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三阶超对称非线性 Schrödinger 方程的延拓结构

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摘要: 超对称的 Heisenberg 铁磁连模型是一类非常重要的可积系统，它与固体物理中的电子强关联 Hubbard 模型有着紧密的联系. 文章主要利用超对称延拓结构理论的方法, 分析高阶超对称非线性Schrödinger方程, 进行研究得到了该方程延拓代数对应的 Lax 对.

关键词: 非线性 Schrödinger 方程; 超对称; 李代数; 延拓结构; Lax 对; 线性谱问题

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Prolongation structure of the third-order supersymmetric nonlinear Schrödinger equation

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Abstract: The Heisenberg supermagnet model is an supersymmetric system and has a close relationship with the strong electron correlated Hubbard model. In this paper, the supersymmetric prolongation structure was used to analyze the high order supersymmetric nonlinear Schrödinger equation. Its Lax representation of prolongation algebra was constructed.

Key words: nonlinear Schrödinger equation; supersymmetric; Lie algebra; prolongation structure; Lax pair linear spectral problems

1 引 言

随着对孤立子在大量非线性物理问题中的深入研究, 孤立子在数学上已经形成了比较完整的理论体系. 已有的结论显示, 对于可积的非线性系统必定存在孤立子解, 而且有时候也能在不可积系统中得到孤立子解. 对于一个非线性系统的可积性, 目前人们还无法给出一个确切的概念; 因此, 称一个非线性系统可积的时候往往是指不同意义下的可积. 早在 1975 年, Wahlquist 和 Estabrook 基于外微分形式系统及李代数表示理论建立了 1+1 维非线性演化方程的延拓结构理论, 给出了一个系统求解非线性演化方程线性谱问题的有效方法. 该理论主要是将要研究的(1+1)维可积非线性微分方程表达为一组外微分形式 2-形式, 使得这些

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外微分形式构成闭理想, 然后引进势或伪势和与之相联系的外微分 1-形式, 并要求引入的外微分 1-形式与原来的外微分形式 2-形式构成新的闭理想, 从而成功给出可积方程的 Lax 表示以及贝克隆变换. 他们将该理论应用于 KdV 方程, 系统地得到了 KdV 方程的线性谱问题和 B$\ddot{a}$cklund 变换. 随后, 该理论被广泛应用于研究 (1+1) 可积非线性演化方程[1-5]. 最近人们利用该理论对海森堡铁磁链方程[6-10], 高阶非线性薛定谔方程[11-15], 反应扩散方程[16-19]进行了深入分析和研究.

近年来对于各种超对称非线性 Schrödinger 方程[20-22]的研究得到人们的普遍关注. 在本文中, 我们将主要运用延拓结构理论对超对称非线性 Schrödinger 方程进行研究, 讨论其延拓代数结构, 并给出它们的 Lax 对.

2　三阶超对称非线性 Schrödinger 方程

方程(1)表示的是一个重要的物理模型, 它表示经典的连续铁磁自旋系统的非线性动力学情况, 是一个非常重要的可积系统. 目前人们对其经典以及量子行为进行了大量研究, 它在拓扑场论以及超弦理论中都有重要应用.

$$
\begin{aligned}
&\mathrm{i}\varphi_t+\varphi_{xx}+2(\varphi\overline{\varphi}+\psi\overline{\psi})\varphi-2\mathrm{i}\varepsilon(\varphi\overline{\varphi}\varphi+\psi\overline{\psi}\varphi)_x-\mathrm{i}\varepsilon\varphi_{xxx}\\
&\quad+\mathrm{i}\varepsilon(2\varphi\overline{\varphi}_x\varphi-2\varphi_x\overline{\varphi}\varphi-\psi_x\overline{\psi}\varphi+2\psi\overline{\psi}_x\varphi-\psi\overline{\psi}\varphi_x)-\mathrm{i}h\varphi_x=0,\\
&\mathrm{i}\psi_t+\psi_{xx}+2\varphi\overline{\varphi}\psi-2\mathrm{i}\varepsilon(\varphi\overline{\varphi}\psi)_x-\mathrm{i}\varepsilon\psi_{xxx}+\mathrm{i}\varepsilon(2\varphi\overline{\varphi_x}\psi-\varphi\overline{\varphi}\psi_x-\varphi_x\overline{\varphi}\psi)-\mathrm{i}h\psi_x=0,\\
&\mathrm{i}\overline{\varphi}_t-\overline{\varphi}_{xx}-2(\overline{\varphi}\varphi+\overline{\psi}\psi)\overline{\varphi}-2\mathrm{i}\varepsilon(\overline{\varphi}\varphi\overline{\varphi}+\overline{\psi}\psi\overline{\varphi})_x-\mathrm{i}\varepsilon\overline{\varphi}_{xxx}\\
&\quad+\mathrm{i}\varepsilon(2\overline{\varphi}\varphi_x\overline{\varphi}-2\overline{\varphi}_x\varphi\overline{\varphi}-\overline{\psi}_x\psi\overline{\varphi}+2\overline{\psi}\psi_x\overline{\varphi}-\overline{\psi}\psi\overline{\varphi}_x)-\mathrm{i}h\overline{\varphi}_x=0,\\
&\mathrm{i}\overline{\psi}_t-\overline{\psi}_{xx}-2\overline{\varphi}\varphi\overline{\psi}-2\mathrm{i}\varepsilon(\overline{\varphi}\varphi\overline{\psi})_x-\mathrm{i}\varepsilon\overline{\psi}_{xxx}-\mathrm{i}\varepsilon(2\overline{\varphi}\varphi_x\overline{\psi}-\overline{\varphi}\varphi\overline{\psi}_x-\overline{\varphi}_x\varphi\overline{\psi})-\mathrm{i}h\overline{\psi}_x=0,
\end{aligned}\tag{1}
$$

其中 ε为常数, $\varphi(x,t)$是玻色函数, $\psi(x,t)$是费米项. 费米函数ψ,$\overline{\psi}$的计算规则为$\psi^2=\overline{\psi}^2$=0, $\psi\overline{\psi}=-\overline{\psi}\psi$ 我们首先引进新的独立变量 $\varphi_x,\varphi_{xx},\psi_x,\psi_{xx},\overline{\varphi}_x,\overline{\varphi}_{xx},\overline{\psi}_x,\overline{\psi}_{xx}$. 则可以在流形

$$
U=\{x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\varphi_{xx},\psi_x,\psi_{xx},\overline{\varphi}_x,\overline{\varphi}_{xx},\overline{\psi}_x,\overline{\psi}_{xx}\}\tag{2}
$$

上定义一组外微分 2-形式如下:

$$
\begin{aligned}
&\alpha_1=\mathrm{d}t\wedge\mathrm{d}\varphi+\mathrm{d}x\wedge\mathrm{d}t\varphi_x,\\
&\alpha_2=\mathrm{d}t\wedge\mathrm{d}\overline{\varphi}+\mathrm{d}x\wedge\mathrm{d}t\ \overline{\varphi}_x,\\
&\alpha_3=\mathrm{d}t\wedge\mathrm{d}\varphi_x+\mathrm{d}x\wedge\mathrm{d}t\varphi_{xx},\\
&\alpha_4=\mathrm{d}t\wedge\mathrm{d}\overline{\varphi}_x+\mathrm{d}x\wedge\mathrm{d}t\overline{\varphi}_{xx}.\\
&\alpha_5=\mathrm{i}\varepsilon\mathrm{d}t\wedge\mathrm{d}\varphi_{xx}+\mathrm{d}x\wedge\mathrm{d}t(\varphi_{xx}-3\mathrm{i}\varepsilon[(\varphi_x\overline{\varphi}+\psi_x\overline{\psi})\varphi+(\varphi\overline{\varphi}+\psi\overline{\psi})\varphi_x]\\
&\qquad+2(\varphi\overline{\varphi}+\psi\overline{\psi})\varphi-\mathrm{i}h\varphi_x)+\mathrm{i}\mathrm{d}x\wedge\mathrm{d}\varphi,\\
&\alpha_6=\mathrm{i}\varepsilon\mathrm{d}t\wedge\mathrm{d}\overline{\varphi}_{xx}-\mathrm{d}x\wedge\mathrm{d}t(\overline{\varphi}_{xx}-3\mathrm{i}\varepsilon[(\overline{\varphi}_x\varphi+\overline{\psi}_x\psi)\overline{\varphi}+(\overline{\varphi}\varphi+\overline{\psi}\psi)\overline{\varphi}_x]\\
&\qquad+2(\overline{\varphi}\varphi+\overline{\psi}\psi)\overline{\varphi}+\mathrm{i}h\overline{\varphi}_x)+\mathrm{i}\mathrm{d}x\wedge\mathrm{d}\overline{\varphi},
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 &= \mathrm{d}t\wedge \mathrm{d}\psi + \mathrm{d}x\wedge \mathrm{d}t\psi_x,\\
\beta_2 &= \mathrm{d}t\wedge \mathrm{d}\overline{\psi} + \mathrm{d}x\wedge \mathrm{d}t\overline{\psi}_x,\\
\beta_3 &= \mathrm{d}t\wedge \mathrm{d}\psi_x + \mathrm{d}x\wedge \mathrm{d}t\psi_{xx},\\
\beta_4 &= \mathrm{d}t\wedge \mathrm{d}\overline{\psi}_x + \mathrm{d}x\wedge \mathrm{d}t\overline{\psi}_{xx},\\
\beta_5 &= \mathrm{i}\varepsilon \mathrm{d}t\wedge \mathrm{d}\psi_{xx} + \mathrm{d}x\wedge \mathrm{d}t(\psi_{xx} - 3\mathrm{i}\varepsilon[(\varphi_x\overline{\varphi} + \psi_x\overline{\psi})\psi + (\varphi\overline{\varphi} + \psi\overline{\psi})\psi_x]\\
&\quad + 2(\varphi\overline{\varphi} + \psi\overline{\psi})\psi - \mathrm{i}h\psi_x) + \mathrm{i}\mathrm{d}x\wedge \mathrm{d}\psi,\\
\beta_6 &= \varepsilon \mathrm{i}\mathrm{d}t\wedge \mathrm{d}\overline{\psi}_{xx} - \mathrm{d}x\wedge \mathrm{d}t(\overline{\psi}_{xx} - 3\mathrm{i}\varepsilon[(\overline{\varphi}_x\varphi + \overline{\psi}_x\psi)\overline{\psi} + (\overline{\varphi}\varphi + \overline{\psi}\psi)\overline{\psi}_x]\\
&\quad + 2(\overline{\varphi}\varphi + \overline{\psi}\psi)\overline{\psi} + \mathrm{i}h\overline{\psi}_x) + \mathrm{i}\mathrm{d}x\wedge \mathrm{d}\overline{\psi}.
\end{aligned} \tag{3}
$$

其中 d 表示外导数, $\wedge$ 表示外积. 方程组 (3) α_1,α_2,α_3,α_4,β_1,β_2,β_3,β_4对应于引入新变元的项, α_5,α_6,β_5,β_6则对应于原始方程的项. 易证外微分形式构成的集合 $I=\{\alpha_i,\beta_i,i=1,2,3,4,5,6\}$ 在流形 I 上构成闭理想. 当 2-形式 α_i 限制到解流形

$$
\begin{aligned}
U=\{&x,t,\varphi(x,t),\psi(x,t),\overline{\varphi}(x,t),\overline{\psi}(x,t),\varphi_x(x,t),\psi_x(x,t),\\
&\overline{\varphi}_x(x,t),\overline{\psi}_x(x,t),\varphi_{xx}(x,t),\psi_{xx}(x,t),\overline{\varphi}_{xx}(x,t),\overline{\psi}_{xx}(x,t),\}
\end{aligned} \tag{4}
$$

上为0时, 则可以得到方程 (1). 现在引进 n 个玻色的外微分 1-形式.

$$
\begin{aligned}
w_0^k =&\mathrm{d}y^k + F_0^k(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx}\overline{\psi}_{xx},y^k)\mathrm{d}x\\
&+ G_0^k(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx}\overline{\psi}_{xx},y^k)\mathrm{d}t,\quad (k=1,\cdots,n_0)
\end{aligned} \tag{5}
$$

其中 y^k 为延拓变量, 以及一组新的费米延拓变量 ξ^l 以及新的费米的外微分 1-形式,

$$
\begin{aligned}
w_1^l =&\mathrm{d}\xi^l + F_1^l(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx},\overline{\psi}_{xx},\xi^l)\mathrm{d}x\\
&+ G_1^l(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx},\overline{\psi}_{xx},\xi^l)\mathrm{d}t,\quad (l=1,\cdots,n_1)
\end{aligned} \tag{6}
$$

F_0^k 和 G_0^k 是玻色函数,F_1^l 和 G_1^l 是费米函数. 并要求其与 α_i,β_i 构成一个新的闭理想, 即要求 $w_0^k(k=1,\ldots,n_0)$与 $w_1^l(l=1,\ldots,n_1)$ 满足条件,

$$
\mathrm{d}\omega_0^k + \mathrm{d}\omega_1^l = \sum_{j=1}^{6} f_j^k\alpha_j + \sum_{j=1}^{n_0}\eta_{0,j}^k\wedge\omega_0^j + \sum_{j=1}^{6} g_j^l\beta_j + \sum_{j=1}^{n_1}\eta_{1,j}^l\wedge\omega_1^j, \tag{7}
$$

其中 f_j^k , g_j^l为 0-形式, $\eta_{0,j}^k$, $\eta_{0,j}^l$为 1- 形式, 由方程 (7) 并消去 f_j^k, g_j^l, 可得

$$
\begin{aligned}
&F^k(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx},\overline{\psi}_{xx},y^k),\\
&G^k(x,t,\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx},\overline{\psi}_{xx},y^k),
\end{aligned}
$$

满足的偏微分方程组

$$F_{\varphi_x}=F_{\psi_x}=F_{\overline{\varphi}_x}=F_{\overline{\psi}_x}=F_{\varphi_{xx}}=F_{\psi_{xx}}=F_{\overline{\varphi}_{xx}}=F_{\overline{\psi}_{xx}}=0, \tag{7a}$$

$$\varepsilon F_\varphi-G_{\varphi_{xx}}=\varepsilon F_\psi-G_{\psi_{xx}}=\varepsilon F_{\overline{\varphi}}-G_{\overline{\varphi}_{xx}}=\varepsilon F_{\overline{\psi}}-G_{\overline{\psi}_{xx}}=0. \tag{7b}$$

$$\begin{aligned}&\varphi_xG_\varphi+\psi_xG_\psi+\bar{\varphi}_xG_{\overline{\varphi}}+\psi_{xx}G_{\psi_x}+\overline{\psi_{xx}}G_{\overline{\psi_x}}+\overline{\psi}_xG_\psi+\varphi_{xx}G_{\varphi_x}+\overline{\varphi}_{xx}G_{\overline{\varphi}_x}\\&-(\mathrm{i}\varphi_{xx}+2\mathrm{i}\varphi\overline{\varphi}\varphi+2\mathrm{i}\psi\overline{\psi}\varphi+6\varepsilon\varphi_x\overline{\varphi}\varphi+3\varepsilon\psi_x\overline{\psi}\varphi+3\varepsilon\psi\overline{\psi}\varphi_x+h\varphi_x)F_\varphi\\&-(\mathrm{i}\psi_{xx}+2\mathrm{i}\varphi\overline{\varphi}\psi+3\varepsilon\varphi_x\overline{\varphi}\psi+3\varepsilon\varphi\overline{\varphi}\psi_x+h\psi_x)F_\psi\\&+(\mathrm{i}\overline{\varphi}_{xx}+2\mathrm{i}\overline{\varphi}\varphi\overline{\varphi}+2\mathrm{i}\overline{\psi}\psi\overline{\varphi}-6\varepsilon\overline{\varphi}_x\varphi\overline{\varphi}-3\varepsilon\overline{\psi}_x\psi\overline{\varphi}-3\varepsilon\overline{\psi}\psi\overline{\varphi}_x-h\overline{\varphi}_x)F_{\overline{\varphi}}\\&+(\overline{\mathrm{i}\psi}_{xx}+2\mathrm{i}\overline{\varphi}\varphi\overline{\psi}-3\varepsilon\overline{\varphi}_x\varphi\overline{\psi}-3\varepsilon\overline{\varphi}\varphi\overline{\psi}_x-h\overline{\psi}_x)F_{\overline{\psi}}+[G,F]=0.\end{aligned} \tag{7c}$$

其中

$$F=F_0^k\partial_{y^k}+F_1^l\partial_{\xi^l},\quad G=G_0^{\overline{k}}\partial_{y^{\overline{k}}}+G_1^{\overline{l}}\partial_{\xi^{\overline{l}}}, \tag{8}$$

$$\begin{cases}[F,G]=(F_0^k\partial_{y^k}G_0^{\overline{k}}+F_1^l\partial_{\xi^l}G_0^{\overline{k}})\partial_{y^{\overline{k}}}-(G_0^{\overline{k}}\partial_{y^{\overline{k}}}F_0^k+G_1^{\overline{l}}\partial_{\xi^{\overline{l}}}F_0^k)\partial_{y^k}\\ \quad+(F_0^k\partial_{y^k}G_1^{\overline{l}}+F_1^l\partial_{\xi^l}G_1^{\overline{l}})\partial_{\xi^{\overline{l}}}-(G_0^{\overline{k}}\partial_{y^{\overline{k}}}F_1^l+G_1^{\overline{l}}\partial_{\xi^{\overline{l}}}F_1^l)\partial_{\xi^k}.\end{cases} \tag{9}$$

从方程 (7c) 式和 (9) 式, 可以将 F 和 G 的形式分别记为

$$F=F(\varphi,\psi,\overline{\varphi},\overline{\psi},\overline{y}) \tag{10}$$

$$G=\varepsilon(\varphi_{xx}F_\varphi+\psi_{xx}F_\psi+\overline{\varphi}_{xx}F_{\overline{\varphi}}+\overline{\psi}_{xx}F_{\overline{\psi}})+G_1(\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x,\overline{y}) \tag{11}$$

其中延拓变量, $\overline{y}=y_k\partial_{y_k}+\xi_l\partial_{\xi_l}$ 积分常数 G_1 与变量 $(\varphi_{xx},\psi_{xx},\overline{\varphi}_{xx},\overline{\psi}_{xx})$ 无关. 对方程 G 分别关于变量 $\varphi,\psi,\overline{\varphi},\overline{\psi},\varphi_x,\psi_x,\overline{\varphi}_x,\overline{\psi}_x$ 求导, 就得到

$$\begin{cases}G_\varphi=\varepsilon(\varphi_{xx}F_{\varphi\varphi}+\psi_{xx}F_{\psi\varphi}+\overline{\varphi}_{xx}F_{\overline{\varphi}\varphi}+\overline{\psi}_{xx}F_{\overline{\psi}\varphi})+G_{1\varphi},\\ G_\psi=\varepsilon(\varphi_{xx}F_{\varphi\psi}+\overline{\varphi}_{xx}F_{\overline{\varphi}\psi}-\overline{\psi}_{xx}F_{\overline{\psi}\psi})+G_{1\psi},\\ G_{\overline{\varphi}}=\varepsilon(\varphi_{xx}F_{\varphi\overline{\varphi}}+\psi_{xx}F_{\psi\overline{\varphi}}+\overline{\varphi}_{xx}F_{\overline{\varphi}\overline{\varphi}}+\overline{\psi}_{xx}F_{\overline{\psi}\overline{\varphi}})+G_{1\overline{\varphi}},\\ G_{\overline{\psi}}=\varepsilon(\varphi_{xx}F_{\varphi\overline{\psi}}-\psi_{xx}F_{\psi\overline{\psi}}+\overline{\varphi}_{xx}F_{\overline{\varphi}\overline{\psi}})+G_{1\overline{\psi}},\\ G_{\varphi_x}=G_{1\varphi_x},\quad G_{\psi_x}=G_{1\psi_x},\quad G_{\overline{\varphi}_x}=G_{1\overline{\varphi}_x},\quad G_{\overline{\psi}_x}=G_{1\overline{\psi}_x}.\end{cases} \tag{12}$$

将上式代入到方程(7c)式中, 可得

$$\begin{aligned}&\varepsilon\varphi_x\varphi_{xx}F_{\varphi\varphi}+\varepsilon\varphi_x\varphi_{xx}F_{\psi\varphi}+\varepsilon\varphi_x\overline{\varphi}_{xx}F_{\overline{\varphi}\varphi}+\varepsilon\varphi_x\overline{\psi}_{xx}F_{\overline{\psi}\varphi}+\varphi_xG_{1\varphi}+\varepsilon\psi_x\varphi_{xx}F_{\varphi\psi}\\&+\varepsilon\psi_x\psi_{xx}F_{\psi\psi}+\varepsilon\psi_x\overline{\varphi}_{xx}F_{\overline{\varphi}\psi}+\varepsilon\psi_x\overline{\psi}_{xx}F_{\overline{\psi}\psi}+\psi_xG_{1\psi}+\varepsilon\overline{\varphi}_x\varphi_{xx}F_{\varphi\overline{\varphi}}\\&+\varepsilon\overline{\varphi}_x\psi_{xx}F_{\psi\overline{\varphi}}+\varepsilon\overline{\varphi}_x\overline{\varphi}_{xx}F_{\overline{\varphi\varphi}}+\varepsilon\overline{\varphi}_x\overline{\psi}_{xx}F_{\overline{\psi\varphi}}+\overline{\varphi}_xG_{1\overline{\varphi}}+\varepsilon\overline{\psi}_x\varphi_{xx}F_{\varphi\overline{\psi}}\\&+\varepsilon\overline{\psi}_x\overline{\varphi}_{xx}F_{\overline{\varphi\psi}}+\varepsilon\overline{\psi}_x\overline{\psi}_{xx}F_{\overline{\psi\psi}}+\overline{\psi}_xG_{1\overline{\psi}}+\psi_{xx}G_{1\psi_x}+\overline{\psi}_{xx}G_{1\overline{\psi}_x}\\&+\varphi_{xx}G_{1\varphi_x}+\overline{\varphi}_{xx}G_{1\overline{\varphi}_x}-\mathrm{i}\varphi_{xx}F_\varphi-2\mathrm{i}\varphi\overline{\varphi}\varphi F_\varphi-2\mathrm{i}\psi\overline{\psi}\varphi F_\varphi-6\varepsilon\varphi_x\overline{\varphi}\varphi F_\varphi\\&-3\varepsilon\psi_x\overline{\psi}\varphi F_\varphi-h\varphi_xF_\varphi-\mathrm{i}\psi_{xx}F_\psi-2\mathrm{i}\varphi\overline{\varphi}\psi F_\psi-3\varepsilon\varphi_x\overline{\varphi}\psi F_\psi-3\varepsilon\phi\overline{\phi}\psi_xF_\psi\\&-h\psi_xF_\psi+\mathrm{i}\overline{\phi}_{xx}F_{\overline{\phi}}+2\mathrm{i}\overline{\varphi}\varphi\overline{\varphi}F_{\overline{\varphi}}+2\mathrm{i}\overline{\psi}\psi\overline{\varphi}F_{\overline{\varphi}}-6\varepsilon\overline{\varphi}_x\varphi\overline{\varphi}F_{\overline{\varphi}}-3\varepsilon\overline{\psi}_x\psi\overline{\varphi}F_{\overline{\varphi}}\\&-3\varepsilon\overline{\psi}\psi\overline{\varphi}_xF_{\overline{\varphi}}-h\overline{\varphi}_xF_{\overline{\varphi}}+\mathrm{i}\overline{\psi}_{xx}F_{\overline{\psi}}+2\mathrm{i}\overline{\varphi}\varphi\overline{\psi}F_{\overline{\psi}}-3\varepsilon\overline{\varphi}_x\varphi\overline{\psi}F_{\overline{\psi}}-3\varepsilon\overline{\varphi}\varphi\overline{\psi}_xF_{\overline{\psi}}\\&-h\overline{\psi}_xF_{\overline{\psi}}+\varepsilon\varphi_{xx}[F_\varphi,F]+\varepsilon\psi_{xx}[F_\psi,F]+\varepsilon\overline{\varphi}_{xx}[F_{\overline{\varphi}},F]+\varepsilon\overline{\psi}_{xx}[F_{\overline{\psi}},F]+[G_1,F]=0.\end{aligned} \tag{13}$$

为了得到 G_1 的表达式, 对(13)式方程分别关于变量 φ_{xx},ψ_{xx},$\overline{\varphi}_{xx}$,$\overline{\psi}_{xx}$ 求导, 得

$$\begin{aligned}
G_{1\varphi_x} &= \mathrm{i}F_\varphi - \varepsilon(\varphi_x F_{\varphi\varphi} + \psi_x F_{\varphi\psi} + \overline{\varphi}_x F_{\varphi\overline{\varphi}} + \overline{\psi}_x F_{\varphi\overline{\psi}}) - \varepsilon[F_\varphi, F],\\
G_{1\psi_x} &= \mathrm{i}F_\psi - \varepsilon(\varphi_x F_{\psi\varphi} + \overline{\varphi}_x F_{\psi\overline{\varphi}} + \overline{\psi}_x F_{\psi\overline{\psi}}) - \varepsilon[F_\psi, F],\\
G_{1\overline{\varphi}_x} &= -\mathrm{i}F_{\overline{\psi}} - \varepsilon(\varphi_x F_{\overline{\varphi}\varphi} + \psi_x F_{\overline{\varphi}\psi} + \overline{\varphi}_x F_{\overline{\varphi}\overline{\varphi}} + \overline{\psi}_x F_{\overline{\varphi}\overline{\psi}}) - \varepsilon[F_{\overline{\varphi}}, F],\\
G_{1\overline{\psi}_x} &= -\mathrm{i}F_{\overline{\psi}} - \varepsilon(\varphi_x F_{\overline{\psi}\varphi} + \psi_x F_{\overline{\psi}\psi} + \overline{\varphi}_x F_{\overline{\psi}\overline{\varphi}}) - \varepsilon[F_{\overline{\psi}}, F].
\end{aligned} \tag{14}$$

方程组（14）中的四个方程分别对 φ_x,ψ_x,$\overline{\varphi}_x$,$\overline{\psi}_x$ 积分, 得

$$\begin{aligned}
G_1 = {}&\mathrm{i}(\varphi_x F_\varphi + \psi_x F_\psi - \overline{\psi}_x F_{\overline{\varphi}} - \overline{\psi}_x F_{\overline{\psi}}) - \varepsilon\Big(\frac{1}{2}\varphi_{x^2} F_{\varphi\varphi} + \varphi_x\psi_x F_{\varphi\psi} + \varphi_x\overline{\varphi}_x F_{\varphi\overline{\varphi}} + \varphi_x\overline{\psi}_x F_{\varphi\overline{\psi}}\\
&+ \psi_x\overline{\varphi}_x F_{\psi\overline{\varphi}} + \psi_x\overline{\psi}_x F_{\psi}\overline{\psi} + \frac{1}{2}\overline{\varphi}_{x^2} F_{\overline{\varphi}\overline{\varphi}} + \overline{\varphi}_x\overline{\psi}_x F_{\overline{\varphi}\overline{\psi}} + \varphi_x[F_\varphi, F] + \psi_x[F_\psi, F] + \overline{\varphi}_x[F_{\overline{\varphi}}, F]\\
&+ \overline{\psi}_x[F_{\overline{\varphi}}, F] + \overline{\psi}_x[F_{\overline{\psi}}, F]\Big) + G_2(\varphi, \overline{\varphi}, \psi, \overline{\psi}, \overline{y}).
\end{aligned} \tag{15}$$

其中, 积分常数 G_2 仅依赖于变量 $\varphi, \overline{\varphi}, \psi, \overline{\psi}, \overline{y}$. 重复求解 G_1 的过程我们将(15)式中 G_1 的表达式代入到(14)式中, 对所得到的 G 分别关于 $\varphi, \psi, \overline{\varphi}, \overline{\psi}, \varphi_x$,$\psi_x$,$\overline{\varphi}_x$,$\overline{\psi}_x$ 求导后代入到(15)式并展开, 比较各独立单项式的系数, 可得下列方程组

$$\begin{gathered}
F_{\varphi\varphi\varphi} = 0, \quad F_{\varphi\overline{\varphi}\varphi} = 0, \quad F_{\varphi\psi\varphi} = 0, \quad F_{\varphi\overline{\psi}\varphi} = 0, \quad F_{\overline{\varphi}\overline{\varphi}\varphi} = 0, \quad F_{\psi\overline{\varphi}\varphi} = 0,\\
F_{\overline{\varphi}\overline{\psi}\varphi} = 0, \quad F_{\psi\overline{\psi}\varphi} = 0, \quad F_{\overline{\varphi}\overline{\varphi}\overline{\varphi}} = 0, \quad F_{\overline{\varphi}\overline{\varphi}\psi} = 0, \quad F_{\overline{\varphi}\overline{\varphi}\overline{\psi}} = 0, \quad F_{\overline{\varphi}\psi\overline{\psi}} = 0,\\
\mathrm{i}F_{\varphi\varphi} - \frac{3}{2}\varepsilon[F_{\varphi\varphi}, F] = 0, \quad \mathrm{i}F_{\varphi\overline{\varphi}} + \frac{3}{2}\varepsilon[F_{\varphi\overline{\varphi}}, F] = 0, \quad \mathrm{i}F_{\varphi\psi} - \frac{3}{2}\varepsilon[F_{\varphi\psi}, F] = 0,\\
\mathrm{i}F_{\varphi\overline{\psi}} + \frac{3}{2}\varepsilon[F_{\varphi\overline{\psi}}, F] = 0, \quad \mathrm{i}F_{\overline{\varphi}\overline{\varphi}} + \frac{3}{2}\varepsilon[F_{\overline{\varphi}\overline{\varphi}}, F] = 0, \quad \mathrm{i}F_{\overline{\varphi}\psi} + \frac{3}{2}\varepsilon[F_{\psi\overline{\psi}}, F] = 0,\\
-3\varepsilon(2\overline{\varphi}\varphi+\psi\overline{\psi})F_\varphi+\overline{\varphi}\psi F_\psi+G_{2\varphi}+\mathrm{i}[F_\varphi, F]-\varepsilon[[F_\varphi, F], F] = 0, \quad -3\varepsilon(2\varphi\overline{\varphi}+\overline{\psi}\psi)F_{\overline{\varphi}}+\varphi\overline{\psi}F_{\overline{\varphi}}+G_{2\overline{\varphi}}\\
-\mathrm{i}[F_{\overline{\psi}}, F] - \varepsilon[[F_{\overline{\varphi}}, F], F] = 0, \quad -3\varepsilon(\overline{\psi}\varphi F_\varphi + \varphi\overline{\varphi})F_\psi + G_{2\overline{\psi}} + \mathrm{i}[F_\psi, F] - \varepsilon[[F_{\overline{\psi}}, F], F] = 0,\\
-2\mathrm{i}(\varphi\overline{\varphi} + \psi\overline{\psi})\varphi F_\varphi + (\varphi\overline{\varphi} + \psi\overline{\psi})\psi F_\psi + (\overline{\varphi}\varphi + \overline{\psi}\psi)\overline{\varphi}F_{\overline{\varphi}} + (\varphi\overline{\varphi} + \psi\overline{\psi})\psi F_{\overline{\psi}} + [G_2, F] = 0.
\end{gathered} \tag{16}$$

通过分析方程组(16)式, 我们可以得出如下形式, 其中待定函数 $F(\varphi, \overline{\varphi}, \psi, \overline{\psi}, \overline{y})$ 中只含有 $\varphi, \overline{\varphi}, \psi, \overline{\psi}$ 的一次项, 因此可以将 F 取作下列的形式.

$$F = \mathrm{i}(\varphi X_1 + \overline{\varphi}X_2 + \psi X_{-1} + \overline{\psi}X_{-2} + X_0), \tag{17}$$

其中 $X_i(i = 0, 1, -1, 2, -2,)$ 为仅仅依赖于延拓变量 $\overline{y}$ 的生成元, 下标的正负分别代表 X_i 为偶生成元和奇生成元. 再由(17)式可以得到 G_2 对于 $\varphi, \overline{\varphi}, \psi, \overline{\psi}$ 的导数如下.

$$\begin{aligned}
G_{2\varphi} &= 3\varepsilon\{(2\overline{\varphi}\varphi + \psi\overline{\psi})F_\varphi + \overline{\varphi}\psi F_\psi\} - \mathrm{i}[F_\varphi, F] + \varepsilon[[F_\varphi, F], F],\\
G_{2\overline{\varphi}} &= 3\varepsilon\{(2\varphi\overline{\varphi} + \overline{\psi}\psi)F_{\overline{\varphi}} + \varphi\overline{\psi}F_{\overline{\psi}}\} + \mathrm{i}[F_{\overline{\varphi}}, F] + \varepsilon[[F_{\overline{\varphi}}, F], F],\\
G_{2\psi} &= 3\varepsilon\{\overline{\psi}\varphi F_\varphi + \varphi\overline{\varphi}F_\psi\} - \mathrm{i}[F_\psi, F] + \varepsilon[[F_\psi, F], F],\\
G_{2\overline{\psi}} &= 3\varepsilon\{\psi\overline{\varphi}F_{\overline{\varphi}} + \overline{\varphi}\varphi F_{\overline{\psi}}\} + \mathrm{i}[F_{\overline{\psi}}, F] + \varepsilon[[F_{\overline{\psi}}, F], F].
\end{aligned} \tag{18}$$

为了利用(18)式得到 G_2 的具体表达式, 需要利用以下关系式：

$$
\begin{aligned}
&G_{2\varphi\overline{\varphi}}=G_{2\overline{\varphi}\varphi},\quad G_{2\varphi\psi}=G_{2\psi\varphi},\quad G_{2\varphi\overline{\psi}}=G_{2\overline{\psi}\varphi},\\
&G_{2\overline{\varphi}\psi}=G_{2\psi\overline{\varphi}},\quad G_{2\overline{\varphi}\overline{\psi}}=G_{2\overline{\psi}\overline{\varphi}},\quad G_{2\psi\overline{\psi}}=G_{2\overline{\psi}\psi},
\end{aligned}\tag{19}
$$

将 F 的表达式(17)式代入到 G_2 对于 $\varphi,\overline{\varphi},\psi,\overline{\psi}$ 的导数的表达式(18)式中, 经过直接的运算整理, 由(19)式中的六个关系式可以得到方程组:

$$
\begin{aligned}
&\varphi(2X_1-[[X_1,X_2],X_1])-\overline{\varphi}(2X_2+[[X_1,X_2],X_2])+\psi(X_{-1}-[[X_1,X_2],X_{-1}])-\overline{\psi}(X_{-2}\\
&+[[X_1,X_2],X_{-2}])-[[X_1,X_2],X_0]=0,\quad 2[X_1,X_{-1}]-3\varphi[[X_1,X_{-1}],X_1])+\overline{\varphi}[[X_1,X_{-1}],X_2]\\
&+\psi[[X_1,X_{-1}],X_{-1}]+\overline{\psi}[[X_1,X_{-1}],X_{-2}]-[[X_1,X_2],X_0]\}=0,\quad \varphi([[X_1,X_{-2}],X_1])\\
&-\overline{\varphi}(X_{-2}+[[X_1,X_{-2}],X_2])+\psi(X_1-[[X_1,X_{-2}],X_{-1}])-\overline{\psi}([[X_1,X_{-2}],X_{-2}])\\
&-[[X_1,X_{-2}],X_0]=0,\quad -\varphi(X_{-1}+[[X_2,X_{-1}],X_1])-\overline{\varphi}[[X_2,X_{-1}],X_2])+\psi[[X_2,X_{-1}],X_{-1}])\\
&+\overline{\psi}(X_2-[[X_2,X_{-1}],X_{-2}])-[[X_2,X_{-1}],X_0]=0,2[X_2,X_{-2}]-3\varphi[[X_2,X_{-2}],X_1])\\
&+\overline{\varphi}[[X_2,X_{-2}],X_2]+\psi[[X_2,X_{-2}],X_{-1}]+\overline{\psi}[[X_2,X_{-2}],X_{-2}]-[[X_2,X_{-2}],X_0]\}=0,\\
&\varphi(X_1-[[X_{-1},X_{-2}],X_1])-\overline{\varphi}(X_2+[[X_{-1},X_{-2}],X_2])+\psi[[X_{-1},X_{-2}],X_{-1}])\\
&-\overline{\psi}[[X_{-1},X_{-2}],X_{-2}])-[[X_{-1},X_{-2}],X_0]=0.
\end{aligned}\tag{20}
$$

比较(20)式的六组方程中各独立单项式的系数, 我们可得下列对易关系式.

$$
\begin{cases}
\varphi:[[X_1,X_2],X_1]=2X_1,\\
\overline{\varphi}:[[X_1,X_2],X_2]=-2X_2,\\
\psi:[[X_1,X_2],X_{-1}]=X_{-1},\\
\overline{\psi}:[[X_1,X_2],X_{-2}]=-X_{-2},\\
1:[[X_1,X_2],X_0]=0.
\end{cases}\tag{21}
$$

$$
\begin{cases}
\varphi:[[X_1,X_{-1}],X_1]=0\\
\overline{\varphi}:[[X_1,X_{-1}],X_2]=0,\\
\psi:[[X_1,X_{-1}],X_{-1}]=0,\\
\overline{\psi}:[[X_1,X_{-1}],X_{-2}]=0,\\
1:2[X_1,X_{-1}]+3[[X_1,X_{-1}],X_0]=0.
\end{cases}\tag{22}
$$

$$
\begin{cases}
\varphi:[[X_1,X_{-2}],X_1]=0\\
\overline{\varphi}:[[X_1,X_{-2}],X_2]=-X_{-2},\\
\psi:[[X_1,X_{-2}],X_{-1}]=X_1,\\
\overline{\psi}:[[X_1,X_{-2}],X_{-2}]=0,\\
1:[[X_1,X_{-2}],X_0]=0.
\end{cases}\tag{23}
$$

$$
\begin{cases}
\varphi:[[X_2,X_{-1}],X_1]=-X_{-1}\\
\overline{\varphi}:[[X_2,X_{-1}],X_2]=0,\\
\psi:[[X_2,X_{-1}],X_{-1}]=0,\\
\overline{\psi}:[[X_2,X_{-1}],X_{-2}]=X_2,\\
1:[[X_2,X_{-1}],X_0]=0.
\end{cases}\tag{24}
$$

$$\begin{cases} \varphi : [[X_2, X_{-2}], X_1] = 0 \\ \overline{\varphi} : [[X_2, X_{-2}], X_2] = 0, \\ \psi : [[X_2, X_{-2}], X_{-1}] = 0, \\ \overline{\psi} : [[X_2, X_{-2}], X_{-2}] = 0, \\ 1 : 2[X_2, X_{-2}] + 3[[X_2, X_{-2}], X_0] = 0. \end{cases} \tag{25}$$

$$\begin{cases} \varphi : [[X_{-1}, X_{-2}], X_1] = X_1 \\ \overline{\varphi} : [[X_{-1}, X_{-2}], X_2] = -X_2, \\ \psi : [[X_{-1}, X_{-2}], X_{-1}] = 0, \\ \overline{\psi} : [[X_{-1}, X_{-2}], X_{-2}] = 0, \\ 1 : [[X_{-1}, X_{-2}], X_0] = 0. \end{cases} \tag{26}$$

其中上式运算关系式中的两个算子函数均为费米变量(X_i,i 为负整数), 则它们之间为反对易关系, 仍然采用 [,]来表示, 例如

$$[X_{-1}, X_{-2}] = X_{-1}X_{-2} + X_{-2}X_{-1}. \tag{27}$$

下面就通过分析式(21)–(27), 来确定这些生成元之间所有的对易关系, 并给出它们的一组表示. 由式(21)–(27), 可以得出：

$$[X_1, X_{-1}] = 0, \quad [X_2, X_{-2}] = 0. \tag{28}$$

为了运算的方便, 定义一些新的代数生成元如下.

$$[X_1, X_2] \equiv X_3, \quad [X_1, X_{-2}] \equiv X_{-4}, \quad [X_1, X_0] \equiv X_5,$$

$$[X_{-1}, X_2] \equiv -X_{-6}, \quad [X_{-1}, X_{-2}] \equiv X_7, \quad [X_{-1}, X_0] \equiv X_{-8},$$

$$[X_2, X_0] \equiv X_9, \quad [X_{-2}, X_0] \equiv X_{-10}. \tag{29}$$

则式 (27)–(29)中的各个对易关系式与反对易关系式可以重新记为

$$[X_3, X_1] = 2X_1, \quad [X_3, X_{-1}] = X_{-1}, \quad [X_3, X_2] = -2X_2,$$

$$[X_3, X_{-2}] = -X_{-2}, \quad [X_3, X_0] = 0, \quad [X_{-4}, X_1] = 0,$$

$$[X_{-4}, X_{-1}] = X_1, \quad [X_{-4}, X_2] = -X_{-2}, \quad [X_{-4}, X_{-2}] = 0,$$

$$[X_{-4}, X_0] = 0, \quad [X_{-6}, X_1] = -X_{-1}, \quad [X_{-6}, X_{-1}] = 0,$$

$$[X_{-6}, X_{-2}] = X_2, \quad [X_{-6}, X_0] = 0, \quad [X_{-6}, X_2] = 0,$$

$$[X_7, X_1] = X_1, \quad [X_7, X_{-1}] = 0, \quad [X_7, X_2] = -X_2,$$

$$[X_7, X_{-2}] = 0, \quad [X_7, X_0] = 0. \tag{30}$$

利用 Jacobi 恒等式以及式(29)中结果, 还可以得到如下的关系式.

$$[X_5, X_1] = 0, \quad [X_5, X_2] = -X_3, \quad [X_5, X_{-1}] = 0,$$

$$
\begin{gathered}
[X_5,X_{-2}]=X_{-4},\quad [X_5,X_0]=X_1,\quad [X_9,X_1]=-X_3,\\
[X_9,X_2]=0,\quad [X_9,X_{-1}]=-X_{-1},\quad [X_9,X_{-2}]=0,\\
[X_9,X_0]=X_2,\quad [X_{-8},X_1]=0,\quad [X_{-8},X_2]=\lambda X_{-6},\\
[X_{-8},X_{-1}]=0,\quad [X_{-8},X_{-2}]=-\lambda X_7,[X_{-8},X_0]=-\lambda X_{-8},\\
[X_{-10},X_1]=-\lambda X_{-4},[X_{-10},X_2]=0,\quad [X_{-10},X_{-1}]=-\lambda X_7,\\
[X_{-10},X_{-2}]=X_2,\quad [X_{-10},X_0]=-\lambda X_{-10}.
\end{gathered}
\tag{31}
$$

分析(30)式和(31)式中的代数关系, 利用李代数表示理论, 得到生成元的一组矩阵表示如下, 其中玻色成元为

$$
\begin{aligned}
X_0&=\begin{pmatrix}0&0&0\\0&-\lambda&0\\0&0&-\lambda\end{pmatrix},\quad X_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\\
X_2&=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad X_3=\begin{pmatrix}1&0&0\\0&-1&0\\1&0&0\end{pmatrix},\\
X_5&=\begin{pmatrix}0&-\lambda&0\\0&0&0\\0&0&0\end{pmatrix},\quad X_7=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix},\\
X_9&=\begin{pmatrix}0&0&0\\\lambda&0&0\\0&0&0\end{pmatrix}.
\end{aligned}
\tag{32}
$$

费米成元为

$$
\begin{aligned}
X_{-1}&=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad X_{-2}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\\
X_{-4}&=\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix},\quad X_{-6}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\\
X_{-8}&=\begin{pmatrix}0&0&-\lambda\\0&0&0\\0&0&0\end{pmatrix},\quad X_{-10}=\begin{pmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{pmatrix}.
\end{aligned}
\tag{33}
$$

考察式(32)和式(33)中的表示矩阵与超代数 usp1 的关系, 容易验证

$$
\begin{aligned}
&X_0=\lambda(T_3-T_4),\quad X_1=T_1+\mathrm{i}T_2,\quad X_2=T_1-\mathrm{i}T_2,\\
&X_{-1}=T_5+\mathrm{i}T_6,\quad X_{-2}=T_5-\mathrm{i}T_6,\quad X_3=2T_3,\\
&X_{-4}=-T_7+\mathrm{i}T_8,\quad X_5=-\lambda(T_1+\mathrm{i}T_2),\quad X_{-6}=T_7+\mathrm{i}T_8,\\
&X_7=T_3+T_4,\quad X_8=-\lambda(T_5+\mathrm{i}T_6),\quad X_9=\lambda(T_1-\mathrm{i}T_2),\\
&X_{10}=\lambda(T_5-\mathrm{i}T_6),
\end{aligned}
\tag{34}
$$

$$\overrightarrow{T}=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},T_1=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},T_2=\frac{\mathrm{i}}{2}\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix},$$
$$T_3=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\1&0&0\end{pmatrix},T_4=\frac{1}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix},T_5=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\tag{35}$$
$$T_6=\frac{\mathrm{i}}{2}\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},T_7=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},T_8=\frac{\mathrm{i}}{2}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},$$

其中 $\overrightarrow{\sigma}=\sigma_1,\sigma_2,\sigma_3$ 是 Pauli 矩阵, I_2 是幺正矩阵.

利用在(34)式与(35)式中所求得的生成元之间的对易关系式, 可以将(18)式中 G_2 关于 φ,ψ,$\overline{\varphi}$,$\overline{\psi}$ 的导数的表达式重新写作

$$\begin{cases}G_{2\varphi}=2\mathrm{i}\varepsilon[(2\varphi\overline{\varphi}+\psi\overline{\psi})X_1+\overline{\varphi}\psi X_{-1}+\overline{\varphi}^2X_2+\overline{\psi}\overline{\varphi}X_{-2}]+\mathrm{i}(1+\lambda\varepsilon)(\overline{\varphi}X_3+\overline{\psi}X_{-4}+X_5)\\G_{2\overline{\varphi}}=2\mathrm{i}\varepsilon[\varphi^2X_1+\varphi\psi X_{-1}+(2\varphi\overline{\varphi}+\psi\overline{\psi})X_2+\overline{\psi}\varphi X_{-2}]+\mathrm{i}(1+\lambda\varepsilon)(\varphi X_3-\psi X_{-6}-X_9),\\G_{2\psi}=2\mathrm{i}\varepsilon[\overline{\psi}\varphi X_1+\varphi\overline{\varphi}X_{-1}+\overline{\varphi}\overline{\psi}X_2]+\mathrm{i}(1+\lambda\varepsilon)(-\overline{\varphi}X_{-6}+\overline{\psi}X_7+X_{-8}),\\G_{2\overline{\psi}}=2\mathrm{i}\varepsilon[-\psi\varphi X_1-\overline{\varphi}\psi X_2+\varphi\overline{\varphi}X_{-2}]+\mathrm{i}(1+\lambda\varepsilon)(\varphi X_{-4}-\psi X_7-X_{-10}).\end{cases}\tag{36}$$

将以上方程组(23)中的四个方程分别对变量 φ,ψ,$\overline{\varphi}$,$\overline{\psi}$ 积分, 并利用我们比较各独立系数所得到的方程组(36) 中的最后一式确定积分常数, 可以得到

$$G_2=2\mathrm{i}\varepsilon[(\varphi\overline{\varphi}+\psi\overline{\psi})\varphi X_1+\varphi\overline{\varphi}\psi X_{-1}+\overline{\varphi}(\varphi\overline{\varphi}+\psi\overline{\psi})X_2+\overline{\psi}\varphi\overline{\varphi}X_{-2}]+\mathrm{i}h(\varphi X_1+\psi X_{-1}+\overline{\varphi}X_2+\overline{\psi}X_{-2})$$
$$+\mathrm{i}(1+\lambda\varepsilon)(\varphi\overline{\varphi}X_3+\varphi\overline{\psi}X_{-4}+\varphi X_5-\psi\overline{\varphi}X_{-6}+\psi\overline{\psi}X_7+\psi X_{-8}-\overline{\varphi}X_9-\overline{\psi}X_{-10})+(\mathrm{i}\varepsilon\lambda^2-\lambda)X_0.\tag{37}$$

因此, 我们所求得的一组三阶超对称非线性 Schrödinger 方程 (1) 的 Lax 表示为

$$\begin{aligned}F=&\mathrm{i}(\varphi X_1+\psi X_{-1}+\overline{\varphi}X_{-2}+X_0),\\G=&\mathrm{i}\{[\varepsilon\varphi_{xx}+\mathrm{i}\varphi_x+2\varepsilon(\varphi\overline{\varphi}+\psi\overline{\psi})\varphi]X_1+[\varepsilon\psi_{xx}+\mathrm{i}\psi_x+2\varepsilon(\varphi\overline{\varphi}+\psi\overline{\psi})\psi]X_{-1}\\&+[\varepsilon\overline{\varphi}_{xx}-\mathrm{i}\overline{\varphi}_X+2\varepsilon\overline{\varphi}(\varphi\overline{\varphi}+\psi\overline{\psi})]X_2+[\varepsilon\overline{\psi}_{xx}-\mathrm{i}\overline{\psi}_x+2\varepsilon\overline{\psi}(\varphi\overline{\varphi}+\psi\overline{\psi})]X_{-2}\\&+[\mathrm{i}\varepsilon(-\varphi_x\overline{\varphi}+\overline{\varphi}_x\varphi)+(1+\lambda\varepsilon)\varphi\overline{\varphi}]X_3+[\mathrm{i}\varepsilon(-\varphi_x\overline{\psi}+\overline{\psi}_x\varphi)+(1+\lambda\varepsilon)\varphi\overline{\psi}]X_{-4}\\&+[-\mathrm{i}\varepsilon\varphi_x+(1+\lambda\varepsilon)\varphi]X_5+[\mathrm{i}\varepsilon(\psi_x\overline{\varphi}-\overline{\varphi}_x\psi)-(1+\lambda\varepsilon)\psi\overline{\varphi}]X_{-6}+[-\mathrm{i}\varepsilon(\psi_x\overline{\psi}+\overline{\psi}_x\psi)\\&+(1+\lambda\varepsilon)\psi\overline{\psi}]X_7+[-\mathrm{i}\varepsilon\psi_x+(1+\lambda\varepsilon)\psi]X_{-8}+[-\mathrm{i}\varepsilon\overline{\varphi}_x-(1+\lambda\varepsilon)\overline{\varphi}]X_9\\&+[-\mathrm{i}\varepsilon\overline{\psi}_x-(1+\lambda\varepsilon)\overline{\psi}]X_{-10}\}+[\mathrm{i}\varepsilon\lambda^2-\lambda]X_0.\end{aligned}\tag{39}$$

为了构造一组三阶超对称非线性 Schrödinger 方程解的 Bäcklund 变换, 我们将表示矩阵(31)和(35)代入 F 和 G, 得到方程(1)的 Lax 表示

若要求 $\omega^k|_U = 0$, 可以得到方程 (1) 的 Lax 表示,

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_x = \mathrm{i}\begin{pmatrix} 0 & \varphi & \psi \\ \overline{\varphi} & -\lambda & 0 \\ \overline{\psi} & 0 & -\lambda \end{pmatrix}, \quad \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_x = F\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix},$$

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_t = \mathrm{i}\begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}, \quad \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_t = G\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}, \tag{40}$$

其中

$$\begin{cases} G_{11} = \mathrm{i}\varepsilon(\varphi\overline{\varphi}_x - \varphi_x\overline{\varphi} + \psi\overline{\psi}_x - \psi_x\overline{\psi}) + (1+\lambda\varepsilon)(\varphi\overline{\varphi} + \psi\overline{\psi}), \\ G_{12} = \varepsilon\varphi_{xx} + \mathrm{i}(1+\lambda\varepsilon)\varphi_x + 2\varepsilon(\varphi\overline{\varphi} + \psi\overline{\psi})\varphi - (1+\lambda\varepsilon)\lambda\varphi, \\ G_{13} = \varepsilon\psi_{xx} + \mathrm{i}(1+\lambda\varepsilon)\psi_x + 2\varepsilon(\varphi\overline{\varphi} + \psi\overline{\psi})\psi - (1+\lambda\varepsilon)\lambda\psi, \\ G_{21} = \varepsilon\overline{\varphi}_{xx} - \mathrm{i}(1+\lambda\varepsilon)\overline{\varphi}_x + 2\varepsilon\overline{\varphi}(\varphi\overline{\varphi} + \psi\overline{\psi}) - (1+\lambda\varepsilon)\lambda\overline{\varphi}, \\ G_{22} = \mathrm{i}\varepsilon(\overline{\varphi}\varphi_x - \overline{\varphi}_x\varphi) - (1+\lambda\varepsilon)\overline{\varphi}\varphi + \lambda^2 - \mathrm{i}\varepsilon\lambda^3, \\ G_{23} = \mathrm{i}\varepsilon(\overline{\varphi}\psi_x - \overline{\varphi}_x\psi) - (1+\lambda\varepsilon)\overline{\varphi}\psi, \\ G_{31} = \varepsilon\overline{\psi}_{xx} - \mathrm{i}(1+\lambda\varepsilon)\overline{\psi}_x + 2\varepsilon\overline{\psi}(\varphi\overline{\varphi} + \psi\overline{\psi}) - (1+\lambda\varepsilon)\lambda\overline{\psi}, \\ G_{32} = \mathrm{i}\varepsilon(\overline{\psi}\varphi_x - \overline{\psi}_x\varphi) - (1+\lambda\varepsilon)\overline{\psi}\varphi, \\ G_{33} = \mathrm{i}\varepsilon(\overline{\psi}\psi_x - \overline{\psi}_x\psi) - (1+\lambda\varepsilon)\overline{\psi}\psi + \lambda^2 - \mathrm{i}\varepsilon\lambda^3. \end{cases} \tag{41}$$

由相容性关系 $y_{xt} = y_{tx}$, 很容易得到原方程 (1).

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