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Notes on Chen's inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection

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Abstract: By using algebraic techniques, we proved Chen's general inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection, which generalized a result of C. Özgür and A. Mihai. Also, a mistake of their paper has been modified.

Key words: Chen's general inequalities; Chen-Ricci inequalities; real space form; semi-symmetric non-metric connection

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关于具有半对称非度量联络的实空间形式中子流形的 Chen 不等式的注记

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摘要: 利用代数技巧, 得到了具有半对称非度量联络的实空间形式中的子流形的 Chen 广义不等式, 推广了 C. Özgür 和 A. Mihai 的一个结果. 并订正了他们文章中的一个错误.

关键词: Chen 广义不等式; Chen-Ricci 不等式; 实空间形式; 半对称非度量联络

0 Introduction

According to B.-Y. Chen[1], one of the most important problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Related with famous Nash embedding theorem[2], B.-Y. Chen introduced a new type of Riemannian invariants, known as δ-*invariants*[3-5]. The author's original motivation was to provide answers to a question raised by S. S. Chern concerning the existence of minimal isometric immersions into Euclidean space[6]. Therefore, B.-Y. Chen

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obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the squared mean curvature[7]. Later, he established general inequalities relating $\delta(n_1, \cdots, n_k)$ and the squared mean curvature for submanifolds in real space forms[8]. Similar inequalities also hold for Lagrangian submanifolds of complex space forms. In [9], B.-Y. Chen proved that, for any $\delta(n_1, \cdots, n_k)$, the equality case holds if and only if the Lagrangian submanifold is minimal. This interesting phenomenon inspired people to look for a more sharp inequality. In 2007, T. Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms[10]. Recently, B.-Y. Chen and F. Dillen established general inequalities for submanifolds in complex space forms and provided some examples showing these new improved inequalities are best possible[11]. However, it was pointed out[12] that the proof of the general inequality given[11] is incorrect when $\sum_{i=1}^{k} \frac{1}{2+n_i} > \frac{1}{3}$. In [13], B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken corrected the proof of the general inequality in the case $n_1 + \cdots + n_k < n$ and showed that the inequality can be improved in the case $n_1 + \cdots + n_k = n$.

Such invariants and inequalities have many nice applications to several areas in mathematics[14]. Afterwards, many papers studied similar problems for different submanifolds in various ambient spaces, like complex space forms[15], Sasakian space forms[16], (κ, μ)-contact space forms[17], Lorentzian manifold[18], Euclidean space[19] and locally conformal almost cosymplectic manifolds[20].

Recently, C. Özgür and A. Mihai proved Chen's inequalities for submanifolds of real space forms endowed with a semi-symmetric non-metric connection[21]. In this paper, we generalize a result of paper [21]. Moreover, we show that a result of C. Özgür and A. Mihai [21, Theorem 4.1] is incorrect. For the sake of correcting the result, we establish Chen-Ricci inequalities for submanifolds of real space forms endowed with a semi-symmetric non-metric connection in Section 3.

1 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of Riemannian manifolds endowed with a semi-symmetric non-metric connection are briefly presented. A more complete elementary treatment can be found in [23,24].

Let N^{n+p} be an $(n+p)-$dimensional Riemannian manifold with Riemannian metric g and the linear connection $\overline{\nabla}$. For vector fields $\overline{X}, \overline{Y}$ on N^{n+p} the torsion tensor field $\overline{T}$ of the linear connection $\overline{\nabla}$ is defined by $\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]$. A liner connection $\overline{\nabla}$ is said to be a semi-symmetric connection if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies $\overline{T}(\overline{X}, \overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$, where ϕ is a 1-form on N^{n+p}. Further, if $\overline{\nabla}$ satisfies $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection[22]. If $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is called a semi-symmetric non-metric connection[23]. Suppose $\hat{\overline{\nabla}}$ is the Levi-Civita connection of N. Following [23], we define a semi-symmetric connection $\overline{\nabla}$ given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \hat{\overline{\nabla}}_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X},$$

where ϕ is a 1-form on N. This clearly is a semi-symmetric non-metric connection. As ϕ is a 1-form we can introduce a dual vector field P by

$$g(P,\overline{X})=\phi(\overline{X}). \tag{1.1}$$

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional real space form N^{n+p} of constant sectional curvature c endowed with the semi-symmetric non-metric connection $\overline{\nabla}$ and the Levi-Civita connection $\widehat{\overline{\nabla}}$. On M^n we consider the induced semi-symmetric non-metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\widehat{\nabla}$. Let $\overline{R}$ be the curvature tensor of N^{n+p} with respect to $\overline{\nabla}$ and $\widehat{\overline{R}}$ the curvature tensor of N^{n+p} with respect to $\widehat{\overline{\nabla}}$. We also denote by R and $\widehat{R}$ the curvature tensors associated to ∇ and $\widehat{\nabla}$.

The Gauss formulas with respect to ∇, respectively $\widehat{\nabla}$, can be written as the following

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y),\quad \widehat{\overline{\nabla}}_XY=\widehat{\nabla}_XY+\widehat{h}(X,Y),$$

for any vector fields X, Y on M^n, where h is a $(0,2)$ symmetric tensor on M^n and $\widehat{h}$ is the second fundamental form associated to Levi-Civita connection $\widehat{\nabla}$. According to formula (3.4) in [24] we have

$$h=\widehat{h}. \tag{1.2}$$

The curvature tensor $\widehat{\overline{R}}$ with respect to $\widehat{\overline{\nabla}}$ on N^{n+p} is expressed by

$$\widehat{\overline{R}}(\overline{X},\overline{Y},\overline{Z},\overline{W})=c[g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})-g(\overline{Y},\overline{Z})g(\overline{X},\overline{W})], \tag{1.3}$$

and the curvature tensor $\overline{R}$ with respect to $\overline{\nabla}$ on N^{n+p} can be written as[23]

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W})=\widehat{\overline{R}}(\overline{X},\overline{Y},\overline{Z},\overline{W})+s(\overline{Y},\overline{Z})g(\overline{X},\overline{W})-s(\overline{X},\overline{Z})g(\overline{Y},\overline{W}), \tag{1.4}$$

for any vector fields $\overline{X},\overline{Y},\overline{Z},\overline{W}$ on N, where s is a $(0,2)$-tensor field defined by

$$s(\overline{X},\overline{Y})=(\widehat{\overline{\nabla}}_{\overline{X}}\phi)\overline{Y}-\phi(\overline{X})\phi(\overline{Y}).$$

From (1.3) and (1.4) it follows that the curvature tensor $\overline{R}$ can be expressed as

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W})=&c[g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})-g(\overline{Y},\overline{Z})g(\overline{X},\overline{W})]\\&+s(\overline{Y},\overline{Z})g(\overline{X},\overline{W})-s(\overline{X},\overline{Z})g(\overline{Y},\overline{W}).\end{aligned} \tag{1.5}$$

Decompsing the vector field P on M uniquely into its tangent and normal components P^T and $P^\perp$, respectively, we have $P=P^T+P^\perp$. From [24], for any vector fields X, Y, Z, W on M^n the Gauss equation with respect to the semi-symmetric non-metric connection is

$$\begin{aligned}R(X,Y,Z,W)=&\overline{R}(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z))\\&+g(P^\perp,h(Y,Z))g(X,W)-g(P^\perp,h(X,Z))g(Y,W).\end{aligned} \tag{1.6}$$

In N^{n+p} we can choose a local orthonormal frame $e_1,\cdots,e_n,e_{n+1},\cdots,e_{n+p}$, such that, restricting to M^n, $e_1,e_2,\cdots,e_n$ are tangent to M^n. We write $h_{ij}^r=g(h(e_i,e_j),e_r)$. The squared length of h is $\|h\|^2 = \sum\limits_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$ and the mean curvature vector is $H = \frac{1}{n}\sum\limits_{i=1}^{n} h(e_i, e_i)$. We denote $\lambda = \sum\limits_{i=1}^{n} s(e_i, e_i)$.

Let $\pi \subset T_xM^n,\ x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the *sectional curvature* of M^n respect to the induced semi-symmetric non-metric connection ∇. Then the *scalar curvature* of M^n with respect to ∇ is given by

$$\tau = \sum_{1 \leqslant i < j \leqslant n} K(e_i \wedge e_j). \tag{1.7}$$

Let L be an l-dimensional subspace of T_xM, $x \in M$, $l \geqslant 2$ and $\{e_1, \cdots, e_l\}$ an orthonormal basis of L. We define the *scalar curvature* $\tau(L)$ of the l-plane L by

$$\tau(L) = \sum_{1 \leqslant \alpha < \beta \leqslant l} K(e_\alpha \wedge e_\beta). \tag{1.8}$$

For simplicity we put

$$\Psi(L) = \frac{1}{2}\sum_{i=1}^{l-1}[s(e_i, e_i) + \phi(h(e_i, e_i))]. \tag{1.9}$$

For an integer $k \geqslant 0$ we denote by $S(n, k)$ the set of k-tuples $(n_1, \cdots, n_k)$ of integers $\geqslant 2$ satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leqslant n$. We denote by $S(n)$ the set of unordered k-tuples with $k \geqslant 0$ for a fixed n. For each k-tuples $(n_1, \cdots, n_k) \in S(n)$, B.-Y. Chen defined a *Riemannian invariant* $\delta(n_1, \cdots, n_k)$ as follows[8]

$$\delta(n_1, \cdots, n_k)(x) = \tau(x) - S(n_1, \cdots, n_k)(x), \tag{1.10}$$

where $S(n_1, \cdots, n_k)(x) = \inf\{\tau(L_1) + \cdots + \tau(L_k)\}$ and $L_1, \cdots, L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j,\ j \in \{1, \cdots, k\}$. In particular, we have $\delta(2) = \tau(x) - \inf K$, where K is the sectional curvature. For each $(n_1, \cdots, n_k) \in S(n)$, we put

$$c(n_1, \cdots, n_k) = \frac{n^2\Big(n + k - 1 - \sum\limits_{j=1}^{k} n_j\Big)}{2\Big(n + k - \sum\limits_{j=1}^{k} n_j\Big)}, \quad d(n_1, \cdots, n_k) = \frac{1}{2}\Big[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big].$$

We shall use the following lemmas.

Lemma 1.1[7]　Let $a_1, a_2, \cdots, a_n, b$ be $(n+1)(n \geqslant 2)$ real numbers such that

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big),$$

then $2a_1a_2 \geqslant b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Lemma 1.2　Let $f(x_1, x_2, \cdots, x_n)$ be a function in $\mathbb{R}^n$ defined by $f(x_1, x_2, \cdots, x_n) = x_1\sum\limits_{i=2}^{n} x_i$. If $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, then we have $f(x_1, x_2, \cdots, x_n) \leqslant \varepsilon^2$, with equality holding if and only if $x_1 = x_2 + x_3 + \cdots + x_n = \varepsilon$.

Proof From $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, we have

$$\sum_{i=2}^{n} x_i = 2\varepsilon - x_1.$$

It follows that

$$f(x_1, x_2, \cdots, x_n) = x_1(2\varepsilon - x_1) = -(x_1 - \varepsilon)^2 + \varepsilon^2,$$

which represents Lemma 1.2 to prove.

2 Chen's general inequalities

Theorem 2.1 Let M^n, $n \geqslant 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form N^{n+p} of constant sectional curvature c endowed with a semi-symmetric non-metric connection, then we have

$$\begin{aligned}\delta(n_1, \cdots, n_k) \leqslant & c(n_1, \cdots, n_k) \parallel H \parallel^2 + d(n_1, \cdots, n_k)c \\ & - \frac{(n-1)}{2}\lambda - \frac{n^2-n}{2}\phi(H) + \sum_{j=1}^{k} \Psi(L_j), \end{aligned} \tag{2.1}$$

for any k-tuples $(n_1, \cdots, n_k) \in S(n)$. The equality case of (2.1) holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, \cdots, e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \mu_r I \end{pmatrix}, \quad r = n+2, \cdots, n+p,$$

where $a_1, \cdots, a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying $\mathrm{trace}(A_1^r) = \cdots = \mathrm{trace}(A_k^r) = \mu_r$. I is an identity matrix.

Remark 2.2 For $\delta(2)$, inequality (2.1) is due to Özgür and Mihai[21, Theorem 3.1].

Proof Let $x \in M^n$ and $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^{\perp}M^n$, respectively, such that the mean curvature vector H is in the direction of the normal vector to e_{n+1}. For convenience, we set

$$\begin{aligned} & a_i = h_{ii}^{n+1}, \quad i = 1, 2, \cdots, n, \\ & b_1 = a_1, \quad b_2 = a_2 + \cdots + a_{n_1}, \quad b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \cdots, \\ & b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \\ & b_{k+2} = a_{n_1+\cdots+n_k+1}, \cdots, b_{\gamma+1} = a_n, \\ & \Delta_1 = \{1, \cdots, n_1\}, \cdots, \Delta_k = \{(n_1 + \cdots + n_{k-1}) + 1, \cdots, n_1 + \cdots + n_k\}, \\ & \Delta_{k+1} = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k). \end{aligned}$$

Let $L_1, \cdots, L_k$ be mutually orthogonal subspaces of T_xM with $\dim L_j = n_j$, defined by

$$L_j = Span\{e_{n_1+\cdots+n_{j-1}+1}, \cdots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \cdots, k.$$

From (1.1), (1.5)—(1.9) we have

$$\tau(L_j) = \frac{n_j(n_j-1)}{2}c - \Psi(L_j) + \sum_{r=n+1}^{n+p}\sum_{\alpha_j<\beta_j}[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j} - (h^r_{\alpha_j\beta_j})^2], \tag{2.2}$$

$$2\tau = n(n-1)c - (n-1)\lambda + n^2 \parallel H \parallel^2 - (n^2-n)\phi(H) - \parallel h \parallel^2. \tag{2.3}$$

We can rewrite (2.3) as

$$n^2 \parallel H \parallel^2 = (\parallel h \parallel^2 + \eta)\gamma,$$

or equivalently,

$$(\sum_{i=1}^{n} h^{n+1}_{ii})^2 = \gamma[\sum_{i=1}^{n}(h^{n+1}_{ii})^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 + \eta], \tag{2.4}$$

where

$$\eta = 2\tau - 2c(n_1, \cdots, n_k)H^2 - n(n-1)c + (n-1)\lambda + (n^2-n)\phi(H), \tag{2.5}$$

$$\gamma = n + k - \sum_{j=1}^{k} n_j.$$

From (2.4) we have

$$(\sum_{i=1}^{\gamma+1} b_i)^2 = \gamma[\eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 - 2\sum_{j=1}^{k}\sum_{\mu_j<\nu_j} a_{\mu_j}a_{\nu_j}],$$

where $\mu_j, \nu_j \in \Delta_j$, for all $j = 1, \cdots, k$. Applying Lemma 1.1, we derive

$$\sum_{j=1}^{k}\sum_{\mu_j<\nu_j} a_{\mu_j}a_{\nu_j} \geqslant \frac{1}{2}[\eta + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2],$$

it follows that

$$\begin{aligned}\sum_{j=1}^{k}\sum_{r=n+1}^{n+p}\sum_{\mu_j<\nu_j}[h^r_{\mu_j\mu_j}h^r_{\nu_j\nu_j} - (h^r_{\mu_j\nu_j})^2] &\geqslant \frac{\eta}{2} + \frac{1}{2}\sum_{r=n+1}^{n+p}\sum_{(\mu,\nu)\notin\Delta_{k+1}}(h^r_{\mu\nu})^2 + \sum_{r=n+2}^{n+p}\sum_{\mu_j\in\Delta_j}(h^r_{\mu_j\mu_j})^2 \\ &\geqslant \frac{\eta}{2}.\end{aligned} \tag{2.6}$$

From (2.2) and (2.6) we have

$$\sum_{j=1}^{k}\tau(L_j) \geqslant \sum_{j=1}^{k}[\frac{n_j(n_j-1)}{2}c - \Psi(L_j)] + \frac{1}{2}\eta. \tag{2.7}$$

Using (1.10), (2.5) and (2.7), we derive the desired inequality.

The equality case of (2.1) at a point $x \in M$ holds if and only if we have the equality in all the previous inequalities and also in Lemma 1.1. Hence, from (1.2) the shape operators take the desired forms.

As an application of Theorem 2.1, we have the following

Corollary 2.3 If a Riemannian n-manifold M^n $(n \geqslant 3)$ admits an isometric immersion into a real space form N^{n+p} of constant curvature c endowed with a semi-symmetric non-metric connection whose δ-invariant satisfies

$$\delta(n_1, \cdots, n_k) > d(n_1, \cdots, n_k)c - \frac{(n-1)}{2}\lambda + \sum_{j=1}^{k}\Psi(L_j)$$

at some points in M^n for some $(n_1, \cdots, n_k) \in S(n)$, then M^n is not minimal.

3 Chen-Ricci inequalities

In [25], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any n-dimensional Riemannian submanifold of a real space form $R^m(c)$ of constant sectional curvature c as follows

Theorem 3.1(See [25, Theorem 4]) Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then the following statements are true.

(1) For each unit vector $X \in T_pM$, we have

$$\|H\|^2 \geqslant \frac{4}{n^2}[\mathrm{Ric}(X) - (n-1)c]. \tag{3.1}$$

(2) If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of (3.1) if and only if X belongs to the relative null space $\mathcal{N}(p)$ given by

$$\mathcal{N}(p) = \{X \in T_pM \mid h(X, Y) = 0, \forall Y \in T_pM\}.$$

(3) The equality case of (3.1) holds for all unit vectors $X \in T_pM$ if and only if either p is a geodesic point or $n = 2$ and p is an umbilical point.

Afterwards, many papers studied similar problems for different submanifolds in various ambient manifolds[26-28], one proves the results similar to that of Theorem 3.1. In [23], C. Özgür and A. Mihai proved that

Theorem 3.2(See [21, Theorem 4.1]) Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ endowed with a semi-symmetric non-metric connection. Then

(i) For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leqslant (n-1)c + \frac{n^2\| H \|^2}{4} - \frac{n-1}{2}\lambda + \frac{(n-1)(n-2)}{2}s(X, X) - \frac{n^2-n}{2}\phi(H). \tag{3.2}$$

(ii) If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of (3.2) if and only if $X \in \mathcal{N}(x) = \{X \in T_xM^n \mid h(X, Y) = 0, \forall Y \in T_xM^n\}$.

(iii) The equality case of inequality (3.2) holds for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.

Remark 3.3 For $n \neq 2$, if the equality case of (3.2) holds for all unit tangent vectors X at x, from Theorem 3.2, we know that $h^r_{ij} = 0, \forall i, j, r$. Further, using (1.5) and (1.6) we have

$$\mathrm{Ric}(X) = \sum_{i=2}^{n} R_{1i1i} = (n-1)c - (n-1)s(X,X) - (n-1)\phi(h(X,X)),$$

here is a contradiction with the equality case of (3.2).

Remark 3.4 For $n = 2$, if the equality case of (3.2) holds for all unit tangent vectors X at x, from Theorem 3.2, we know that $h^r_{11} = h^r_{22}, h^r_{12} = 0, r = 3, \cdots, 2+p$. Further, using (1.5) and (1.6) we have

$$\mathrm{Ric}(X) = R_{1212} = c - s(X,X) - \phi(h(X,X)) + \| H \|^2,$$

here is also a contradiction with the equality case of (3.2).

Remark 3.5 In the proof of Theorem 4.1 in [21], they wrote

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i,e_j,e_j,e_i) + g(h(e_i,e_i),h(e_j,e_j)) - g(h(e_i,e_j),h(e_i,e_j)) \\ &= c - s(e_j,e_j) + \sum_{r=n+1}^{n+p} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \end{aligned}$$

But according to the formula (3.2) and (3.3) in [21], we get

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i,e_j,e_j,e_i) + g(h(e_i,e_i),h(e_j,e_j)) - g(h(e_i,e_j),h(e_i,e_j)) - \phi(h(e_j,e_j)) \\ &= c - s(e_j,e_j) + \sum_{r=n+1}^{n+p} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] - \phi(h(e_j,e_j)). \end{aligned}$$

This is the reason they made a mistake. We should notice that the Gauss equation with respect to the semi-symmetric non-metric connection is very different from the Gauss equation with respect to the Levi-Civita connection.

Under these circumstances it becomes necessary to give a theorem, which could present a sharp inequality between the Ricci-curvature and the squared mean curvature with respect to the semi-symmetric non-metric connection.

According to the equation (3.1) in [21], denote by

$$\Omega(X) = s(X,X) + g(P^{\perp}, h(X,X)) \tag{3.3}$$

for a unit vector X tangent to M^n at a point x. We remark that Ω doesn't depend on X and denote it simply by Ω. Detailed explanations were given in the proof of Theorem 3.1 in [21]. In this paper we prove

Theorem 3.6 Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional real space form N^{n+p} of constant sectional curvature c, endowed with a semi-symmetric non-metric connection. For each unit vector X in T_xM we have

$$Ric(X) \leqslant (n-1)(c-\Omega) + \frac{n^2}{4} \| H \|^2 . \tag{3.4}$$

The equality holds for all unit tangent vectors on M^n if and only if either M^n is a totally geodesic submanifold in N^{n+p} or $n=2$ and M^2 is a totally umbilical submanifold.

Remark 3.7 We should point out that our approach is different from B.-Y. Chen's.

Proof Let $x \in M^n$ and $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^{\perp}M^n$, respectively, such that $X=e_1$. From the equation (1.5), (1.6) and (3.3) we have

$$\begin{aligned} R_{ijij} &= c - s(e_i,e_i) - g(P^{\perp}, h(e_i,e_i)) + \sum_{r=n+1}^{n+p}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ &= c - \Omega(e_i) + \sum_{r=n+1}^{n+p}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ &= c - \Omega + \sum_{r=n+1}^{n+p}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \end{aligned} \tag{3.5}$$

Using (3.5) one gets

$$\mathrm{Ric}(X) = \sum_{i=2}^{n} R_{1i1i} \leqslant (n-1)(c-\Omega) + \sum_{r=n+1}^{n+p}\sum_{i=2}^{n} h_{11}^r h_{ii}^r. \tag{3.6}$$

Let us consider the quadratic forms $f_r : \mathbb{R}^n \to \mathbb{R}$, defined by $f_r(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r) = \sum\limits_{i=2}^{n} h_{11}^r h_{ii}^r$. We consider the problem $\max f_r$, subject to $\Gamma : h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, where k^r is a real constant. From Lemma 1.2, we see that the solution $(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r)$ of the problem in question must satisfy

$$h_{11}^r = \sum_{j=2}^{n} h_{jj}^r = \frac{k^r}{2}, \tag{3.7}$$

with the following holds

$$f_r \leqslant \frac{(k^r)^2}{4}. \tag{3.8}$$

From (3.6) and (3.8) we have

$$\mathrm{Ric}(X) \leqslant (n-1)(c-\Omega) + \sum_{r=n+1}^{n+p}\frac{(k^r)^2}{4} = (n-1)(c-\Omega) + \frac{n^2}{4} \parallel H \parallel^2 .$$

Next, we shall study the equality case. For each unit tangent vector X at x, if the equality case of inequality (3.4) holds, from (3.6) and (3.7) we have

$$h_{1i}^r = 0, \quad i \neq 1, \ \forall\, r, \tag{3.9}$$

$$h_{11}^r + h_{22}^r + \cdots + h_{nn}^r - 2h_{11}^r = 0, \quad \forall\, r. \tag{3.10}$$

For all unit tangent vectors at x, if the equality case of inequality (3.4) holds, by computing $Ric(e_i), i = 2, 3, \cdots, n$ and combining (3.9) and (3.10) we have

$$h_{ij}^r = 0, \quad i \neq j, \quad \forall\, r; \quad h_{11}^r + h_{22}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \quad \forall\, i, r.$$

We can distinguish two cases:

(1) $n \neq 2$, $h_{ij}^r = 0, i, j = 1, 2, \cdots, n, r = n+1, \cdots, n+p$ or

(2) $n = 2$, $h_{11}^r = h_{22}^r, h_{12}^r = 0, r = 3, \cdots, 2+p$.

Then the equality case holds for all unit tangent vectors on M^n if and only if either M^n is a totally geodesic submanifold in N^{n+p} or $n = 2$ and M^2 is a totally umbilical submanifold.

Theorem 3.8　If $H(x) = 0$, then a unit vector $X \in T_xM$ satisfies the equality case of (3.4) if and only if X belongs to the relative null space $\mathcal{N}(x)$ given by

$$\mathcal{N}(x) = \{X \in T_xM \mid h(X, Y) = 0, \forall Y \in T_xM\}.$$

Proof　Assume $H(x) = 0$. For each unit vector $X \in T_xM$, equality holds in (3.4) if and only if (3.7) and (3.9) holds. Then $h_{1i}^r = 0, \forall i, r$, i.e. $X \in \mathcal{N}(x)$.

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(下转第 26 页)