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A new proof of the generalized Craig-Sakamoto theorem

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Abstract: We gave a new proof of the generalized Craig-Sakamoto theorem, which asserts that two normal matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy $\det(\boldsymbol{I}-a\boldsymbol{A}-b\boldsymbol{B})=\det(\boldsymbol{I}-a\boldsymbol{A})\det(\boldsymbol{I}-b\boldsymbol{B})$ for all complex numbers a and b if and only if $\boldsymbol{AB}=O$.

Key words: eigenvalues; singular values; normal matrices; quadratic forms

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推广的Craig-Sakamoto 定理的一个新证明

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摘要: 给出了推广的Craig-Sakamoto定理的一个新证明. 设$\boldsymbol{A},\boldsymbol{B}$是两个正规矩阵，对于任意的复数$a,b$满足$\det(\boldsymbol{I}-a\boldsymbol{A}-b\boldsymbol{B})=\det(\boldsymbol{I}-a\boldsymbol{A})\det(\boldsymbol{I}-b\boldsymbol{B})$ 当且仅当$\boldsymbol{AB}=O$.

关键词: 特征值; 奇异值; 正规矩阵; 二次型

0 Introduction

Let $\mathbf{C}$ be the field of complex numbers. We denote by $M_{m,n}$ the set of all $m\times n$ matrices over $\mathbf{C}$. $M_{n,n}$ will be abbreviated as M_n. In particular, $\mathbf{C}^n := M_{n,1}$, which means the set of all n-tuples column vectors. The operator norm on $M_{m,n}$ induced by the Euclidean norm is called the spectral norm, denoted by

$$\|\boldsymbol{A}\|_\infty=\max\{\|\boldsymbol{Ax}\|_2:\|\boldsymbol{x}\|_2=1,\boldsymbol{x}\in\mathbf{C}^n\},$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbf{C}^n$. It is well known that the spectral norm of $\|\boldsymbol{A}\|_\infty$ equals to the largest singular value of $\boldsymbol{A}$. Let $\boldsymbol{A}\in M_{m,n}$. If $m\geqslant n$, then the singular values of $\boldsymbol{A}$ are defined to be the nonnegative square roots of the eigenvalues of $\boldsymbol{A}^*\boldsymbol{A}$. If $m<n$, then the singular values of $\boldsymbol{A}$ are defined to be the nonnegative square roots of the eigenvalues of $\boldsymbol{AA}^*$. We always

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denote the singular values of $\boldsymbol{A} \in M_n$ in decreasing order by $s_1(\boldsymbol{A}) \geqslant s_2(\boldsymbol{A}) \geqslant \cdots \geqslant s_n(\boldsymbol{A})$. The Craig-Sakamoto theorem on the independence of two quadratic forms can be stated as follows.

Theorem 0.1 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ real symmetric matrices. Then $\det(\boldsymbol{I} - a\boldsymbol{A} - b\boldsymbol{B}) = \det(\boldsymbol{I} - a\boldsymbol{A})\det(\boldsymbol{I} - b\boldsymbol{B})$ for all real numbers a and b if and only if $\boldsymbol{AB} = O$.

One may see [1, 2] for the history and the importance of this result. Many proofs of Theorem 0.1 can be found in [3-8]. Using the similar idea due to Zhang and Yi[8], we provide a new version of Craig-Sakamoto theorem and give the proof.

Theorem 0.2[9] Let $\boldsymbol{A}, \boldsymbol{B} \in M_n$ be normal matrices. Then $\det(\boldsymbol{I} - a\boldsymbol{A} - b\boldsymbol{B}) = \det(\boldsymbol{I} - a\boldsymbol{A})\det(\boldsymbol{I} - b\boldsymbol{B})$ for all complex numbers a and b if and only if $\boldsymbol{AB} = O$.

1 Main result

Our proof depends on Lemma 1.1, whose proof can be found in the Theorem 4.5 of [10].

Lemma 1.1[10] If $\boldsymbol{B}$ is an $r \times t$ submatrix of $\boldsymbol{A} \in M_n$, then $s_i(\boldsymbol{A}) \geqslant s_i(\boldsymbol{B})$, where $1 \leqslant i \leqslant \min\{r, t\}$. In particular, $\|\boldsymbol{A}\|_\infty \geqslant \|\boldsymbol{B}\|_\infty$.

Proof of Theorem 0.2 The part ($\Leftarrow$) is clear. We only need show the converse. Suppose $\det(\boldsymbol{I} - a\boldsymbol{A} - b\boldsymbol{B}) = \det(\boldsymbol{I} - a\boldsymbol{A})\ \det(\boldsymbol{I} - b\boldsymbol{B})$ for all complex numbers a and b. Hence

$$\det(\boldsymbol{I} - c(\boldsymbol{A} + \boldsymbol{B})) = \det(\boldsymbol{I} - c\boldsymbol{A})\det(\boldsymbol{I} - c\boldsymbol{B})$$

for any nonzero complex numbers c. Thus

$$\det(z\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{B})) = \frac{1}{z^n}\det(z\boldsymbol{I} - \boldsymbol{A})\det(z\boldsymbol{I} - \boldsymbol{B}) \tag{1.1}$$

for any nonzero complex numbers z. Suppose $\boldsymbol{A}$ has $a_1, a_2, \cdots, a_s$ as its nonzero eigenvalues and $\boldsymbol{B}$ has $b_1, b_2, \cdots, b_t$ as nonzero eigenvalues. By equation (1.1), it follows that the nonzero eigenvalues of $\boldsymbol{A} + \boldsymbol{B}$ are $\{a_1, a_2, \cdots, a_s, b_1, b_2, \cdots, b_t\}$. Since the characteristic polynomial's degrees of matrix $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{A} + \boldsymbol{B}$ are all n, equation (1.1) shows that the sum number of the zero roots of the characteristic polynomial $\det(z\boldsymbol{I} - \boldsymbol{A})$ and $\det(z\boldsymbol{I} - \boldsymbol{B})$ is greater than or equals to n, therefore, the number of their nonzero roots $s + t$ is less than or equals to n. Thus $s + t \leqslant n$.

Case $s + t = n$: Let $\boldsymbol{A}_1 = \mathrm{diag}(a_1, a_2, \cdots, a_s)$ and $\boldsymbol{B}_1 = \mathrm{diag}(b_1, b_2, \cdots, b_t)$. Without loss of generality, we assume that $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & O \\ O & O \end{pmatrix}$, $\boldsymbol{B} = \boldsymbol{U}\begin{pmatrix} O & O \\ O & \boldsymbol{B}_1 \end{pmatrix}\boldsymbol{U}^*$ and $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} \\ \boldsymbol{U}_{21} & \boldsymbol{U}_{22} \end{pmatrix}$ is unitary. We will show that $\boldsymbol{U}_{12} = O_{s,t}$ and $\boldsymbol{U}_{21} = O_{t,s}$.

Since $\boldsymbol{U}$ is unitary, it follows that $\boldsymbol{U}_{12}^*\boldsymbol{U}_{12} + \boldsymbol{U}_{22}^*\boldsymbol{U}_{22} = \boldsymbol{I}_t$ and $\boldsymbol{U}_{21}\boldsymbol{U}_{21}^* + \boldsymbol{U}_{22}\boldsymbol{U}_{22}^* = \boldsymbol{I}_t$. By straight calculation, we have

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{U}_{12} \\ O & \boldsymbol{U}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_1 & O \\ O & \boldsymbol{B}_1 \end{pmatrix}\begin{pmatrix} \boldsymbol{I} & O \\ \boldsymbol{U}_{12}^* & \boldsymbol{U}_{22}^* \end{pmatrix}.$$

Thus

$$\det(\boldsymbol{A} + \boldsymbol{B}) = \det(\boldsymbol{A}_1)\det(\boldsymbol{B}_1)\det(\boldsymbol{U}_{22}^*)\det(\boldsymbol{U}_{22}).$$

Hence

$$\det(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = \det(\boldsymbol{U}_{22}\boldsymbol{U}_{22}^{*}) = \det(\boldsymbol{U}_{22}^{*})\det(\boldsymbol{U}_{22}) = 1.$$

Since $\boldsymbol{U}_{22}$ is a submatrix of $\boldsymbol{U}$, it follows that $1 = \|\boldsymbol{U}\|_\infty \geqslant \|\boldsymbol{U}_{22}\|_\infty$ and $\|\boldsymbol{U}_{22}\|_\infty^2 = \|\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}\|_\infty = \|\boldsymbol{U}_{22}\boldsymbol{U}_{22}^{*}\|_\infty = s_1(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = s_1(\boldsymbol{U}_{22}\boldsymbol{U}_{22}^{*}) \leqslant 1$. Note that for positive semidefinite matrices, singular values and eigenvalues are the same. In fact, $\det(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = \prod_{i=1}^{n} s_i(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22})$. Thus $s_1(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = s_2(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = \cdots = s_t(\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}) = 1$. Thus each eigenvalue of $\boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}$ equals to 1. Therefore each eigenvalue of $\boldsymbol{I}_t - \boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}$ equals to 0. Since

$$\boldsymbol{U}_{12}^{*}\boldsymbol{U}_{12} = \boldsymbol{I}_t - \boldsymbol{U}_{22}^{*}\boldsymbol{U}_{22}$$

is positive semidefinite, it follows that $\boldsymbol{U}_{12} = O_{s,t}$. Similar, we have $\boldsymbol{U}_{21} = O_{t,s}$. Therefore $\boldsymbol{AB} = O$.

Case $s+t<n$: Note that the rank of a normal matrix equals the number of its nonzero eigenvalues, since the singular values of a normal matrix are the module of its eigenvalues. Thus $\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) = s+t$. Note that

$$\operatorname{rank}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix} \leqslant \operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) = s+t<n.$$

Thus there exists a unitary matrix $\boldsymbol{U}_0 = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n)$ such that $\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\boldsymbol{u}_i = O, i = 1, 2, \cdots, n-s-t$. Note that if $\boldsymbol{C} \in M_n$ is normal, then the null space $\mathcal{N}(\boldsymbol{C})$ equals to the null space $\mathcal{N}(\boldsymbol{C}^{*})$. Thus $\begin{pmatrix}\boldsymbol{A}^{*}\\ \boldsymbol{B}^{*}\end{pmatrix}\boldsymbol{u}_i = O, i = 1, 2, \cdots, n-s-t$. By straight calculation, we have

$$\boldsymbol{U}_0^{*}\boldsymbol{A}\boldsymbol{U}_0 = \begin{pmatrix} O & O\\ O & \boldsymbol{A}_0\end{pmatrix}, \boldsymbol{U}_0^{*}\boldsymbol{B}\boldsymbol{U}_0 = \begin{pmatrix} O & O\\ O & \boldsymbol{B}_0\end{pmatrix}, \tag{1.2}$$

where $\boldsymbol{A}_0, \boldsymbol{B}_0 \in M_{s+t}$. It is easy to check that $\boldsymbol{A}_0, \boldsymbol{B}_0$ are normal matrices. On one hand, by equation (1.2), we have $\boldsymbol{A}$ and $\boldsymbol{A}_0$ have the same nonzero eigenvalues $a_1, a_2, \cdots, a_s$ and $\boldsymbol{B}$ and $\boldsymbol{B}_0$ have the same nonzero eigenvalues $b_1, b_2, \cdots, b_t$. On the other hand, by equation (1.1), it follows that the nonzero eigenvalues of $\boldsymbol{A}_0 + \boldsymbol{B}_0$ are $\{a_1, a_2, \cdots, a_s, b_1, b_2, \cdots, b_t\}$ and $\boldsymbol{A}_0, \boldsymbol{B}_0$ and $\boldsymbol{A}_0 + \boldsymbol{B}_0$ satisfy equation (1.1) and Case (1). Thus $\boldsymbol{A}_0\boldsymbol{B}_0 = O$ and so $\boldsymbol{AB} = O$.

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