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Biharmonic hypersurfaces with at most two distinct principal curvatures in $\mathbb{E}_s^{n+1}$

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Abstract: In this paper, we studied biharmonic hypersurfaces of pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ with at most two distinct principal curvatures. Assume the shape operator is diagonalizable, we proved that such hypersurfaces are minimal.

Key words: pseudo-Euclidean space; biharmonic hypersurface; shape operator; diagonalizable

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$\mathbb{E}_s^{n+1}$ 中具有至多两个不同主曲率的 2-调和超曲面

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摘要: 研究了伪欧氏空间$\mathbb{E}_s^{n+1}$中具有至多两个不同主曲率的2-调和超曲面. 在假设形状算子可对角化的前提下，证明了这样的超曲面是极小的.

关键词: 伪欧氏空间; 2-调和超曲面; 形状算子; 可对角化

0 Introduction

Let $x : M_r^m \to \mathbb{E}_s^n$ be an isometric immersion of an m-dimensional submanifold M_r^m of a pseudo-Euclidean space $\mathbb{E}_s^n$. Denote by $\overrightarrow{H}$ and Δ the mean curvature vector field of M_r^m and the Laplace operator of M_r^m with respect to the induced metric. The submanifold M_r^m is said to be *biharmonic* if it satisfies the equation

$$\Delta\overrightarrow{H} = 0. \tag{1}$$

If the mean curvature vector field $\overrightarrow{H}$ of the submanifold M_r^m vanishes identically, then M_r^m is called minimal. Clearly, every minimal submanifold of $\mathbb{E}_s^n$ satisfies (1).

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As remarked, minimal submanifolds are biharmonic ones. Conversely, the natural question is whether any biharmonic submanifold is minimal. In fact, [1-3] gave examples of nonminimal biharmonic surfaces in pseudo-Euclidean spaces.

However, biharmonicity implies minimality in some special cases. Indeed, it was proved in [3] that any biharmonic surface in $\mathbb{E}_s^3$ $(s = 1, 2)$ is minimal. It was shown in [4] that every biharmonic hypersurface M_r^3 of $\mathbb{E}_s^4$ $(s = 0, 1, 2, 3, 4)$ whose shape operator is diagonal is minimal.

Naturally, we want to consider the same problem in n-dimensional pseudo-Euclidean space $\mathbb{E}_s^n$ $(s = 0, 1, \cdots, n)$. In this paper, we study the minimality of biharmonic hypersurface M_r^n with at most two distinct principal curvatures in $(n + 1)$-dimensional pseudo-Euclidean space $\mathbb{E}_s^{n+1}$, and prove the following theorem.

Theorem *Let M_r^n be a nondegenerate biharmonic hypersurface with at most two distinct principal curvatures of the $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{E}_s^{n+1}$. Assume that the shape operator of M_r^n is diagonalizable. Then M_r^n must be minimal.*

Remark A shape operator of a Riemannian submanifold is always diagonalizable, but for pseudo-Riemannian submanifolds, there may be other forms for A (cf. [5]).

1 Preliminaries

Let $x : M_r^n \to \mathbb{E}_s^{n+1}$ be an isometric immersion of a hypersurface M_r^n $(r = 0, 1, \cdots, n)$ in $\mathbb{E}_s^{n+1}(s = 0, 1, \cdots, n+1)$, $r \leqslant s$. The hypersurface M_r^n can itself be endowed with a Riemannian or a pseudo-Riemannian metric structure, depending on whether the metric induced on M_r^n from the pseudo-Riemannian space $\mathbb{E}_s^{n+1}$, is positive-definite or indefinite.

Let ξ denote a unit normal vector field with $\langle \xi, \xi \rangle = \varepsilon$, $\varepsilon = \pm 1$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M_r^n and $\mathbb{E}_s^{n+1}$ respectively. For any vector fields X, Y tangent to M_r^n, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$

where h is the scalar-valued second fundamental form. If we denote by A the shape operator of M_r^n associated to ξ, the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A(X),$$

where $\langle A(X), Y \rangle = \varepsilon h(X, Y)$. The mean curvature vector $\overrightarrow{H} = H\xi$ with $H = \frac{1}{n}\varepsilon \mathrm{tr} A$, determines a well defined normal vector field to M_r^n in $\mathbb{E}_s^{n+1}$. The Codazzi and Gauss equations are given by (cf. [5])

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2}$$

$$R(X, Y)Z = \langle A(Y), Z \rangle A(X) - \langle A(X), Z \rangle A(Y), \tag{3}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4}$$

A hypersurface M_r^n of $\mathbb{E}_s^{n+1}$ is said to be biharmonic, if

$$\Delta \overrightarrow{H} = 0.$$

The condition is equivalent to (cf. [1])

$$\Delta\overrightarrow{H} = \{2A(\nabla H) + n\varepsilon H(\nabla H)\} + \{\Delta H + \varepsilon H \mathrm{tr} A^2\}\xi = 0.$$

By comparing the vertical and horizontal parts of the above equation, this is equivalent to the conditions

$$A(\nabla H) = -\frac{n}{2}\varepsilon H(\nabla H), \tag{5}$$

$$\Delta H + \varepsilon H \mathrm{tr} A^2 = 0, \tag{6}$$

where the Laplace operator Δ acting on scalar-valued function f is given by (cf. [1])

$$\Delta f = -\sum_{i=1}^{n} \varepsilon_i (e_i e_i f - \nabla_{e_i} e_i f), \tag{7}$$

where $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of $T_p(M_r^n)$ with $\langle e_i, e_i \rangle = \varepsilon_i = \pm 1$.

2 Proof of the theorem

If H is a constant, then (6) implies that $H \mathrm{tr}\, A^2 = 0$. If H is zero, the result follows. Otherwise, $\mathrm{tr}\, \boldsymbol{A}^2 = 0$ implies that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = 0$, so $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Since $\mathrm{tr}\, \boldsymbol{A} = \lambda_1 + \lambda_2 + \cdots + \lambda_n = n\varepsilon H$, we obtain that $H = 0$.

Now, assume that H is not a constant, we will end up with a contradiction.

When H is not a constant, $\nabla H \neq 0$. According to (5), ∇H is an eigenvector of the shape operator A. Without loss generality, we can choose ∇H in the direction of e_1, and therefore the shape operator of M_r^n takes the form with respect to a suitable orthonormal frame $\{e_1, e_2, \cdots, e_n\}$

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where $\lambda_1 = -\frac{n}{2}\varepsilon H$. If the shape operator $\boldsymbol{A}$ has only one principal curvature, i.e.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = -\frac{n}{2}\varepsilon H,$$

then $\mathrm{tr}\, A = -\frac{n^2}{2}\varepsilon H$. On the other hand, $\mathrm{tr}\, A = n\varepsilon H$. So $-\frac{n^2}{2}\varepsilon H = n\varepsilon H$, which is a contradiction as H is not a constant.

From now on, we assume that the shape operator $\boldsymbol{A}$ has two different principal curvatures. Let us express ∇H as

$$\nabla H = \varepsilon_1 e_1(H) e_1 + \varepsilon_2 e_2(H) e_2 + \cdots + \varepsilon_n e_n(H) e_n.$$

Since we choose ∇H in the direction of e_1, it follows that

$$e_1(H) \neq 0, \quad e_2(H) = e_3(H) = \cdots = e_n(H) = 0. \tag{8}$$

For any $i, j = 1, 2, \cdots, n$, let $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k$. By using compatibility conditions to $\nabla_{e_k} \langle e_i, e_i \rangle = 0$ and $\nabla_{e_k} \langle e_i, e_j \rangle = 0$, we obtain

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j = -\varepsilon_i \varepsilon_j \omega_{kj}^i, \tag{9}$$

for $i \neq j$ and $i, j, k = 1, 2, \cdots, n$. Since $\lambda_1 = -\frac{n}{2}\varepsilon H$, we get

$$e_1(\lambda_1) \neq 0, \quad e_2(\lambda_1) = e_3(\lambda_1) = \cdots = e_n(\lambda_1) = 0. \tag{10}$$

The Codazzi equation (2) for hypersurfaces implies the equations

$$\langle (\nabla_{e_i} A) e_j, e_j \rangle = \langle (\nabla_{e_j} A) e_i, e_j \rangle \text{ and } \langle (\nabla_{e_i} A) e_j, e_k \rangle = \langle (\nabla_{e_j} A) e_i, e_k \rangle.$$

A straightforward calculation gives

$$e_i(\lambda_j) = (\lambda_i - \lambda_j) \omega_{ji}^j, \tag{11}$$

$$(\lambda_i - \lambda_j) \omega_{ki}^j = (\lambda_k - \lambda_j) \omega_{ik}^j, \tag{12}$$

for distinct $i, j, k = 1, 2, \cdots, n$.

We claim that $\lambda_j \neq \lambda_1$ for $j = 2, \cdots, n$. Indeed, if $\lambda_j = \lambda_1$, we have from (11) that

$$0 = (\lambda_1 - \lambda_j) \omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts to (10).

Taking into account the fact $\lambda_j \neq \lambda_1$ for $j = 2, \cdots, n$ and the assumption that the hypersurfaces M_r^n have two distinct principal curvatures, we denote $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \mu$ and $\mu \neq \lambda_1$. Since $H = \frac{1}{n}\varepsilon \mathrm{tr} A$, it follows that

$$\mu = \frac{3n}{2(n-1)} \varepsilon H.$$

Consider equations (11) for $j = 1, i \neq 1$, combining (9) and (10), we get

$$\omega_{1i}^1 = \omega_{11}^i = 0, \quad i = 1, 2, \cdots, n.$$

For $i = 1, j \neq 1$ in (11), combining (9) we obtain

$$\omega_{j1}^j = -\frac{3e_1(H)}{(n+2)H}, \quad \omega_{jj}^1 = \varepsilon_1 \varepsilon_j \frac{3e_1(H)}{(n+2)H}. \tag{13}$$

Using equation (12) for $i = 1, j \neq k$ and $k, j = 2, 3, \cdots, n$, combining (9), we have

$$\omega_{k1}^j = \omega_{kj}^1 = 0.$$

Applying the above equations, we find that

$$\begin{gathered}
\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_i = \sum_{k \neq 1, i} \omega_{1i}^k e_k, \quad \nabla_{e_i} e_1 = \omega_{21}^2 e_i, \\
\nabla_{e_i} e_i = -\varepsilon_1 \varepsilon_i \omega_{21}^2 e_1 + \sum_{k \neq 1, i} \omega_{ii}^k e_k, \quad \nabla_{e_i} e_j = \sum_{k \neq 1, j} \omega_{ij}^k e_k,
\end{gathered}$$

where $\omega_{21}^2=-\frac{3e_1(H)}{(n+2)H}$, for distinct i,j and $i,j=2,3,\cdots,n$.

Applying Gauss equation (3) and the definition (4) of the curvature tensor to $\langle R(e_1,e_2)e_1,e_2\rangle$, it is easy to check that

$$e_1(\omega_{21}^2)=\frac{3n^2}{4(n-1)}\varepsilon_1 H^2-(\omega_{21}^2)^2. \tag{14}$$

Using (7), (8) and the formulas of $\nabla_{e_i}e_j$, it follows from (6) that

$$-\varepsilon_1 e_1e_1(H)-(n-1)\varepsilon_1\omega_{21}^2e_1(H)+\varepsilon\frac{(n+8)n^2H^3}{4(n-1)}=0. \tag{15}$$

By differentiating (13) with $j=2$ along e_1, and using (14), we get

$$e_1e_1(H)=\frac{(n+2)(n+5)}{9}H(\omega_{21}^2)^2-\varepsilon_1\frac{n^2(n+2)}{4(n-1)}H^3. \tag{16}$$

Substituting (16) into (15), combining (13), we have

$$H\Big[\varepsilon_1\frac{(n+2)(-2n+8)}{9}(\omega_{21}^2)^2-\frac{n^2(n+2)+\varepsilon n^2(n+8)}{4(n-1)}H^2\Big]=0,$$

and as $H\neq 0$, it follows that

$$\varepsilon_1\frac{(n+2)(-2n+8)}{9}(\omega_{21}^2)^2-\frac{n^2(n+2)+\varepsilon n^2(n+8)}{4(n-1)}H^2=0. \tag{17}$$

Acting on (17) with e_1 and using (13) and (14), then

$$\varepsilon_1\frac{(n+2)(-2n+8)}{9}(\omega_{21}^2)^2-\frac{n^2(n+2)(-n+10+\varepsilon(n+8))}{12(n-1)}H^2=0. \tag{18}$$

Eliminating ω_{21}^2 from (17) and (18), we obtain that

$$H^2=0,$$

which leads to $H=0$, a contradiction.

[References]

[1] CHEN B Y, ISHIKAWA S. Biharmonic surfaces in pseudo-Euclidean spaces [J]. Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics, 1991, 45(2): 323-347.

[2] ISHIKAWA S. Biharmonic W-surfaces in 4-dimensional pesudo-Euclidean space [J]. Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics, 1992, 46(2): 269-286.

[3] CHEN B Y, ISHIKAWA S. Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces [J]. Kyushu Journal of Mathematics, 1998, 52(1): 167-185.

[4] DEFEVER F, KAIMAKAMIS G, PAPANTONIOU V. Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space E_s^4 [J]. Journal of Mathematical Analysis and Applications, 2006, 315(1): 276-286.

[5] O'NEILL B. Semi-Riemannian Geometry with Applications to Relativity [M]. San Diego, CA: Academic press, 1983.

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