_0+(c_2s^2+c_1s+c_0)\mathrm{e}^{-s\tau_2}=0. \tag{20}$$

令$s=\mathrm{i}\omega\ (\omega>0)$是方程(20)的一个纯虚根, 分离其实部与虚部得

$$\begin{cases}\omega^4-a_2\omega^2+(a_0+b_0)=-c_1\omega\sin(\omega\tau_2)+(c_2\omega^2-c_0)\cos(\omega\tau_2),\\ -a_3\omega^3+(a_1+b_1)\omega=-c_1\omega\cos(\omega\tau_2)-(c_2\omega^2-c_0)\sin(\omega\tau_2).\end{cases} \tag{21}$$

将方程组(21)的两个方程分别平方后再相加得

$$\omega^8+m_3\omega^6+m_2\omega^4+m_1\omega^2+m_0=0. \tag{22}$$

其中:

$$m_0=(a_0+b_0)^2-c_0^2;\quad m_1=(a_1+b_1)^2-2(a_0+b_0)a_2-c_1^2+2c_0c_2;$$
$$m_2=a_2^2+2(a_0+b_0)-2(a_1+b_1)a_3-c_2^2;$$
$$m_3=a_3^2-2a_2=(e+\delta)^2+a^2+u^2+\left(d+\frac{\beta v^*}{1+\alpha v^*}\right)^2.$$

令$\omega^2=z$, 则方程(22)变为

$$z^4+m_3z^3+m_2z^2+m_1z+m_0=0. \tag{23}$$

经计算,

$$\Delta_1=m_3>0,\quad \Delta_2=\begin{vmatrix} m_3 & 1\\ m_1 & m_2\end{vmatrix}>0,\quad \Delta_3=\begin{vmatrix} m_3 & 1 & 0\\ m_1 & m_2 & m_3\\ 0 & m_0 & m_1\end{vmatrix}>0,\quad \Delta_4=m_0\Delta_3>0.$$

由 Routh-Hurwitz 判别准则知, 方程(23)的所有特征值均具有负实部, 这与$z=\omega^2>0$相矛盾, 于是方程(20)的任意根均具有负实部. 所以, 当$\tau_1=0,\tau_2\neq 0$时, 慢性感染平衡点$P^*(x^*,\omega^*,y^*,v^*)$局部渐近稳定.

(ii) $\tau_1\neq 0,\tau_2=0$

此时, 方程(19)变为

$$s^4+a_3s^3+(a_2+c_2)s^2+(a_1+c_1)s+a_0+c_0+(b_1s+b_0)\mathrm{e}^{-s\tau_1}=0. \tag{24}$$

类似于情形(i)的讨论可知, 方程(24)的任意根均具有负实部. 所以, 当$\tau_1\neq 0,\tau_2=0$时, 慢性感染平衡点$P^*(x^*,\omega^*,y^*,v^*)$局部渐近稳定.

(iii) $\tau_1=\tau_2\neq 0$

令$\tau_1=\tau_2=\bar{\tau}\ (\bar{\tau}\neq 0)$, 此时, 方程(19)变为

$$s^4+a_3s^3+a_2s^2+a_1s+a_0+\left[c_2s^2+(b_1+c_1)s+(b_0+c_0)\right]\mathrm{e}^{-s\bar{\tau}}=0. \tag{25}$$

类似于情形(i)的讨论可知, 方程(25)的任意根均具有负实部. 所以, 当$\tau_1=\tau_2\neq 0$时, 慢性感染平衡点$P^*(x^*,\omega^*,y^*,v^*)$局部渐近稳定.

综上可知, 当$R_0>1$, 且满足(i)、(ii)、(iii)中任一种情况时, 系统(3)的慢性感染平衡点$P^*(x^*,\omega^*,y^*,v^*)$是局部渐近稳定的.

3　平衡点的全局稳定性

定理 3　当$R_0\leqslant 1$时, 对任意$\tau_1\geqslant 0,\tau_2\geqslant 0$, 系统(3)的无病平衡点$P_0(x_0,0,0,0)$是全局渐近稳定的.

证　明　构造 Lyapunov 函数

$$\begin{aligned}V_1(t)=&x_0\left(\frac{x(t)}{x_0}-1-\ln\frac{x(t)}{x_0}\right)+\frac{k\lambda\beta\delta}{adu(e+\delta)}\omega(t)+\frac{k\lambda\beta}{adu}y(t)+\frac{\lambda\beta}{du}v(t)\\&+\frac{k\lambda\beta\delta}{adu(e+\delta)}\int_{t-\tau_1}^{t}\frac{(1-q)\beta x(\theta)v(\theta)}{1+\alpha v(\theta)}\mathrm{d}\theta+\frac{k\lambda\beta}{adu}\int_{t-\tau_2}^{t}\frac{q\beta x(\theta)v(\theta)}{1+\alpha v(\theta)}\mathrm{d}\theta.\end{aligned}$$

计算函数$V_1(t)$沿系统(3)的全导数, 得

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)=&\left(1-\frac{x_0}{x(t)}\right)\left(\lambda-dx(t)-\frac{\beta x(t)v(t)}{1+\alpha v(t)}\right)\\
&+\frac{k\lambda\beta\delta}{adu(e+\delta)}\left(\frac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)}-(e+\delta)\omega(t)\right)\\
&+\frac{k\lambda\beta}{adu}\left(\frac{q\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)}-ay(t)+\delta\omega(t)\right)+\frac{\lambda\beta}{du}[ky(t)-uv(t)]\\
&+\frac{k\lambda\beta\delta}{adu(e+\delta)}\cdot\frac{(1-q)\beta x(t)v(t)}{1+\alpha v(t)}-\frac{k\lambda\beta\delta}{adu(e+\delta)}\cdot\frac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)}\\
&+\frac{k\lambda\beta}{adu}\cdot\frac{q\beta x(t)v(t)}{1+\alpha v(t)}-\frac{k\lambda\beta}{adu}\cdot\frac{q\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)}.
\end{aligned}\tag{26}
$$

由于$\lambda=dx_0$, $R_0=\frac{k\lambda\beta(eq+\delta)}{adu(e+\delta)}$, 则式(26)变为

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)=&-\frac{d}{x(t)}\left(x(t)-x_0\right)^2+(R_0-1)\frac{\beta x(t)v(t)}{1+\alpha v(t)}+\left(\frac{1}{1+\alpha v(t)}-1\right)\beta x_0 v(t)\\
=&-\frac{d}{x(t)}\left(x(t)-x_0\right)^2+(R_0-1)\frac{\beta x(t)v(t)}{1+\alpha v(t)}-\frac{\alpha\beta x_0 v^2(t)}{1+\alpha v(t)}.
\end{aligned}
$$

若$R_0\leqslant 1$, $\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)\leqslant 0$, 当且仅当$x=x_0, v(t)=0$时, $\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)=0$. 当$v(t)=0$时, 由系统(3)的第四个方程得$y(t)=0$, 此时再由系统(3)的第三个方程得$\omega(t)=0$. 因此, 当且仅当$(x,\omega,y,v)=(x_0,0,0,0)$时, $\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)=0$, 所以系统(3)的最大正向不变集 M 是单点集$\{P_0(x_0,0,0,0)\}$. 因此, 由 LaSalle 不变集原理知, 当$R_0\leqslant 1$时, 无病平衡点$\{P_0\}$是全局渐近稳定的.

定 理 4 当$R_0>1$时, 对任意$\tau_1\geqslant 0, \tau_2\geqslant 0$, 系统(3)的慢性感染平衡点$P^*(x^*,\ \omega^*,\ y^*,\ v^*)$是全局渐近稳定的.

证 明 定义函数

$$
F:\mathbf{R}(>0)\to\mathbf{R}(\geqslant 0),\quad F(z)=z-1-\ln z,
$$

易知, 对$\forall z>0$, $F(z)\geqslant 0$, 且有$F_{\min}=F(1)=0$.

构造 Lyapunov 函数

$$
\begin{aligned}
V_2(t)=&x^*F\left(\frac{x(t)}{x^*}\right)+\frac{\delta}{eq+\delta}\omega^*F\left(\frac{\omega(t)}{\omega^*}\right)+\frac{e+\delta}{eq+\delta}y^*F\left(\frac{y(t)}{y^*}\right)+\frac{a(e+\delta)}{k(eq+\delta)}v^*F\left(\frac{v(t)}{v^*}\right)\\
&+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\int_{t-\tau_1}^{t}F\left(\frac{(1+\alpha v^*)x(\theta)v(\theta)}{x^*v^*(1+\alpha v(\theta))}\right)\mathrm{d}\theta\\
&+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\int_{t-\tau_2}^{t}F\left(\frac{(1+\alpha v^*)x(\theta)v(\theta)}{x^*v^*(1+\alpha v(\theta))}\right)\mathrm{d}\theta.
\end{aligned}
$$

计算函数$V_2(t)$沿系统(3)的全导数, 得

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) =& \left(1-\frac{x^*}{x(t)}\right)\left(\lambda - dx(t) - \frac{\beta x(t)v(t)}{1+\alpha v(t)}\right) \\
&+ \frac{\delta}{eq+\delta}\left(1-\frac{\omega^*}{\omega(t)}\right)\left(\frac{(1-q)\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)} - (e+\delta)\omega(t)\right) \\
&+ \frac{e+\delta}{eq+\delta}\left(1-\frac{y^*}{y(t)}\right)\left(\frac{q\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)} - ay(t) + \delta\omega(t)\right) \\
&+ \frac{a(e+\delta)}{k(eq+\delta)}\left(1-\frac{v^*}{v(t)}\right)[ky(t)-uv(t)] \\
&+ \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x(t)v(t)}{1+\alpha v(t)} - \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)} \\
&+ \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)} \\
&+ \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x(t)v(t)}{1+\alpha v(t)} - \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)} \\
&+ \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}.
\end{aligned}\tag{27}$$

将$\lambda = dx^* + \frac{\beta x^*v^*}{1+\alpha v^*}$代入式(27), 整理得

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) =& \left(1-\frac{x^*}{x(t)}\right)\left(-d\left(x(t)-x^*\right) + \frac{\beta x^*v^*}{1+\alpha v^*}\right) - \frac{\beta x(t)v(t)}{1+\alpha v(t)} + \frac{\beta x^*v(t)}{1+\alpha v(t)} \\
&- \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\omega^*}{\omega(t)}\cdot\frac{\beta x(t-\tau_1)v(t-\tau_1)}{1+\alpha v(t-\tau_1)} + \frac{\delta(e+\delta)}{eq+\delta}\omega^* \\
&- \frac{(e+\delta)q}{eq+\delta}\cdot\frac{y^*}{y(t)}\cdot\frac{\beta x(t-\tau_2)v(t-\tau_2)}{1+\alpha v(t-\tau_2)} + \frac{a(e+\delta)}{eq+\delta}y^* + \frac{\delta(e+\delta)}{eq+\delta}\cdot\frac{y^*}{y(t)}\omega(t) \\
&- \frac{a(e+\delta)u}{k(eq+\delta)}v(t) - \frac{a(e+\delta)}{(eq+\delta)}\cdot\frac{v^*}{v(t)}y(t) + \frac{a(e+\delta)}{k(eq+\delta)}uv^* \\
&+ \frac{\delta}{eq+\delta}\cdot\frac{(1-q)\beta x(t)v(t)}{1+\alpha v(t)} + \frac{e+\delta}{eq+\delta}\cdot\frac{q\beta x(t)v(t)}{1+\alpha v(t)} \\
&+ \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)} \\
&+ \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}.
\end{aligned}\tag{28}$$

由于

$$\frac{\delta}{eq+\delta}\cdot\frac{(1-q)\beta x(t)v(t)}{1+\alpha v(t)} + \frac{e+\delta}{eq+\delta}\cdot\frac{q\beta x(t)v(t)}{1+\alpha v(t)} = \frac{\beta x(t)v(t)}{1+\alpha v(t)},$$

所以式(28)可化为

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = & -\frac{d\left(x(t)-x^*\right)^2}{x(t)} + \frac{\beta x^* v^*}{1+\alpha v^*} - \frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{x^*}{x(t)} + \frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{(1+\alpha v^*)v(t)}{(1+\alpha v(t))v^*} \\
& - \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{\omega^*}{\omega(t)}\cdot\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)}{x^* v^*\left(1+\alpha v(t-\tau_1)\right)} + \frac{\delta(e+\delta)}{eq+\delta}\omega^* \\
& - \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)}{x^* v^*\left(1+\alpha v(t-\tau_2)\right)} + \frac{a(e+\delta)}{eq+\delta}y^* \\
& + \frac{\delta(e+\delta)\omega^*}{eq+\delta}\cdot\frac{y^*}{y(t)}\cdot\frac{\omega(t)}{\omega^*} - \frac{a(e+\delta)uv^*}{k(eq+\delta)}\cdot\frac{v(t)}{v^*} \\
& - \frac{ay^*(e+\delta)}{(eq+\delta)}\cdot\frac{v^*}{v(t)}\cdot\frac{y(t)}{y^*} + \frac{a(e+\delta)}{k(eq+\delta)}uv^* \\
& + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)} \\
& + \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}.
\end{aligned}$$

注意到

$$(e+\delta)\omega^* = \frac{(1-q)\beta x^* v^*}{1+\alpha v^*},\quad ay^* = \frac{q\beta x^* v^*}{1+\alpha v^*} + \delta\omega^*,\quad y^* = \frac{uv^*}{k},$$

则

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = & -\frac{d\left(x(t)-x^*\right)^2}{x(t)} + \frac{\beta x^* v^*}{1+\alpha v^*} - \frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{x^*}{x(t)} + \frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{(1+\alpha v^*)v(t)}{(1+\alpha v(t))v^*} \\
& - \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{\omega^*}{\omega(t)}\cdot\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)}{x^* v^*\left(1+\alpha v(t-\tau_1)\right)} + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*} \\
& - \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)}{x^* v^*\left(1+\alpha v(t-\tau_2)\right)} + \frac{e+\delta}{eq+\delta}\cdot\left(\frac{q\beta x^* v^*}{1+\alpha v^*}+\delta\omega^*\right) \\
& + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{\omega(t)}{\omega^*} - \frac{e+\delta}{eq+\delta}\cdot\left(\frac{q\beta x^* v^*}{1+\alpha v^*}+\delta\omega^*\right)\cdot\frac{v(t)}{v^*} \\
& - \frac{e+\delta}{eq+\delta}\cdot\left(\frac{q\beta x^* v^*}{1+\alpha v^*}+\delta\omega^*\right)\cdot\frac{v^*}{v(t)}\cdot\frac{y(t)}{y^*} + \frac{e+\delta}{eq+\delta}\cdot\left(\frac{q\beta x^* v^*}{1+\alpha v^*}+\delta\omega^*\right) \\
& + \frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)} \\
& + \frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^* v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{d\left(x(t)-x^*\right)^2}{x(t)}+\frac{3\beta x^*v^*}{1+\alpha v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{x^*}{x(t)}+\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{(1+\alpha v^*)v(t)}{(1+\alpha v(t))v^*}\\
&\quad-\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{\omega^*}{\omega(t)}\cdot\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)}{x^*v^*\left(1+\alpha v(t-\tau_1)\right)}+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\\
&\quad-\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)}{x^*v^*\left(1+\alpha v(t-\tau_2)\right)}\\
&\quad+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{y^*}{y(t)}\cdot\frac{\omega(t)}{\omega^*}-\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{v(t)}{v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{v^*}{v(t)}\cdot\frac{y(t)}{y^*}\\
&\quad+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)}\\
&\quad+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}.
\end{aligned}
$$

由于

$$
\frac{\beta x^*v^*}{1+\alpha v^*}=\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*},
$$

那么

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t)&=-\frac{d\left(x(t)-x^*\right)^2}{x(t)}+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\left(4-\frac{x^*}{x(t)}-\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)y^*}{x^*v^*\left(1+\alpha v(t-\tau_2)\right)y(t)}\right.\\
&\quad\left.-\frac{v^*y(t)}{v(t)y^*}-\frac{1+\alpha v(t)}{1+\alpha v^*}\right)+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\left(5-\frac{x^*}{x(t)}\right.\\
&\quad\left.-\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)\omega^*}{x^*v^*\left(1+\alpha v(t-\tau_1)\right)\omega(t)}-\frac{y^*\omega(t)}{y(t)\omega^*}-\frac{v^*y(t)}{v(t)y^*}-\frac{1+\alpha v(t)}{1+\alpha v^*}\right)\\
&\quad+\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{1+\alpha v(t)}{1+\alpha v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}+\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{(1+\alpha v^*)v(t)}{(1+\alpha v(t))v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{v(t)}{v^*}\\
&\quad+\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)}\\
&\quad+\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)}.
\end{aligned}\tag{29}
$$

注意到

$$
\begin{aligned}
&\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{1+\alpha v(t)}{1+\alpha v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}+\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{(1+\alpha v^*)v(t)}{(1+\alpha v(t))v^*}-\frac{\beta x^*v^*}{1+\alpha v^*}\cdot\frac{v(t)}{v^*}\\
&=-\frac{\alpha\beta x^*(v-v^*)^2}{(1+\alpha v^*)^2(1+\alpha v(t))},
\end{aligned}
$$

$$
\begin{aligned}
\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_1)\right)} &= \ln\frac{x^*}{x(t)}+\ln\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)\omega^*}{x^*v^*\left(1+\alpha v(t-\tau_1)\right)\omega(t)}+\ln\frac{y^*\omega(t)}{y(t)\omega^*}\\
&\quad+\ln\frac{v^*y(t)}{v(t)y^*}+\ln\frac{1+\alpha v(t)}{1+\alpha v^*},\\
\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v(t)\right)}{x(t)v(t)\left(1+\alpha v(t-\tau_2)\right)} &= \ln\frac{x^*}{x(t)}+\ln\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)y^*}{x^*v^*\left(1+\alpha v(t-\tau_2)\right)y(t)}\\
&\quad+\ln\frac{v^*y(t)}{v(t)y^*}+\ln\frac{1+\alpha v(t)}{1+\alpha v^*}.
\end{aligned}
$$

于是式(29)化为

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = &-\frac{d\left(x(t)-x^*\right)^2}{x(t)}-\frac{\alpha\beta x^*(v-v^*)^2}{(1+\alpha v^*)^2(1+\alpha v(t))}\\
&-\frac{(e+\delta)q}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\left[F\left(\frac{x^*}{x(t)}\right)+F\left(\frac{x(t-\tau_2)v(t-\tau_2)\left(1+\alpha v^*\right)y^*}{x^*v^*\left(1+\alpha v(t-\tau_2)\right)y(t)}\right)\right.\\
&\left.+F\left(\frac{v^*y(t)}{v(t)y^*}\right)+F\left(\frac{1+\alpha v(t)}{1+\alpha v^*}\right)\right]\\
&-\frac{\delta(1-q)}{eq+\delta}\cdot\frac{\beta x^*v^*}{1+\alpha v^*}\left[F\left(\frac{x^*}{x(t)}\right)+F\left(\frac{x(t-\tau_1)v(t-\tau_1)\left(1+\alpha v^*\right)\omega^*}{x^*v^*\left(1+\alpha v(t-\tau_1)\right)\omega(t)}\right)+F\left(\frac{y^*\omega(t)}{y(t)\omega^*}\right)\right.\\
&\left.+F\left(\frac{v^*y(t)}{v(t)y^*}\right)+F\left(\frac{1+\alpha v(t)}{1+\alpha v^*}\right)\right].
\end{aligned}
$$

由于$\forall z>0$, $F(z)\geqslant 0$, 且有$F_{\min}=F(1)=0$. 所以$\frac{\mathrm{d}}{\mathrm{d}t}V_2(t)\leqslant 0$, 当且仅当$x=x^*$, $\omega=\omega^*$, $v=v^*, y=y^*$时, $\frac{\mathrm{d}}{\mathrm{d}t}V_2(t)=0$. 通过类似于证明定理 3 的方法和 LaSalle 不变集原理知, 当$R_0>1$时, 系统(3)的慢性感染平衡点$P^*(x^*,\ \omega^*,\ y^*,\ v^*)$是全局渐近稳定的.

4 数值模拟

在系统(3)中, 若令参数

$$
\begin{aligned}
&\lambda=10,\quad d=0.1,\quad \beta=0.001,\quad \alpha=0.01,\quad q=0.8,\quad e=0.3,\\
&\delta=0.5,\quad a=0.5,\quad k=0.4,\quad u=3,\quad \tau_1=3,\quad \tau_2=8,
\end{aligned}
$$

则基本再生数

$$
R_0=\frac{k\lambda\beta(eq+\delta)}{adu(e+\delta)}\approx 0.024\,7<1,
$$

由定理 3 知, 系统(3)的无病平衡点$P_0(x_0,0,0,0)$是全局渐近稳定的, 其中$x_0=\frac{\lambda}{d}=100$, 此时

数值模拟验证了所得结论(见图 1).

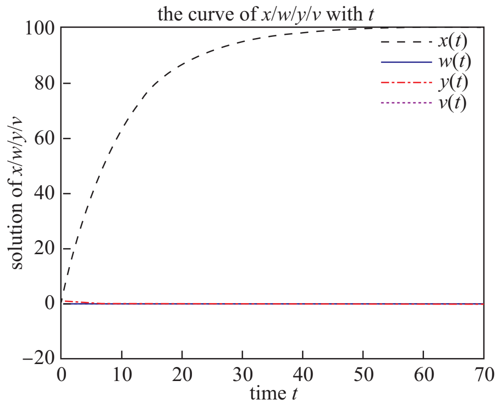

图 1　当$R_0 < 1$时, 无病平衡点$P_0(x_0,\ 0,\ 0,\ 0)$是全局渐近稳定的

Fig. 1　If $R_0 < 1$, the disease-free equilibrium $P_0(x_0, 0, 0, 0)$ is globally asymptotically stable

若令参数

$$\lambda = 250,\quad d = 0.005,\quad \beta = 0.001,\quad \alpha = 0.003,\quad q = 0.6,\quad e = 0.3,\quad \delta = 0.5,$$
$$a = 0.5,\quad k = 0.4,\quad u = 0.3,\quad \tau_1 = 3,\quad \tau_2 = 8,$$

此时基本再生数

$$R_0 = \frac{k\lambda\beta(eq + \delta)}{adu(e + \delta)} \approx 113.33 > 1,$$

系统(3)有唯一的慢性平衡点

$$P^*(1173.6,\ 324.43,\ 415.02,\ 553.37),$$

根据定理4可知, $P^*(x^*,\ \omega^*,\ y^*,\ v^*)$是全局渐近稳定的, 数值模拟验证了上述结论(见图 2).

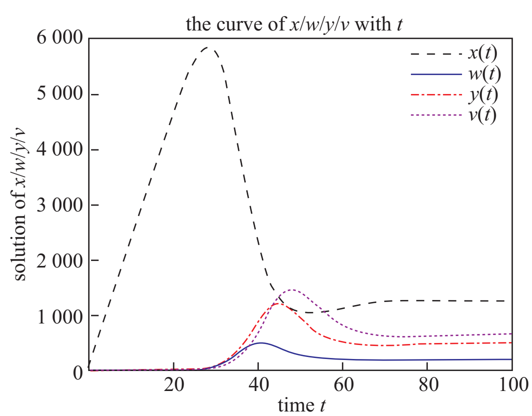

图 2　当$R_0 > 1$时, 正平衡点$P^*(x^*,\ \omega^*,\ y^*,\ v^*)$是全局渐近稳定的

Fig. 2　If $R_0 > 1$, the positive equilibrium$P^*(x^*,\ \omega^*,\ y^*,\ v^*)$ is globally asymptotically stable

5 结　论

本文研究了一类具有潜伏感染细胞和饱和发生率的时滞 HIV-1 传染病模型, 讨论了系统(3)的无病平衡点$P_0(x_0, 0, 0, 0)$和正平衡点$P^*(x^*, \omega^*, y^*, v^*)$的局部稳定性和全局稳定性. 结论表明: ①当$R_0 < 1$, 且时滞$\tau_1, \tau_2$之一为零或两者均不为零但相等时, 无病平衡点$P_0(x_0, 0, 0, 0)$是局部渐近稳定的; ②当$R_0 \leqslant 1$时, 对任意$\tau_1 \geqslant 0, \tau_2 \geqslant 0$, 无病平衡点$P_0(x_0, 0, 0, 0)$是全局渐近稳定的, 即在这种情况下, 细胞没有被传染; ③当$R_0 > 1$, 且时滞τ_1, τ_2其中之一为零或两者均不为零但相等时, 慢性感染平衡点$P^*(x^*, \omega^*, y^*, v^*)$是局部渐近稳定的; ④当$R_0 > 1$时, 对任意$\tau_1 \geqslant 0, \tau_2 \geqslant 0$, 慢性感染平衡点$P^*(x^*, \omega^*, y^*, v^*)$是全局渐近稳定的, 即 CD4+T 细胞被病毒感染后, 一部分被激活进入染病阶段, 但另一部分在被激活之后长时间保持静止, 这种持续潜伏的状态成为细胞从感染中恢复的障碍, 应引起大家的重视. 关于局部稳定性的讨论中, 时滞τ_1, τ_2两者均不为零且不相等的情形, 本文未给出理论分析, 有待进一步研究.

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