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Cells of the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$

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Abstract: The affine Coxeter group $(\widetilde{B}_3, S)$ can be realized as the fixed point set of the affine Coxeter group $(\widetilde{D}_4, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$. Let $\widetilde{\ell}$ be the length function of $\widetilde{D}_4$. We gave an explicit description for all the left cells of the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$. Also，we showed that in the the weighted Coxeter groups $(\widetilde{D}_4, \widetilde{\ell})$ and $(\widetilde{B}_3, \widetilde{\ell})$, each left (respectively, two-sided) cell was left-connected (respectively, two-sided-connected).

Key words: weighted Coxeter group; quasi-split case; cells; left-connectedness

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加权 Coxeter 群$(\widetilde{B}_3, \widetilde{\ell})$的胞腔

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摘要: 仿射 Coxeter 群 $(\widetilde{B}_3, S)$ 可以被看做仿射 Coxeter 群 $(\widetilde{D}_4, \widetilde{S})$ 在满足条件 $\alpha(\widetilde{S}) = \widetilde{S}$ 的某种群自同构 α 下的不动点集合. 设 $\widetilde{\ell}$ 是 $\widetilde{D}_4$ 的长度函数. 本文明显地刻画了加权 Coxeter 群 $(\widetilde{B}_3, \widetilde{\ell})$ 的所有左胞腔. 同时证明了: 加权 Coxeter 群 $(\widetilde{D}_4, \widetilde{\ell})$ 和 $(\widetilde{B}_3, \widetilde{\ell})$ 的所有左胞腔都是左连通的, 所有双边胞腔都是双边连通的.

关键词: 加权 Coxeter 群; 拟分裂情形; 胞腔; 左连通性

0 Introduction

In his book [1], Lusztig introduced a weighted Coxeter group (W, L), which is, by definition, a Coxeter system (W, S) together with a weight function $L : W \longrightarrow \mathbb{Z}$. He proposed a bundle

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of conjectures, intending to generalize many results on cells of W in the equal parameter case to the unequal parameters case. The most successful part for such a generalization is when (W, L) is in a certain quasi-split case, that is, W can be realized as the fixed point set of a finite or an affine Coxeter system $(\widetilde{W}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$, and the weight function L is the restriction to W of the length function $\widetilde{\ell}$ of $\widetilde{W}$ (see [2–5]). Lusztig conjectured in [6] that any left cell of an affine Weyl group is left-connected in the split case. The left-connectedness is a good structural property for a left cell. We now extend this conjecture to any weighted Coxeter group by proposing the following conjecture:

Conjecture A *The left (respectively, two-sided) cells of any weighted Coxeter group are left-connected (respectively, two-sided-connected).*

Though it has been verified in many split cases (see [7–10]), as well as in certain quasi-split cases (see [5]) by Shi, Conjecture A still remains open up to now.

In the present paper, we consider the affine Coxeter group $W = \widetilde{B}_3$ in the quasi-split case where W is realized as the fixed point set of the affine Coxeter group $\widetilde{W} = \widetilde{D}_4$ under the group automorphism α determined by $\alpha(s_i) = s_i$, $0 \leqslant i \leqslant 2$ and $\alpha(s_j) = s_k$ for $j \neq k$ in $\{3, 4\}$, where the Coxeter generator set $\widetilde{S} = \{s_i \mid 0 \leqslant i \leqslant 4\}$ of $\widetilde{D}_4$ satisfies $o(s_i s_2) = 3$ and $o(s_i s_j) = 2$ for $i \neq j$ in $\{0, 1, 3, 4\}$.

Shi described the left cells of $\widetilde{D}_4$ in [11]. Later he designed some algorithms and provided some criteria in his study of left-connectedness of left cells in [10,12]. Based on these results, we shall give a description for all the left cells of the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$ and prove that all the left (respectively, two-sided) cells of the weighted Coxeter groups $(\widetilde{D}_4, \widetilde{\ell})$ and $(\widetilde{B}_3, \widetilde{\ell})$ are left-connected (respectively, two-sided-connected), where $\widetilde{\ell}$ is the length function of the Coxeter system $(\widetilde{D}_4, \widetilde{S})$.

The contents of the paper are organized as follows. Section 1 is served as preliminaries, where we collect some concepts, terms and known results. Then we introduce some known results on the groups $\widetilde{D}_4$ and $\widetilde{B}_3$ in Section 2. We prove the left-connectedness for all the left cells in $\widetilde{D}_4$ in Section 3. Finally, we give an explicit description for all the cells of $(\widetilde{B}_3, \widetilde{\ell})$ and show that each left (respectively, two-sided) cell of the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$ is left-connected (respectively, two-sided-connected) in Section 4.

1 Preliminaries

The results in 1.1-1.5 and 1.7 follow Lusztig in [1].

1.1 Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called reduced if $r = \ell(w)$. By a weight function on W, we mean a map L from W to the integer set $\mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ in W. (W, L) is called a weighted Coxeter group.

A weighted Coxeter group (W, S) is called in the split case if $L = \ell$.

Suppose that there exists a group automorphism $\alpha : W \to W$ with $\alpha(S) = S$. Let $W^\alpha = \{w \in W | \alpha(w) = w\}$. For any α-orbit J on S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J whenever the cardinal $|W_J|$ of the set W_J is finite. Let S_α be the set of elements w_J with J ranging over all such α-orbits in S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function $\ell : W \to \mathbb{N}$ is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the quasi-split case.

1.2 Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. Denote $v_w = v^{L(w)}$ for any $w \in W$. Define a ring involution $a \mapsto \overline{a}$ of $\mathcal{A}$ by setting $\overline{\sum_i a_i v^i} = \sum_i a_i v^{-i}$, where $a_i \in \mathbb{Z}$. Define $\mathcal{A}_{<m} = \{f \in \mathcal{A} | \deg f < m\}$ for any $m \in \mathbb{Z}$.

1.3 For any $w, x, y, z \in W$ and $s \in S$ with $sx < x < y < sy$, define $p_{z,w}, M^s_{x,y} \in \mathcal{A}$ recurrently by the following requirements:

(1.3.1) $p_{z,w} = 0$ if $z \nleqslant w, p_{w,w} = 1$ and $p_{z,w} \in \mathcal{A}_{<0}$ if $z < w$;

(1.3.2) $p_{z,w} = v_s^\epsilon p_{z,sw} + p_{sz,sw} - \sum_{z \leqslant z' < sw, sz' < z'} M^s_{z',sw} p_{z,z'}$ for $z < w$ and $sw < w$, where $\epsilon = 1$ if $sz < z$, and $\epsilon = -1$ if $sz > z$ (see [1, The proof of Theorem 6.6]);

(1.3.3) $\sum_{x \leqslant z < y, sz < z} M^s_{z,y} p_{x,z} \equiv v_s p_{x,y} (\bmod \mathcal{A}_{<0})$;

(1.3.4) $\overline{M^s_{x,y}} = M^s_{x,y}$.

The condition (1.3.3) determines the coefficients of v^k in $M^s_{x,y}$ for all $k \geqslant 0$; then (1.3.4) determines all the other coefficients (see [1, Proposition 6.3]).

1.4 Define a preorder $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$) on W which is transitively generated by the relation $y \underset{L}{\leftarrow} w$ (respectively, $y \underset{R}{\leftarrow} w$), where $w < sw$, and either $y = sw$ or $M^s_{y,w} \neq 0$ (respectively, $w < ws$, and either $y = ws$ or $M^s_{y^{-1},w^{-1}} \neq 0$) holds for some $s \in S$. The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (respectively, $\underset{R}{\sim}$). The corresponding equivalence classes in W are called left cells (respectively, right cells) of W. Write $y \underset{LR}{\leqslant} w$ in W, if there exists a sequence $y_0 = y, y_1, \cdots, y_r = w$ in W with some $r > 0$ such that for every $1 \leqslant i \leqslant r$, either $y_{i-1} \underset{L}{\leqslant} y_i$ or $y_{i-1} \underset{R}{\leqslant} y_i$ holds. The equivalence relation associated to the preorder $\underset{LR}{\leqslant}$ is denoted by $\underset{LR}{\sim}$ and the corresponding equivalence classes in W are called two-sided cells of W.

1.5 For $w \in W$, define $\mathcal{L}(w) = \{s \in S | sw < w\}$ and $\mathcal{R}(w) = \{s \in S | ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) = \mathcal{L}(w)$) (see [1, Lemma 8.6]).

1.6 In [7, Chapter 13], Lusztig defined a function $a : W \to \mathbb{N} \cup \{\infty\}$ in terms of structural coefficients of the Hecke algebra associated to (W, L). Then he proved the following results (1)-(2) when W is either a finite or an affine Coxeter group and when (W, L) is either in the split case or in the quasi-split case in [1, Chapters 14-16].

(1) $y \underset{LR}{\leqslant} w$ in W implies $a(w) \leqslant a(y)$. Hence $y \underset{LR}{\sim} w$ in W implies $a(w) = a(y)$.

(2) If $w, y \in W$ satisfy $a(w) = a(y)$ and $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$, $y \underset{LR}{\leqslant} w$), then $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$, $y \underset{LR}{\sim} w$).

In [13], Lusztig proved the following results (3)-(4) when W is either a finite or an affine Coxeter group and when (W, L) is in the split case.

(3) For any $I \subseteq S$, let W_I be the subgroup of W generated by I. If W_I is finite, let w_I be the longest element of W_I, then $a(w_I) = \ell(w_I)$.

(4) For any nonnegative integer i, let $W_{(i)}=\{w\in W|a(w)=i\}$, then $W_{(i)}$ is either empty or a union of some two-sided cells of W.

1.7 For $w\in W$, we denote by $\Delta(w)$ the nonnegative integer defined by

$$p_{e,w}=n_w v^{-\Delta(w)}+\text{strictly smaller degree terms in}v,\text{with}n_w\in\mathbb{Z}-\{0\}.$$

Note that $\Delta(e)=0, 0<\Delta(w)\leqslant L(w)$ for $w\neq e$. Let $\mathcal{D}=\{w\in W|a(w)=\Delta(w)\}$.

Lusztig called the elements of $\mathcal{D}$ by distinguished involutions and proved that each left cell of W contains a unique distinguished involution when W is either in the split case or in the quasi-split case.

1.8 Let K be a non-empty subset of W. Two elements $x,y\in K$ are called *left-connected (respectively, right-connected, two-sided-connected)* in K, written $x\underset{K_L}{\text{——}}y$ (respectively, $x\underset{K_R}{\text{——}}y$, $x\underset{K_{LR}}{\text{——}}y$), if there exists a sequence $x_0=x, x_1, ..., x_r=y$ in K with some $r\geqslant 0$ such that $x_{i-1}x_i^{-1}\in S$ (respectively, $x_i^{-1}x_{i-1}\in S$, either $x_{i-1}x_i^{-1}\in S$ or $x_i^{-1}x_{i-1}\in S$) for every $1\leqslant i\leqslant r$. This defines an equivalence relation on K. Each equivalence class of K with respect to $\underset{K_L}{\text{——}}$ (respectively, $x\underset{K_R}{\text{——}}y$, $x\underset{K_{LR}}{\text{——}}y$) is called a *left-connected (respectively, right-connected, two-sided-connected) component* of K. The set K is called *left-connected (respectively, right-connected, two-sided-connected)*, if K consists of a single left-connected (respectively, right-connected, two-sided-connected) component.

1.9 Let $s,t\in S$ satisfy $o(st)=3$. By a *right* $\{s,t\}$*-string*, we mean a set $\{ys, yst\}$ with $y\in W$ satisfying $\mathcal{R}(y)\bigcap\{s,t\}=\emptyset$; by a *left* $\{s,t\}$*-string*, we mean a set $\{sy, tsy\}$ with $y\in W$ satisfying $\mathcal{L}(y)\bigcap\{s,t\}=\emptyset$.

We say that x is obtained from w by a *left* (respectively, *right*) $\{s,t\}$*-star operation*, if $\{x,w\}$ is a left (respectively, right) $\{s,t\}$-string. Note that the resulting element x for a left (respectively, right) $\{s,t\}$-star operation on w is always unique whenever it exists.

Sometimes we call a right $\{s,t\}$-string and a right $\{s,t\}$-star operation simply by a right string and a right star operation, respectively. Similarly for the left version of those terms.

We have the following results 1.10–1.12 when (W,L) is in the split case:

Lemma 1.10 (see [14]) *Let* $s,t\in S$ *be with* $o(st)=3$*. Suppose that* $\{x_1,x_2\}$ *and* $\{y_1,y_2\}$ *are two right (respectively, left)* $\{s,t\}$*-strings. Then*

(a) $x_1\text{——}y_1\Leftrightarrow x_2\text{——}y_2$;

(b) $x_1\underset{L}{\sim}y_1\Leftrightarrow x_2\underset{L}{\sim}y_2$ *(respectively,* $x_1\underset{R}{\sim}y_1\Leftrightarrow x_2\underset{R}{\sim}y_2$*).*

1.11 We say that $x,y\in W$ form a *right primitive pair*, if there exist two sequences $x_0=x, x_1, \cdots, x_n$ and $y_0=y, y_1, \cdots, y_n$ in W satisfying:

(a) For any $1\leqslant i\leqslant n$, there exist some $s_i,t_i\in S$ with $o(s_it_i)=3$ such that both $\{x_{i-1},x_i\}$ and $\{y_{i-1},y_i\}$ are right $\{s_i,t_i\}$-strings.

(b) $x_i\text{——}y_i$ for some (hence all) i, $0\leqslant i\leqslant n$.

(c) Either $\mathcal{R}(x)\nsubseteq\mathcal{R}(y)$ and $\mathcal{R}(y_n)\nsubseteq\mathcal{R}(x_n)$, or $\mathcal{R}(y)\nsubseteq\mathcal{R}(x)$ and $\mathcal{R}(x_n)\nsubseteq\mathcal{R}(y_n)$.

Note that any right string x,y of W form a right primitive pair with $n=0$ in the above definition.

Similarly, we can define a left primitive pair in W.

Lemma 1.12 (see [15]) *If x, y is a right (respectively, left) primitive pair, then $x \underset{R}{\sim} y$ (respectively, $x \underset{L}{\sim} y$).*

2 Some known results on the group $\widetilde{D}_4$ and $\widetilde{B}_3$

2.1 Let $\widetilde{S} = \{s_i \mid 0 \leqslant i \leqslant 4\}$ be the generator set of $\widetilde{D}_4$ with $o(s_i s_2) = 3$ and $o(s_i s_j) = 2$ for $i \neq j$ in $\{0, 1, 3, 4\}$. Let $\alpha : \widetilde{D}_4 \to \widetilde{D}_4$ be the group automorphism determined by $\alpha(s_i) = s_i$ for $0 \leqslant i \leqslant 2$ and $\alpha(s_j) = s_k$ for $j \neq k$ in $\{3, 4\}$. Then the affine Weyl group $\widetilde{B}_3$ can be realized as the fixed point set of $\widetilde{D}_4$ under α. Let $S = \{t_i | 0 \leqslant i \leqslant 3\}$ be the Coxeter generator set of $\widetilde{B}_3$, where $t_i = s_i$ for $0 \leqslant i \leqslant 2$ and $t_3 = s_3 s_4$.

Let $\widetilde{\ell}, \ell$ be the length functions on the Coxeter systems $(\widetilde{D}_4, \widetilde{S}), (\widetilde{B}_3, S)$, respectively. By the definition in 1.1, the weighted Coxeter group $(\widetilde{D}_4, \widetilde{\ell})$ is in the split case, while $(\widetilde{B}_3, \widetilde{\ell})$ is in the quasi-split case (see [1, Lemma 16.2]).

From now on, we concentrate ourselves to the weighted Coxeter groups $(\widetilde{D}_4, \widetilde{\ell})$ and $(\widetilde{B}_3, \widetilde{\ell})$. We preserve the notation $\leqslant, \mathcal{L}(w), \mathcal{R}(w), a(w), \Delta(w), \mathcal{D}$ for the group $(\widetilde{B}_3, \widetilde{\ell})$, but denote them by $\widetilde{\leqslant}, \widetilde{\mathcal{L}}(w), \widetilde{\mathcal{R}}(w), \widetilde{a}(w), \widetilde{\Delta}(w), \widetilde{\mathcal{D}}$, respectively for the group $(\widetilde{D}_4, \widetilde{\ell})$.

2.2 Since the condition $x \leqslant y$ is equivalent to $x \,\widetilde{\leqslant}\, y$ for any $x, y \in \widetilde{B}_3$, it will cause no confusion if we use the notation $\leqslant$ in the place of $\widetilde{\leqslant}$. Hence from now on we shall use $\leqslant$ for both $\leqslant$ and $\widetilde{\leqslant}$.

The following fact can be checked easily.

For any $w, y \in \widetilde{B}_3$ and $0 \leqslant i \leqslant 2$, we see that $t_i \in \mathcal{L}(w)$ if and only if $s_i \in \widetilde{\mathcal{L}}(w)$ and that $t_i \in \mathcal{R}(w)$ if and only if $s_i \in \widetilde{\mathcal{R}}(w)$. Also, $t_3 \in \mathcal{L}(w)$ if and only if $s_3 \in \widetilde{\mathcal{L}}(w)$ if and only if $s_4 \in \widetilde{\mathcal{L}}(w)$; $t_3 \in \mathcal{R}(w)$ if and only if $s_3 \in \widetilde{\mathcal{R}}(w)$ if and only if $s_4 \in \widetilde{\mathcal{R}}(w)$.

Lemma 2.3 (see [1, Lemma 16.5]) *$a(w) = \widetilde{a}(w)$ for any $w \in \widetilde{B}_3$.*

Lemma 2.4 (see [1, Lemma 16.14]) *Let $x, y \in \widetilde{B}_3$. Then $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{B}_3$ if and only if $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{D}_4$.*

By Lemma 2.4, we can just use the notation $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) for $x, y \in \widetilde{B}_3$ without indicating whether the relation refers to the group $\widetilde{D}_4$ or $\widetilde{B}_3$.

Let Γ be a left cell of $\widetilde{D}_4$. Denote $\Gamma' = \Gamma \cap \widetilde{B}_3$.

Corollary 2.5 *If $\Gamma' \neq \emptyset$, then Γ' is a left cell of $\widetilde{B}_3$.*

Proof It is a direct consequence of Lemma 2.4.

Lemma 2.6 (see [1, Lemma 16.6]) $\mathcal{D} = \widetilde{\mathcal{D}} \cap \widetilde{B}_3$.

Denote the distinguished involution of $\widetilde{D}_4$ in the left cell Γ by d_Γ.

Corollary 2.7 *$\Gamma' \neq \emptyset$ if and only if $\alpha(d_\Gamma) = d_\Gamma$.*

Proof By Corollary 2.5 and Lemma 2.6, we get the result.

3 The left-connectedness of left cells in $\widetilde{D}_4$

In the present section, we want to prove the following theorem.

Theorem 3.1 *Any left cell of $\widetilde{D}_4$ is left-connected.*

For simplifying the notation, we denote $s_i \in \widetilde{S}$ by the boldfaced letter $\mathbf{i}$ for any $0 \leqslant i \leqslant 4$.

Following Shi in [10,12], we define, for any left cell Γ and any two-sided cell Ω of the weighted Coxeter group $(\widetilde{D}_4, \widetilde{\ell})$ or $(\widetilde{B}_3, \widetilde{\ell})$, the following sets

$$\begin{aligned} E(\Gamma) :=& \{w \in \Gamma \mid \widetilde{a}(sw) < \widetilde{a}(w), \forall\, s \in \widetilde{\mathcal{L}}(w)\}, \\ E_{\min}(\Gamma) :=& \{w \in \Gamma \mid \widetilde{\ell}(w) \leqslant \widetilde{\ell}(x), \forall\, x \in \Gamma\}, \\ E(\Omega) :=& \{w \in \Omega \mid \widetilde{a}(sw) < \widetilde{a}(w), \forall\, s \in \widetilde{\mathcal{L}}(w)\}, \\ F(\Omega) :=& \{w \in \Omega \mid \widetilde{a}(sw), \widetilde{a}(wt) < \widetilde{a}(w), \forall\, s \in \widetilde{\mathcal{L}}(w), t \in \widetilde{\mathcal{R}}(w)\}. \end{aligned}$$

Recall the relation $\underset{K_L}{\text{——}}$ on a non-empty set K of W defined in 1.8. The following result is crucial in proving the left-connectedness of a left cell of $\widetilde{D}_4$.

Lemma 3.2 *Let Γ be a left cell of $\widetilde{D}_4$. If $x \underset{\Gamma_L}{\text{——}} y$ for any $x \neq y$ in $E(\Gamma)$ then Γ is left-connected.*

The proof of Lemma 3.2 is the same as that of Lemma 2.3 in [15], hence we omit it here.

For any x, y, z in $\widetilde{D}_4$, we use the notation $z = x \cdot y$ to indicate $z = xy$ and $\widetilde{\ell}(z) = \widetilde{\ell}(x) + \widetilde{\ell}(y)$.

As a consequence of the results in [10, 12, 16], we have

Lemma 3.3 *Let w, Γ, Ω be an element, a left cell and a two-sided cell of $\widetilde{D}_4$ respectively with $\widetilde{a}(w), \widetilde{a}(\Gamma), \widetilde{a}(\Omega) \leqslant 6$. Then*

(a) w has an expression of the form $w = x \cdot w_J \cdot y$ for some $x, y \in \widetilde{D}_4$ and some $J \subseteq S$ with $\widetilde{\ell}(w_J) = \widetilde{a}(w)$.

(b) For any $w \in E(\Gamma)$, write $w = w_J \cdot y$ with $J = \widetilde{\mathcal{L}}(w)$ for some $y \in \widetilde{D}_4$. Then $\widetilde{\ell}(w_J) = \widetilde{a}(w)$.

(c) If $E(\Gamma) = E_{\min}(\Gamma)$ then Γ is left-connected.

(d) $F(\Omega) = \{w_J \in \Omega \mid J \subseteq S\}$.

Let Ω be a two-sided cell of $\widetilde{D}_4$. In [12], Shi designed the following algorithm for finding the set $E(\Omega)$ from $F(\Omega)$.

Algorithm 3.4

```
(1) Set $Y_0 = F(\Omega)$;
Let $k \geqslant 0$. Suppose that the set $Y_k$ has been found.
(2) If $Y_k = \emptyset$, then the algorithm terminates;
(3) If $Y_k \neq \emptyset$, then find the set $Y_{k+1} = \{xs \mid x \in Y_k, s \in S \setminus \widetilde{\mathcal{R}}(x); xs \in E(\Omega)\}$.
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The most technical part in applying Algorithm 3.4 is to determine whether or not an element xs is in the set $E(\Omega)$, that is, to determine if the relations $\widetilde{a}(tws) < \widetilde{a}(ws) = \widetilde{a}(w)$ holds for any $t \in \widetilde{\mathcal{L}}(ws)$. This is easy by using the computer programme MATLAB, since all elements of $\widetilde{D}_4$ have been described explicitly by Shi in [11].

3.5 Let $i \in \mathbb{N}$. Following [11, Section 3], we see that $W_{(i)} \neq \emptyset$ unless $i \in \{0, 1, 2, 3, 4, 6, 7, 12\}$. $W_{(i)}$ is a single two-sided cell of $\widetilde{D}_4$ if $i \in \{0, 1, 3, 4, 7, 12\}$. On the other hand, $W_{(i)}$ is a union of three two-sided cells (written $\Omega_{i,1}$, $\Omega_{i,2}$ and $\Omega_{i,3}$) of $\widetilde{D}_4$ if $i \in \{2, 6\}$, where the two-sided cells Ω_{ij} are determined by the conditions $w_{\mathbf{01}} \in \Omega_{2,1}$, $w_{\mathbf{03}} \in \Omega_{2,2}$, $w_{\mathbf{04}} \in \Omega_{2,3}$, $w_{\mathbf{012}} \in \Omega_{6,1}$, $w_{\mathbf{123}} \in \Omega_{6,2}$, $w_{\mathbf{124}} \in \Omega_{6,3}$, where we denote w_J by $w_{\mathbf{ijk}\cdots}$ for $J = \{s_i, s_j, s_k, \cdots\}$.

Let $\mathfrak{S}$ be the group of all permutations σ on the set $\{0,1,2,3,4\}$ satisfying $\sigma(2)=2$. Let f_σ be the automorphism of $\widetilde{D}_4$ satisfying $f_\sigma(s_i)=s_{\sigma(i)}$ for any $s_i\in\widetilde{S}$. We denote $f_{(ij)}$ simply by f_{ij}, where (ij) is the transposition of i and j for $i\neq j$ in $\{0,1,3,4\}$. We get $\Omega_{2,2}=f_{13}(\Omega_{2,1})$, $\Omega_{2,3}=f_{14}(\Omega_{2,1})$, $\Omega_{6,2}=f_{03}(\Omega_{6,1})$, $\Omega_{6,3}=f_{04}(\Omega_{6,1})$ from [11, Section 4].

3.6 For $i\in\mathbb{N}$, let $\widetilde{\Sigma}_i$ be the set of all left cells Γ of $\widetilde{D}_4$ with $\widetilde{a}(\Gamma)=i$. By Lemma 3.3(d) and the results of Shi in [11], we get

$F(W_{(1)})=\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$, $F(W_{(3)})=\{w_{\mathbf{02}},w_{\mathbf{12}},w_{\mathbf{23}},w_{\mathbf{24}},w_{\mathbf{013}},w_{\mathbf{014}},w_{\mathbf{034}},w_{\mathbf{134}}\}$,

$F(W_{(4)})=\{w_{\mathbf{0134}}\}$, $F(W_{(12)})=\{w_{\mathbf{0123}},w_{\mathbf{0124}},w_{\mathbf{0234}},w_{\mathbf{1234}}\}$,

$F(\Omega_{2,1})=\{w_{\mathbf{01}},w_{\mathbf{34}}\},F(\Omega_{2,2})=f_{13}(F(\Omega_{2,1})),F(\Omega_{2,3})=f_{14}(F(\Omega_{2,1}))$,

$F(\Omega_{6,1})=\{w_{\mathbf{012}},w_{\mathbf{234}}\},F(\Omega_{6,2})=f_{03}(F(\Omega_{6,1})),F(\Omega_{6,3})=f_{04}(F(\Omega_{6,1}))$.

We also have $F(W_{(7)})=\{\mathbf{i2ki2ij2i}\mid \mathbf{i},\mathbf{j},\mathbf{k}\in\{\mathbf{0},1,\mathbf{3},\mathbf{4}\}\ \textit{distinct}\}$ by a result of Shi in [12, Section 4.7].

So we can perform Algorithm 3.4 to get $E(\Omega)$ for all two-sided cell Ω of $\widetilde{D}_4$ with $\widetilde{a}(\Omega)\in\{1,2,3,4,6,7,12\}$ (see Tables 1-7 for the results). Since $E(\Omega_{2,2})=f_{13}(E(\Omega_{2,1})),E(\Omega_{2,3})=f_{14}(E(\Omega_{2,1})),E(\Omega_{6,2})=f_{03}(E(\Omega_{6,1})),E(\Omega_{6,3})=f_{04}(E(\Omega_{6,1}))$, it allows us not to include $E(\Gamma)$ for any $\Gamma\subset\Omega_{i,k}$, $i\in\{2,6\}$ and $k=2,3$ in the list for saving the space.

3.7 In Tables 1-7, if $i\in\{1,3,4,7,12\}$, then we denote all the left cells in $W_{(i)}$ by $\Gamma_{i,j}$, $1\leqslant j\leqslant\widetilde{n}(i)$, where $\widetilde{n}(i)$ stands for the number of left cells in $W_{(i)}$; if $i\in\{2,6\}$, then we denote all the left cells in $\Omega_{i,k}$, $k=1,2,3$, by $\Gamma_{ik,j}$, $1\leqslant j\leqslant\widetilde{n}_k(i)$, where $\widetilde{n}_k(2)=8$ and $\widetilde{n}_k(6)=48$, $k=1,2,3$. For saving the space in the tables, we denote $\{s_i,s_j,s_k,...\}$ simply by $\mathbf{i},\mathbf{j},\mathbf{k},\cdots$ concerning the set $\widetilde{\mathcal{R}}(\Gamma)$. For example, the set $\{s_1,s_2,s_3,s_5\}$ is denoted by $\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{5}$. However, we only include $E(\Gamma_{7,i})$ and $E(\Gamma_{12,k})$ in Tab. 6 and Tab. 7 respectively, because $E(\Gamma_{7,24+i})=f_{14}(E(\Gamma_{7,i}))$, $E(\Gamma_{7,48+i})=f_{34}(E(\Gamma_{7,i}))$, $E(\Gamma_{7,72+i})=f_{014}(E(\Gamma_{7,i}))$, $E(\Gamma_{12,48+k})=f_{14}(E(\Gamma_{12,k}))$, $E(\Gamma_{12,96+k})=f_{04}(E(\Gamma_{12,k}))$, $E(\Gamma_{12,144+k})=f_{34}(E(\Gamma_{12,k}))$, $1\leqslant i\leqslant 24$, $1\leqslant k\leqslant 48$ (see [11, Section 5]).

We observe from Tables 1–5 that all the elements of $E(\Gamma)$ have the same length for any left cell Γ with $\widetilde{a}(\Gamma)\in\{1,2,3,4,6\}$. So for those left cells Γ, we have $E(\Gamma)=E_{\min}(\Gamma)$ and hence Γ is left-connected by Lemma 3.3 (c). Since $E(\Gamma)$ contains only one element for any left cell Γ in $\widetilde{\Sigma}_{12}$ (see Tab. 7), Γ is left-connected obviously. Thus, to show Theorem 3.1, we need only to deal with all the left cells of $\widetilde{D}_4$ in $\widetilde{\Sigma}_7$. By Lemma 3.2, we shall prove the left-connectedness of those left cells Γ by showing that $x\underset{\Gamma_L}{\text{——}}y$ for any $x\neq y$ in $E(\Gamma)$ by a case-by-case argument.

3.8 We proceed our proof by constructing some connected graphs. Those graphs are named by Figures i, $i=1,2,3,4$, respectively. One connected graph (say Figure i) for each left cell Γ in $\widetilde{\Sigma}_7$, each vertex of Figure i represents an element (say z) of $\widetilde{D}_4$ which is labeled by $\widetilde{\mathcal{L}}(z)$, all the elements of $E(\Gamma)$ must occur as vertices in the graph Figure i. Two vertices are joined by a solid edge if they form a left string. The connectedness of the graph Figure i implies that all the elements corresponding to the vertices of Figure i belong to Γ by Lemma 1.12, which implies that Γ is left-connected by Lemma 3.2.

Example 3.9 We take Figure 1 as an example to illustrate how we prove the left-connectedness for the left cell $\Gamma:=\Gamma_{7,1}$. We have $E(\Gamma)=\{a,b,c\}$ with $a=$

$\mathbf{02301240123420124}$, $b = \mathbf{01230123401230124}$ and $c = \mathbf{12301240123420124}$ by Table 6. The elements a, b, c all occur as vertices of Figure 1 with labels $\boxed{\mathbf{0,2,3}}$, $\boxed{\mathbf{0,1,2}}$, $\boxed{\mathbf{1,2,3}}$, respectively. The vertex labeled by $\boxed{\mathbf{0,1,3}}$ in Figure 1 corresponds to the element $a' := \mathbf{1}a = \mathbf{3}b = \mathbf{0}c$.

By the fact that the vertices labeled by $\boxed{0,2,3}$ and $\boxed{\mathbf{0,1,3}}$ in Fig. 1 are joined by a solid edge, we conclude that $\{a, a'\}$ form a left $\{\mathbf{1}, \mathbf{2}\}$-string. The same argument tells us that $\{b, a'\}$ form a left $\{\mathbf{2}, \mathbf{3}\}$-string and that $\{c, a'\}$ form a left $\{\mathbf{0}, \mathbf{2}\}$-string. This implies by Lemma 1.12 that $a \underset{\Gamma_L}{\text{——}} b \underset{\Gamma_L}{\text{——}} c$ for $\Gamma = \Gamma_{7,1}$. Hence $\Gamma_{7,1}$ is left-connected by Lemma 3.2.

Tab. 1 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{1}}$

Γ	$\Gamma_{1,1}$	$\Gamma_{1,2}$	$\Gamma_{1,3}$	$\Gamma_{1,4}$	$\Gamma_{1,5}$
$E(\Gamma)$	**0**	**1**	**2**	**3**	**4**
$\widetilde{\mathcal{R}}(\Gamma)$	**0**	**1**	**2**	**3**	**4**

Tab. 2 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{2}}$

Γ	$\Gamma_{21,1}$	$\Gamma_{21,2}$	$\Gamma_{21,3}$	$\Gamma_{21,4}$	$\Gamma_{21,5}$	$\Gamma_{21,6}$	$\Gamma_{21,7}$	$\Gamma_{21,8}$
$E(\Gamma)$	**01**	**34**	**012**	**342**	**0123**	**0124**	**3420**	**3421**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1**	**3,4**	**2**	**2**	**3**	**4**	**0**	**1**

Tab. 3 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{3}}$

Γ	$\Gamma_{3,1}$	$\Gamma_{3,2}$	$\Gamma_{3,3}$	$\Gamma_{3,4}$	$\Gamma_{3,5}$	$\Gamma_{3,6}$	$\Gamma_{3,7}$	$\Gamma_{3,8}$	$\Gamma_{3,9}$	$\Gamma_{3,10}$
$E(\Gamma)$	**0201,1201**	**020**	**0230,2320**	**121**	**1231,2321**	**232**	**2423,2324**	**1241,2421**	**242**	**0240,2420**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1**	**0,2**	**0,3**	**1,2**	**1,3**	**2,3**	**3,4**	**1,4**	**2,4**	**0,4**

Γ	$\Gamma_{3,11}$	$\Gamma_{3,12}$	$\Gamma_{3,13}$	$\Gamma_{3,14}$	$\Gamma_{3,15}$	$\Gamma_{3,16}$	$\Gamma_{3,17}$	$\Gamma_{3,18}$	$\Gamma_{3,19}$	$\Gamma_{3,20}$	$\Gamma_{3,21}$	$\Gamma_{3,22}$
$E(\Gamma)$	**0132**	**0142**	**0342**	**1342**	**013**	**01324**	**014**	**01423**	**034**	**03421**	**134**	**13420**
$\widetilde{\mathcal{R}}(\Gamma)$	**2**	**2**	**2**	**2**	**0,1,3**	**4**	**0,1,4**	**3**	**0,3,4**	**1**	**1,3,4**	**0**

Tab. 4 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{4}}$

Γ	$\Gamma_{4,1}$	$\Gamma_{4,2}$	$\Gamma_{4,3}$	$\Gamma_{4,4}$	$\Gamma_{4,5}$	$\Gamma_{4,6}$	$\Gamma_{4,7}$	$\Gamma_{4,8}$	$\Gamma_{4,9}$	$\Gamma_{4,10}$
$E(\Gamma)$	**0134**	**01342**	**013420**	**013421**	**013423**	**013424**	**0134201**	**0134230**	**0134240**	**0134231**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1,3,4**	**2**	**0,2**	**1,2**	**2,3**	**2,4**	**0,1**	**0,3**	**0,4**	**1,3**

Γ	$\Gamma_{4,11}$	$\Gamma_{4,12}$	$\Gamma_{4,13}$	$\Gamma_{4,14}$	$\Gamma_{4,15}$	$\Gamma_{4,16}$	$\Gamma_{4,17}$	$\Gamma_{4,18}$
$E(\Gamma)$	**0134241**	**0134234**	**01342301**	**01342401**	**01342340**	**01342341**	**013423012**	**013424012**
$\widetilde{\mathcal{R}}(\Gamma)$	**1,4**	**3,4**	**0,1,3**	**0,1,4**	**0,3,4**	**1,3,4**	**2**	**2**

Γ	$\Gamma_{4,19}$	$\Gamma_{4,20}$	$\Gamma_{4,21}$	$\Gamma_{4,22}$	$\Gamma_{4,23}$	$\Gamma_{4,24}$
$E(\Gamma)$	**013423402**	**013423412**	**0134230124**	**0134240123**	**0134234021**	**0134234120**
$\widetilde{\mathcal{R}}(\Gamma)$	**2**	**2**	**4**	**3**	**1**	**0**

Tab. 5 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{6}}$

Γ	$\Gamma_{61,1}$	$\Gamma_{61,2}$	$\Gamma_{61,3}$	$\Gamma_{61,4}$	$\Gamma_{61,5}$	$\Gamma_{61,6}$	$\Gamma_{61,7}$	$\Gamma_{61,8}$	$\Gamma_{61,9}$
$E(\Gamma)$	**012012**	**234234**	**0120123**	**0120124**	**2342340**	**2342341**	**01230123**	**01201234**	**01240124**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1,2**	**2,3,4**	**0,1,3**	**0,1,4**	**0,3,4**	**1,3,4**	**2,3**	**0,1,3,4**	**2,4**

Γ	$\Gamma_{61,10}$	$\Gamma_{61,11}$	$\Gamma_{61,12}$	$\Gamma_{61,13}$	$\Gamma_{61,14}$	$\Gamma_{61,15}$	$\Gamma_{61,16}$	$\Gamma_{61,17}$
$E(\Gamma)$	**23423401**	**23423402**	**23423412**	**012301234**	**012012342**	**012401234**	**234234012**	**234234021**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1,3,4**	**0,2**	**1,2**	**3,4**	**2**	**3,4**	**2**	**0,1**

Γ	$\Gamma_{61,18}$	$\Gamma_{61,19}$	$\Gamma_{61,20}$	$\Gamma_{61,21}$	$\Gamma_{61,22}$	$\Gamma_{61,23}$	$\Gamma_{61,24}$
$E(\Gamma)$	**234234120**	**0123012342**	**0120123420**	**0120123421**	**0124012342**	**2342340120**	**2342340121**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1**	**2,4**	**0,2**	**1,2**	**2,3**	**0,2**	**1,2**

Continue of Tab. 5

Γ	$\Gamma_{61,25}$	$\Gamma_{61,26}$	$\Gamma_{61,27}$	$\Gamma_{61,28}$	$\Gamma_{61,29}$	$\Gamma_{61,30}$	$\Gamma_{61,31}$
$E(\Gamma)$	2342340123	2342340124	01230123420	01230123421	01201234201	01240123420	01240123421
$\widetilde{\mathcal{R}}(\Gamma)$	2,3	2,4	0,4	1,4	0,1	0,3	1,3

Γ	$\Gamma_{61,32}$	$\Gamma_{61,33}$	$\Gamma_{61,34}$	$\Gamma_{61,35}$	$\Gamma_{61,36}$	$\Gamma_{61,37}$
$E(\Gamma)$	23423401230	23423401240	23423401231	23423401241	23423401234	012301234201
$\widetilde{\mathcal{R}}(\Gamma)$	0,3	0,4	1,3	1,4	3,4	0,1,4

Γ	$\Gamma_{61,38}$	$\Gamma_{61,39}$	$\Gamma_{61,40}$	$\Gamma_{61,41}$	$\Gamma_{61,42}$	$\Gamma_{61,43}$
$E(\Gamma)$	012401234201	234234012340	234234012341	0123012342012	0124012342012	2342340123402
$\widetilde{\mathcal{R}}(\Gamma)$	0,1,3	0,3,4	1,3,4	2	2	2

Γ	$\Gamma_{61,44}$	$\Gamma_{61,45}$	$\Gamma_{61,46}$	$\Gamma_{61,47}$	$\Gamma_{61,48}$
$E(\Gamma)$	2342340123412	01230123420123	01240123420124	23423401234021	23423401234120
$\widetilde{\mathcal{R}}(\Gamma)$	2	3	4	1	0

Tab. 6 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_7$

Γ	$\Gamma_{7,1}$	$\Gamma_{7,2}$	$\Gamma_{7,3}$	$\Gamma_{7,4}$
$E(\Gamma)$	a=02301240123420124 b=01230123401230124 c=12301240123420124	a=0230124012342012 b=0123012340123012 c=1230124012342012	a=023012401234201 b=012301234012301 c=123012401234201	a=023012012342 b=012301234012 c=123012012342
$\widetilde{\mathcal{R}}(\Gamma)$	4	2	0,1,3	2
Figure	Figure 1	Figure 1	Figure 1	Figure 1

Γ	$\Gamma_{7,5}$	$\Gamma_{7,6}$	$\Gamma_{7,7}$	$\Gamma_{7,8}$	$\Gamma_{7,9}$	$\Gamma_{7,10}$
$E(\Gamma)$	a=02301201234 b=01230123401 c=12301201234	a=0230120123 b=0123012301 c=1230120123	a=012301231 b=023020123	a=0123012341 b=0230201234	a=023012012 b=123012012	a=0230120124 b=1230120124
$\widetilde{\mathcal{R}}(\Gamma)$	0,1,3,4	0,1,3	1,2,3	1,3,4	0,1,2	0,1,4
Figure	Figure 1	Figure 1	Figure 2	Figure 2	Figure 3	Figure 3

Γ	$\Gamma_{7,11}$	$\Gamma_{7,12}$	$\Gamma_{7,13}$	$\Gamma_{7,14}$	$\Gamma_{7,15}$
$E(\Gamma)$	a=012301230 b=123120123	a=0123012340 b=1231201234	a=02301240124 b=12301240124	a=01230123402 b=12312012342	a=01230123412 b=02302012342
$\widetilde{\mathcal{R}}(\Gamma)$	0,2,3	0,3,4	2,4	2,4	2,4
Figure	Figure 4	Figure 4	Figure 3	Figure 4	Figure 2

Γ	$\Gamma_{7,16}$	$\Gamma_{7,17}$	$\Gamma_{7,18}$	$\Gamma_{7,19}$	$\Gamma_{7,20}$
$E(\Gamma)$	a=012301234120 b=023020123420	a=023012401234 b=123012401234	a=012301234021 b=123120123421	a=0230120123420 b=0123012340120 c=1230120123420	a=02301240123420 b=01230123401230 c=12301240123420
$\widetilde{\mathcal{R}}(\Gamma)$	0,4	3,4	1,4	0,2	0,3
Figure	Figure 2	Figure 3	Figure 4	Figure 1	Figure 1

Γ	$\Gamma_{7,21}$	$\Gamma_{7,22}$	$\Gamma_{7,23}$	$\Gamma_{7,24}$
$E(\Gamma)$	a=0230124012342 b=0123012340123 c=1230124012342	a=02301240123421 b=01230123401231 c=12301240123421	a=0230120123421 b=0123012340121 c=1230120123421	a=02301201234201 b=01230123401201 c=12301201234201
$\widetilde{\mathcal{R}}(\Gamma)$	2,3	1,3	1,2	0,1
Figure	Figure 1	Figure 1	Figure 1	Figure 1

Tab. 7 Description of left cells in $\widetilde{\boldsymbol{\Sigma}}_{12}$

Γ	$\Gamma_{12,1}$	$\Gamma_{12,2}$	$\Gamma_{12,3}$	$\Gamma_{12,4}$	$\Gamma_{12,5}$	$\Gamma_{12,6}$
$E(\Gamma)$	012301230123	0123012301234	01230123012342	012301230123420	012301230123421	012301234012342
$\widetilde{\mathcal{R}}(\Gamma)$	0,1,2,3	0,1,3,4	2,4	0,2,4	1,2,4	2,3,4

Γ	$\Gamma_{12,7}$	$\Gamma_{12,8}$	$\Gamma_{12,9}$	$\Gamma_{12,10}$	$\Gamma_{12,11}$
$E(\Gamma)$	0123012301234201	0123012340123420	0123012340123421	01230123012342012	01230123401234201
$\widetilde{\mathcal{R}}(\Gamma)$	0,1,4	0,3,4	1,3,4	0,1,2	0,1,3,4

Γ	$\Gamma_{12,12}$	$\Gamma_{12,13}$	$\Gamma_{12,14}$	$\Gamma_{12,15}$	$\Gamma_{12,16}$
$E(\Gamma)$	01230123401234020	01230123401234121	012301230123420123	012301234012342012	012301234012340201
$\widetilde{\mathcal{R}}(\Gamma)$	0,2,3	1,2,3	0,1,3	2	0,1,3

Continue of Tab. 7

Γ	$\Gamma_{12,17}$	$\Gamma_{12,18}$	$\Gamma_{12,19}$	$\Gamma_{12,20}$
$E(\Gamma)$	**012301234012341201**	**0123012340123420123**	**0123012340123412012**	**0123012340123402012**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1,3**	**2,3**	**0,2**	**1,2**
Γ	$\Gamma_{12,21}$	$\Gamma_{12,22}$	$\Gamma_{12,23}$	$\Gamma_{12,24}$
$E(\Gamma)$	**0123012340123420124**	**01230123401234120123**	**01230123401234020123**	**01230123401234201234**
$\widetilde{\mathcal{R}}(\Gamma)$	**2,4**	**0,2,3**	**1,2,3**	**3,4**
Γ	$\Gamma_{12,25}$	$\Gamma_{12,26}$	$\Gamma_{12,27}$	$\Gamma_{12,28}$
$E(\Gamma)$	**01230123401234012012**	**01230123401234120124**	**01230123401234020124**	**01230123401234012012**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,1,2**	**0,4**	**1,4**	**0,1,3**
Γ	$\Gamma_{12,29}$	$\Gamma_{12,30}$	$\Gamma_{12,31}$	$\Gamma_{12,32}$
$E(\Gamma)$	**012301234012341201234**	**012301234012340201234**	**012301234012340120124**	**012301234012340120124**
$\widetilde{\mathcal{R}}(\Gamma)$	**0,3,4**	**1,3,4**	**0,1,4**	**0,1,3,4**
Γ	$\Gamma_{12,33}$	$\Gamma_{12,34}$	$\Gamma_{12,35}$	$\Gamma_{12,36}$
$E(\Gamma)$	**0123012340123412012342**	**0123012340123402012342**	**0123012340123401240124**	**01230123401234012012342**
$\widetilde{\mathcal{R}}(\Gamma)$	2,4	2,4	2,4	2
Γ	$\Gamma_{12,37}$	$\Gamma_{12,38}$	$\Gamma_{12,39}$	$\Gamma_{12,40}$
$E(\Gamma)$	**01230123401234120123421**	**01230123401234020123420**	**01230123401234012401234**	**012301234012340120123420**
$\widetilde{\mathcal{R}}(\Gamma)$	1,4	0,4	3,4	0,2
Γ	$\Gamma_{12,41}$	$\Gamma_{12,42}$	$\Gamma_{12,43}$	
$E(\Gamma)$	**012301234012340120123421**	**012301234012340124012342**	**0123012340123401201234201**	
$\widetilde{\mathcal{R}}(\Gamma)$	1,2	2,3	0,1	
Γ	$\Gamma_{12,47}$	$\Gamma_{12,48}$		
$E(\Gamma)$	**012301234012340124012342012**	**0123012340123401240123420124**		
$\widetilde{\mathcal{R}}(\Gamma)$	2	4		

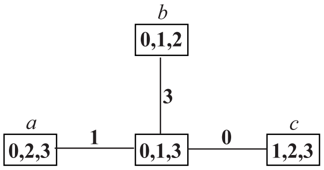

Fig. 1

Fig. 2

Fig. 3

Fig. 4

4 Cells of the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$

4.1 Recall in 3.6 that we defined the set $\widetilde{\Sigma}_i$ in the group $\widetilde{D}_4$ for any $i \in \mathbb{N}$. Let Σ_i be the set of all left cells in the weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$ and let $n_i = |\Sigma_i|$. By Corollary 2.5, we have $\Sigma_i = \{\Gamma \cap \widetilde{B}_3 \mid \Gamma \in \widetilde{\Sigma}_i, \Gamma \cap \widetilde{B}_3 \neq \emptyset\}$ for any $i \in \mathbb{N}$.

Let us use some special notation for the left cells of $(\widetilde{B}_3, \widetilde{\ell})$ as follows: For $i \neq 2, 6$, denote $\Gamma'_{i,j} := \Gamma_{i,j} \cap \widetilde{B}_3$ for any $\Gamma_{i,j} \in \widetilde{\Sigma}_i$ with $\Gamma_{i,j} \cap \widetilde{B}_3 \neq \emptyset$. On the other hand, for $i \in \{2, 6\}$, denote $\Gamma'_{ik,j} := \Gamma_{ik,j} \cap \widetilde{B}_3$ for any $\Gamma_{ik,j} \in \widetilde{\Sigma}_i$ with $\Gamma_{ik,j} \cap \widetilde{B}_3 \neq \emptyset$.

For example, $\Gamma_{4,16} \in \widetilde{\Sigma}_4$ satisfies $\Gamma_{4,16} \cap \widetilde{B}_3 \neq \emptyset$, so $\Gamma'_{4,16} := \Gamma_{4,16} \cap \widetilde{B}_3$ is a left cell of $\widetilde{B}_3$.

In the subsequent discussion, when we mention a left cell Γ' of $\widetilde{B}_3$, we mean $\Gamma' = \Gamma \cap \widetilde{B}_3$ for some left cell Γ of $\widetilde{D}_4$.

Theorem 4.2 $(n_0, n_1, n_2, n_3, n_4, n_6, n_7, n_{12}) = (1, 3, 6, 10, 12, 24, 24, 48)$.

Proof By the knowledge of the set $\widetilde{\mathcal{D}}$ in [17], we can get the set $\mathcal{D} = \{d \in \widetilde{\mathcal{D}} \mid \alpha(d) = d\}$ (see 2.1 for α) by Lemma 2.6. So we get the numbers n_i by a direct counting in $\mathcal{D}$ and by Lemma 2.3, Corollaries 2.5 and 2.7.

The weighted Coxeter group $(\widetilde{B}_3, \widetilde{\ell})$ contains 128 left cells in total by Theorem 4.2. In the proof of Theorem 4.2, we actually get the following results.

$\Sigma_0 = \{\Gamma'_{0,1}\}$, where $\Gamma'_{0,1} = \{e\}$;

$\Sigma_1 = \{\Gamma'_{1,1}, \Gamma'_{1,2}, \Gamma'_{1,3}\}$;

$\Sigma_2 = \{\Gamma'_{21,1}, \Gamma'_{21,2}, \Gamma'_{21,3}, \Gamma'_{21,4}, \Gamma'_{21,7}, \Gamma'_{21,8}\}$;

$\Sigma_3 = \{\Gamma'_{2,1}, \Gamma'_{2,2}, \Gamma'_{2,4}, \Gamma'_{2,7}, \Gamma'_{2,13}, \Gamma'_{2,14}, \Gamma'_{2,19}, \Gamma'_{2,20}, \Gamma'_{2,21}, \Gamma'_{2,22}\}$;

$\Sigma_4 = \{\Gamma'_{4,1}, \Gamma'_{4,2}, \Gamma'_{4,3}, \Gamma'_{4,4}, \Gamma'_{4,7}, \Gamma'_{4,12}, \Gamma'_{4,15}, \Gamma'_{4,16}, \Gamma'_{4,19}, \Gamma'_{4,20}, \Gamma'_{4,23}, \Gamma'_{4,24}\}$;

$\Sigma_6 = \{\Gamma'_{61,1}, \Gamma'_{61,2}, \Gamma'_{61,5}, \Gamma'_{61,6}, \Gamma'_{61,8}, \Gamma'_{61,10}, \Gamma'_{61,11}, \Gamma'_{61,12}, \Gamma'_{61,14}, \Gamma'_{61,16}, \Gamma'_{61,17}, \Gamma'_{61,18}, \Gamma'_{61,20}, \Gamma'_{61,21}, \Gamma'_{61,23}, \Gamma'_{61,24}, \Gamma'_{61,29}, \Gamma'_{61,36}, \Gamma'_{61,39}, \Gamma'_{61,40}, \Gamma'_{61,43}, \Gamma'_{61,44}, \Gamma'_{61,47}, \Gamma'_{61,48}\}$;

$\Sigma_7 = \{\Gamma'_{7,25}, \Gamma'_{7,26}, \Gamma'_{7,27}, \Gamma'_{7,28}, \Gamma'_{7,29}, \Gamma'_{7,30}, \Gamma'_{7,31}, \Gamma'_{7,32}, \Gamma'_{7,39}, \Gamma'_{7,40}, \Gamma'_{7,43}, \Gamma'_{7,46}, \Gamma'_{7,73}, \Gamma'_{7,74}, \Gamma'_{7,75}, \Gamma'_{7,76}, \Gamma'_{7,77}, \Gamma'_{7,78}, \Gamma'_{7,79}, \Gamma'_{7,80}, \Gamma'_{7,87}, \Gamma'_{7,88}, \Gamma'_{7,91}, \Gamma'_{7,94}\}$;

$\Sigma_{12} = \{\Gamma'_{12,49}, \Gamma'_{12,50}, \Gamma'_{12,51}, \Gamma'_{12,52}, \Gamma'_{12,57}, \Gamma'_{12,59}, \Gamma'_{12,61}, \Gamma'_{12,63}, \Gamma'_{12,65}, \Gamma'_{12,67}, \Gamma'_{12,69}, \Gamma'_{12,71}, \Gamma'_{12,74}, \Gamma'_{12,76}, \Gamma'_{12,78}, \Gamma'_{12,80}, \Gamma'_{12,82}, \Gamma'_{12,84}, \Gamma'_{12,86}, \Gamma'_{12,88}, \Gamma'_{12,93}, \Gamma'_{12,94}, \Gamma'_{12,95}, \Gamma'_{12,96}, \Gamma'_{12,97}, \Gamma'_{12,98}, \Gamma'_{12,99}, \Gamma'_{12,101}, \Gamma'_{12,104}, \Gamma'_{12,107}, \Gamma'_{12,108}, \Gamma'_{12,111}, \Gamma'_{12,112}, \Gamma'_{12,116}, \Gamma'_{12,117}, \Gamma'_{12,118}, \Gamma'_{12,123}, \Gamma'_{12,124}, \Gamma'_{12,125}, \Gamma'_{12,128}, \Gamma'_{12,129}, \Gamma'_{12,132}, \Gamma'_{12,133}, \Gamma'_{12,137}, \Gamma'_{12,140}, \Gamma'_{12,142}, \Gamma'_{12,143}, \Gamma'_{12,144}\}$.

4.3 Let $\Gamma' = \Gamma \cap \widetilde{B}_3$ be a left cell of $(\widetilde{B}_3, \widetilde{\ell})$ for some left cell Γ of $\widetilde{D}_4$. Then the set $E_{\min}(\Gamma')$ can be described as follows.

(1) If $\Gamma' \in \Sigma_i$, $i \in \{1, 2, 4, 6, 12\}$, then $E(\Gamma) \subset \widetilde{B}_3$ and $|E(\Gamma)| = 1$. Hence $E_{\min}(\Gamma') = E(\Gamma)$.

(2) If $\Gamma' \in \Sigma_3$, then there are three cases:

(2a) When $\Gamma' = \Gamma'_{3,1}$, we have $E(\Gamma_{3,1}) = \{a, b\} \subset \widetilde{B}_3$ with $a = \mathbf{0201}$, $b = \mathbf{1201}$, where $\mathbf{1} \cdot a = \mathbf{0} \cdot b \in \Gamma'_{3,1}$. Hence $E_{\min}(\Gamma'_{3,1}) = E(\Gamma_{3,1})$.

(2b) When $\Gamma' = \Gamma'_{3,7}$, we have $E(\Gamma_{3,7}) = \{a, b\} \cap \widetilde{B}_3 = \emptyset$ with $a = \mathbf{2423}$, $b = \mathbf{2324}$. Then $c := \mathbf{3} \cdot a = \mathbf{4} \cdot b \in \widetilde{B}_3$. Hence $E_{\min}(\Gamma'_{3,7}) = \{c\}$.

(2c) When $\Gamma' \notin \{\Gamma'_{3,1}, \Gamma'_{3,7}\}$, we have $E(\Gamma) \subset \widetilde{B}_3$ and $|E(\Gamma)| = 1$. Hence $E_{\min}(\Gamma') = E(\Gamma)$.

(3) If $L' \in \Sigma_7$, there are two cases:

(3a) $E(\Gamma) = \{a, b\}$, $E(\Gamma) \cap \widetilde{B}_3 = \emptyset$ and $c := \mathbf{3} \cdot a = \mathbf{4} \cdot b \in \Gamma'$. Hence $E_{\min}(\Gamma') = \{c\}$.

(3b) $E(\Gamma) = \{a, b, c\}$, $E(\Gamma) \cap \widetilde{B}_3 = \{c\}$ and, either $\mathbf{4} \cdot a = \mathbf{3} \cdot b = \mathbf{1} \cdot c \in \Gamma'$ or $\mathbf{4} \cdot a = \mathbf{3} \cdot b = \mathbf{0} \cdot c \in \Gamma'$. Hence $E_{\min}(\Gamma') = \{c\}$.

We display the sets $E_{\min}(\Gamma')$ for all left cells $\Gamma' \in \bigcup_{i \leqslant 12} \Sigma_i$ in Tables 8–14.

Tab. 8 Description of left cells in $\boldsymbol{\Sigma}_0$

Γ'	$\Gamma'_{1,1}$	$\Gamma'_{1,2}$	$\Gamma'_{1,3}$
$E_{\min}(\Gamma')$	**0**	**1**	**2**

Tab. 9 Description of left cells in $\boldsymbol{\Sigma}_2$

Γ'	$\Gamma'_{21,1}$	$\Gamma'_{21,2}$	$\Gamma'_{21,3}$	$\Gamma'_{21,4}$	$\Gamma'_{21,7}$	$\Gamma'_{21,8}$
$E_{\min}(\Gamma')$	**01**	**34**	**012**	**342**	**3420**	**3421**

Tab. 10 Description of left cells in $\boldsymbol{\Sigma}_3$

Γ'	$\Gamma'_{3,1}$	$\Gamma'_{3,2}$	$\Gamma'_{3,4}$	$\Gamma'_{3,7}$	$\Gamma'_{3,13}$	$\Gamma'_{3,14}$	$\Gamma'_{3,19}$	$\Gamma'_{3,20}$	$\Gamma'_{3,21}$	$\Gamma'_{3,22}$
$E_{\min}(\Gamma')$	**0201 , 1201**	**020**	**121**	$\mathbf{3} \cdot a = \mathbf{4} \cdot b = \mathbf{4} \cdot \mathbf{2324}$	**0342**	**1342**	**034**	**03421**	**134**	**13420**

Tab. 11 Description of left cells in $\boldsymbol{\Sigma}_4$

Γ'	$\Gamma'_{4,1}$	$\Gamma'_{4,2}$	$\Gamma'_{4,3}$	$\Gamma'_{4,4}$	$\Gamma'_{4,7}$	$\Gamma'_{4,12}$	$\Gamma'_{4,15}$	$\Gamma'_{4,16}$	$\Gamma'_{4,19}$
$E_{\min}(\Gamma')$	**0134**	**01342**	**013420**	**013421**	**0134201**	**0134234**	**01342340**	**01342341**	**013423402**

Γ'	$\Gamma'_{4,20}$	$\Gamma'_{4,23}$	$\Gamma'_{4,24}$
$E_{\min}(\Gamma')$	**013423412**	**0134234021**	**0134234120**

Tab. 12 Description of left cells in Σ_6

Γ'	$\Gamma'_{61,1}$	$\Gamma'_{61,2}$	$\Gamma'_{61,5}$	$\Gamma'_{61,6}$	$\Gamma'_{61,8}$	$\Gamma'_{61,10}$	$\Gamma'_{61,11}$	$\Gamma'_{61,12}$
$E_{\min}(\Gamma')$	**012012**	**234234**	**2342340**	**2342341**	**01201234**	**23423401**	**23423402**	**23423412**

Γ'	$\Gamma'_{61,14}$	$\Gamma'_{61,16}$	$\Gamma'_{61,17}$	$\Gamma'_{61,18}$	$\Gamma'_{61,20}$	$\Gamma'_{61,21}$	$\Gamma'_{61,23}$
$E_{\min}(\Gamma')$	**012012342**	**234234012**	**234234021**	**234234120**	**0120123420**	**0120123421**	**2342340120**

Γ'	$\Gamma'_{61,24}$	$\Gamma'_{61,29}$	$\Gamma'_{61,36}$	$\Gamma'_{61,39}$	$\Gamma'_{61,40}$	$\Gamma'_{61,43}$
$E_{\min}(\Gamma')$	**2342340121**	**01201234201**	**23423401234**	**234234012340**	**234234012341**	**2342340123402**

Γ'	$\Gamma'_{61,44}$	$\Gamma'_{61,47}$	$\Gamma'_{61,48}$
$E_{\min}(\Gamma')$	**2342340123412**	**23423401234021**	**23423401234120**

Tab. 13 Description of left cells in Σ_7

Γ'	$\Gamma'_{7,25}$	$\Gamma'_{7,26}$	$\Gamma'_{7,27}$	$\Gamma'_{7,28}$	$\Gamma'_{7,29}$
$E_{\min}(\Gamma')$	c=**23423402341234021**	c=**2342340234123402**	c=**234234023412340**	c=**234234023412**	c=**23423402341**

Γ'	$\Gamma'_{7,30}$	$\Gamma'_{7,31}$	$\Gamma'_{7,32}$	$\Gamma'_{7,39}$
$E_{\min}(\Gamma')$	c=**2342340234**	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **023024023**	$3\cdot a=\mathbf{4}\cdot\mathbf{b}=\mathbf{4}\cdot$ **0230240231**	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **02302402312**

Γ'	$\Gamma'_{7,40}$	$\Gamma'_{7,43}$	$\Gamma'_{7,46}$	$\Gamma'_{7,73}$
$E_{\min}(\Gamma')$	$3\cdot a=\mathbf{4}\cdot b=4\cdot$ **023024023120**	c=**2342340234120**	c=**23423402341234**	c=**23423412340234120**

Γ'	$\Gamma'_{7,74}$	$\Gamma'_{7,75}$	$\Gamma'_{7,76}$	$\Gamma'_{7,77}$	$\Gamma'_{7,78}$
$E_{\min}(\Gamma')$	c=**2342341234023412**	c=**234234123402341**	c=**234234123402**	c=**23423412340**	c=**2342341234**

Γ'	$\Gamma'_{7,79}$	$\Gamma'_{7,80}$	$\Gamma'_{7,87}$
$E_{\min}(\Gamma')$	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **123124123**	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **1231241230**	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **12312412302**

Γ'	$\Gamma'_{7,88}$	$\Gamma'_{7,91}$	$\Gamma'_{7,94}$
$E_{\min}(\Gamma')$	$\mathbf{3}\cdot a=\mathbf{4}\cdot b=\mathbf{4}\cdot$ **123124123021**	c=**2342341234021**	c=**23423412340234**

Tab. 14 Description of left cells in Σ_{12}

Γ'	$\Gamma'_{12,49}$	$\Gamma'_{12,50}$	$\Gamma'_{12,51}$	$\Gamma'_{12,52}$	$\Gamma'_{12,57}$
$E_{\min}(\Gamma')$	**023402340234**	**0234023402341**	**02340234023412**	**023402340234120**	**0234023402341234**

Γ'	$\Gamma'_{12,59}$	$\Gamma'_{12,61}$	$\Gamma'_{12,63}$	$\Gamma'_{12,65}$
$E_{\min}(\Gamma')$	**02340234023412340**	**02340234012341234**	**023402340234123402**	**023402340123412340**

Γ'	$\Gamma'_{12,67}$	$\Gamma'_{12,69}$	$\Gamma'_{12,71}$	$\Gamma'_{12,74}$
$E_{\min}(\Gamma')$	**0234023401234123402**	**0234023402341234021**	**0234023402341234 0234** 	**0234023401234123402 1**

Γ'	$\Gamma'_{12,76}$	$\Gamma'_{12,78}$	$\Gamma'_{12,80}$
$E_{\min}(\Gamma')$	**02340234012341234 0234**	**023402340234123402341**	**023402340123412340 2341**

Γ'	$\Gamma'_{12,82}$	$\Gamma'_{12,84}$	$\Gamma'_{12,86}$
$E_{\min}(\Gamma')$	**023402340234123402 3412**	**023402340123412340 23412**	**023402340234123402 34120**

Γ'	$\Gamma'_{12,88}$	$\Gamma'_{12,93}$	$\Gamma'_{12,94}$
$E_{\min}(\Gamma')$	**02340234012341234023412 0**	**02340234012341234023412 34**	**02340234012341234023412 340**

Γ'	$\Gamma'_{12,95}$	$\Gamma'_{12,96}$	$\Gamma'_{12,97}$
$E_{\min}(\Gamma')$	**0234023401234123402341 23402**	**0234023401234123402341 234021**	**123412341234**

Γ'	$\Gamma'_{12,98}$	$\Gamma'_{12,99}$	$\Gamma'_{12,101}$	$\Gamma'_{12,104}$	$\Gamma'_{12,107}$
$E_{\min}(\Gamma')$	**1234123412340**	**12341234123402**	**123412341234021**	**1234123412340234**	**12341234123402341**

Γ'	$\Gamma'_{12,108}$	$\Gamma'_{12,111}$	$\Gamma'_{12,112}$	$\Gamma'_{12,116}$
$E_{\min}(\Gamma')$	**12341234012340234**	**123412341234023412**	**123412340123402341**	**1234123401234023412**

Γ'	$\Gamma'_{12,117}$	$\Gamma'_{12,118}$	$\Gamma'_{12,123}$
$E_{\min}(\Gamma')$	**1234123412340234120**	**12341234123402341234**	**12341234012340234120**

Γ'	$\Gamma'_{12,124}$	$\Gamma'_{12,125}$	$\Gamma'_{12,128}$
$E_{\min}(\Gamma')$	**123412340123402341234**	**1234123412340234 12340**	**123412340123402341 2340**

Γ'	$\Gamma'_{12,129}$	$\Gamma'_{12,132}$	$\Gamma'_{12,133}$
$E_{\min}(\Gamma')$	**1234123412340234123402**	**1234123401234023412 3402**	**1234123412340234123 4021**

Γ'	$\Gamma'_{12,137}$	$\Gamma'_{12,140}$	$\Gamma'_{12,142}$
$E_{\min}(\Gamma')$	**1234123401234023412 34021**	**1234123401234023412 340234**	**1234123401234023412 3402341**

Γ'	$\Gamma'_{12,143}$	$\Gamma'_{12,144}$
$E_{\min}(\Gamma')$	**1234123401234023412 34023412**	**1234123401234023412 340234120**

Lemma 4.4 *Let Γ' be a left cell of $\widetilde{B}_3$ and let $w \in \Gamma'$. Then $w = w_1 \cdot w_0$ for some $w_0 \in E_{\min}(\Gamma')$ and $w_1 \in \widetilde{B}_3$.*

Proof By Tables 8-14, there are three possible cases:

(1) $E_{\min}(\Gamma') = E(\Gamma)$. By the definition of $E(\Gamma)$, we can write $w = w_1 \cdot w_0$ for some $w_0 \in E_{\min}(\Gamma')$ and $w_1 \in \widetilde{D}_4$. In this case, we have $w_1 = ww_0^{-1} \in \widetilde{B}_3$.

(2) $G(L') = \{c\}$ with $c := \mathbf{3} \cdot a = \mathbf{4} \cdot b$ and $E(\Gamma) = \{a, b\}$ for some $a, b \in \widetilde{D}_4 - \widetilde{B}_3$. It is evident that $\{\mathbf{3}, \mathbf{4}\} \subseteq \widetilde{\mathcal{L}}(\mathbf{3} \cdot a)$, $\mathbf{4} \in \widetilde{\mathcal{L}}(a)$ and $\mathbf{3} \in \widetilde{\mathcal{L}}(b)$. Let $y = \mathbf{4}a$. Then $a = \mathbf{4} \cdot y$ and $b = \mathbf{3} \cdot y$. Since $\mathbf{3} \cdot a = \mathbf{34} \cdot y \in \widetilde{B}_3$, we have $y \in \widetilde{B}_3$. Hence for any $w \in \Gamma'$, we have $w \in \{w_2 \cdot \mathbf{4} \cdot y, w_3 \cdot \mathbf{3} \cdot y\}$ with some $w_2, w_3 \in \widetilde{D}_4$ by the definition of $E(\Gamma)$. Denote $x = wy^{-1}$, then $x \in \{w_2 \cdot \mathbf{4}, w_3 \cdot \mathbf{3}\}$. If $x = w_2 \cdot \mathbf{4}$ then $\mathbf{4} \in \widetilde{\mathcal{R}}(x)$. Since $x \in \widetilde{B}_3$, we get $\{\mathbf{3}, \mathbf{4}\} \subseteq \widetilde{\mathcal{R}}(x)$. So $x = x' \cdot \mathbf{34}$ for some $x' \in \widetilde{B}_3$, then $w = x' \cdot 43 \cdot y = x' \cdot 3 \cdot a$. By a similar argument, we see that if $x = w_3 \cdot 3$, then there is some $x'' \in \widetilde{B}_3$ with $w = x'' \cdot \mathbf{43} \cdot y = x'' \cdot \mathbf{3} \cdot a$.

(3) $E_{\min}(\Gamma') = \{c\}$ with $E(\Gamma) = \{a, b, c\}$ and either $\mathbf{4} \cdot a = \mathbf{3} \cdot b = \mathbf{0} \cdot c$ or $4 \cdot a = \mathbf{3} \cdot b = \mathbf{1} \cdot c$. Assume $\mathbf{4} \cdot a = \mathbf{3} \cdot b = \mathbf{0} \cdot c$ (The case $\mathbf{4} \cdot a = \mathbf{3} \cdot b = \mathbf{1} \cdot c$ can be dealt with similarly). Then $\{\mathbf{0}, \mathbf{3}, \mathbf{4}\} \subseteq \widetilde{\mathcal{L}}(\mathbf{4} \cdot a)$, so $\{\mathbf{0}, \mathbf{3}\} \subseteq \widetilde{\mathcal{L}}(a)$, $\{\mathbf{0}, \mathbf{4}\} \subseteq \widetilde{\mathcal{L}}(b)$ and $\{\mathbf{3}, \mathbf{4}\} \subseteq \widetilde{\mathcal{L}}(c)$. Denote $y := 03a$ (hence $y = 04b = 34c$). Then $a = \mathbf{03} \cdot y$, $b = \mathbf{04} \cdot y$ and $c = \mathbf{34} \cdot y$. Since $c = \mathbf{34} \cdot y \in \widetilde{B}_3$, we get $y \in \widetilde{B}_3$. By the definition of $E(\Gamma)$, we have $w \in \{w_2 \cdot \mathbf{03} \cdot y, w_3 \cdot \mathbf{04} \cdot y, w_4 \cdot c\}$ for some $w_2, w_3, w_4 \in \widetilde{D}_4$. If $w = w_4 \cdot c$, then our result is proved. If $w \in \{w_2 \cdot \mathbf{03} \cdot y, w_3 \cdot \mathbf{04} \cdot y\}$, let $x = wy^{-1}$, then $x \in \{w_2 \cdot \mathbf{03}, w_3 \cdot \mathbf{04}\}$. First assume $x = w_2 \cdot \mathbf{03}$. Then $\{\mathbf{0}, \mathbf{3}\} \subseteq \widetilde{\mathcal{R}}(x)$. Since $x \in \widetilde{B}_3$, we have $\{\mathbf{0}, \mathbf{3}, \mathbf{4}\} \subseteq \widetilde{\mathcal{R}}(x)$. So $x = x' \cdot \mathbf{034}$ for some $x' \in \widetilde{B}_3$. Thus $w = x' \cdot \mathbf{034} \cdot y = x' \cdot \mathbf{0} \cdot c$. Similarly, when $x = w_{\mathbf{3}} \cdot \mathbf{04}$, we can find some $x'' \in \widetilde{B}_3$ such that $w = x'' \cdot \mathbf{034} \cdot y = x'' \cdot \mathbf{0} \cdot c$, too.

Therefore, the lemma is proved.

Theorem 4.5 *Any left cell of $(\widetilde{B}_3, \widetilde{\ell})$ is left-connected.*

Proof Recall that $S = \{t_i | 0 \leqslant i \leqslant 3\}$ is the Coxeter generator set of $\widetilde{B}_3$. Let Γ' be a left cell of $\widetilde{B}_3$ with $\Gamma' \neq \Gamma'_{3,1}$. By Tables 8-14, we have $|E_{\min}(\Gamma')| = 1$. Any $w \in \Gamma'$ can be written in the form $w = w_1 \cdot w_0$ with some $w_0 \in E_{\min}(\Gamma')$ and $w_1 \in \widetilde{B}_3$ by Lemma 4.4. Let $w_1 = t'_0 t'_1 t'_2 \cdots t'_r$ be a reduced expression of w_1 with $t'_i \in S$ and $x_i = t'_i t'_{i+1} \cdots t'_r w_0$ for $0 \leqslant i \leqslant r$. Then $x_0 = w, x_1, \cdots, x_r, x_{r+1} = w_0$ is a sequence of elements in Γ' by 1.6(1)(2). We get $w \underset{\Gamma'_L}{\text{——}} w_0$. Hence Γ' is left-connected.

The left cell $\Gamma' = \Gamma'_{3,1}$ satisfies $E_{\min}(\Gamma') = \{a, b\}$ with $a = \mathbf{0201}$ and $b = \mathbf{1201}$. By 4.3(2a), we get $a \underset{\Gamma'_L}{\text{——}} b$. For any $x, y \in \Gamma'$, write $x = x' \cdot x''$ and $y = y' \cdot y''$ for some $x', y' \in \widetilde{B}_3$ and some $x'', y'' \in E_{\min}(\Gamma')$. We have $x \underset{\Gamma'_L}{\text{——}} x''$ and $y \underset{\Gamma'_L}{\text{——}} y''$ by the argument similar to that in the above paragraph. Since $x'' \underset{\Gamma'_L}{\text{——}} y''$, we have $x \underset{\Gamma'_L}{\text{——}} y$. Therefore $\Gamma'_{3,1}$ is left-connected.

Let $I = \{0, 1, 2, 3, 4, 6, 7, 12\}$. For $i \in I$, denote $\mathfrak{g}_i = \bigcup\limits_{\Gamma' \in \Sigma_i} E_{\min}(\Gamma')$, $\sigma_i = \bigcup\limits_{\Gamma' \in \Sigma_i} \Gamma'$.

Lemma 4.6 *We have $x \underset{LR}{\sim} y$ and $x \underset{\sigma_i LR}{\text{——}} y$ for any $x, y \in \mathfrak{g}_i$ with $i \in I$.*

Proof We claim that each $\mathfrak{g}_i$, $i \in I$, is contained in a two-sided cell of $\widetilde{B}_3$ and in a two-sided-connected component of σ_i.

Since $\mathfrak{g}_0 = \{e\}$, the claim in this case is obviously true.

By Tab. 8, we get $\mathfrak{g}_1 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. Since $\mathbf{0} \underset{R}{\sim} \mathbf{02} \underset{R}{\sim} \mathbf{021} \underset{L}{\sim} \mathbf{21} \underset{L}{\sim} \mathbf{1}$ and $\mathbf{02} \underset{L}{\sim} \mathbf{2}$, we have $\mathbf{0} \underset{LR}{\sim} \mathbf{2} \underset{LR}{\sim} \mathbf{1}$ and $\mathbf{0} \underset{\sigma_{\mathbf{1}} LR}{\text{——}} \mathbf{2} \underset{\sigma_1 LR}{\text{——}} \mathbf{1}$.

By Tab. 9, we get $\mathfrak{g}_2 = \{\mathbf{01}, \mathbf{34}, \mathbf{012}, \mathbf{342}, \mathbf{3420}, \mathbf{3421}\}$. It is obvious that $\mathbf{01} \underset{R}{\sim} \mathbf{012}$ and $\mathbf{34} \underset{R}{\sim} \mathbf{342} \underset{R}{\sim} \mathbf{3420}$ and $\mathbf{342} \underset{R}{\sim} \mathbf{3421}$. We have $\mathbf{01} \underset{LR}{\sim} \mathbf{34}$ since $\mathbf{01} \underset{R}{\sim} \mathbf{012} \underset{R}{\sim} \mathbf{01234} \underset{L}{\sim} 1234 \underset{L}{\sim} \mathbf{234} \underset{L}{\sim} \mathbf{34}$ (see [11]). The claim is true for the set $\mathfrak{g}_2$.

We display all the elements of the sets $\mathfrak{g}_3$, $\mathfrak{g}_4$, $\mathfrak{g}_6$, $\mathfrak{g}_7$, $\mathfrak{g}_{12}$ in Tab. 10, Tab. 11, Tab. 12, Tab. 13, Tab. 14, respectively. To save the space, here we shall not reproduce all the elements of $\mathfrak{g}_i$ for $i = 3, 4, 6, 7, 12$.

For $\mathfrak{g}_3$ (see Tab. 10), we have $\mathbf{134} \underset{L}{\sim} \mathbf{2134} \underset{L}{\sim} \mathbf{12134} \underset{R}{\sim} \mathbf{121}$ and $\mathbf{034} \underset{L}{\sim} \mathbf{2034} \underset{L}{\sim} \mathbf{02034} \underset{R}{\sim} \mathbf{020}$ and $\mathbf{134} \underset{L}{\sim} \mathbf{2134} \underset{L}{\sim} \mathbf{342134} \underset{R}{\sim} \mathbf{34234}$ and $\mathbf{121} \underset{R}{\sim} \mathbf{1210} \underset{L}{\sim} \mathbf{01210} \underset{L}{\sim} \mathbf{0201} \underset{R}{\sim} 020$ and $\mathbf{134} \underset{R}{\sim} \mathbf{1342} \underset{R}{\sim} \mathbf{13420}$ and $\mathbf{034} \underset{R}{\sim} \mathbf{0342} \underset{R}{\sim} \mathbf{03421}$. So the claim is true for $\mathfrak{g}_3$.

Observe Tab. 11. All the elements of $\mathfrak{g}_4$ are in the same right-connected component of σ_4 and hence they are in the same right cell of $\widetilde{B}_3$. The claim is proved for $\mathfrak{g}_4$.

From Tab. 12, we see that $\mathfrak{g}_6$ consists of two kinds of elements: the first kind of elements are of the form $\mathbf{012012} \cdot z$, $z \in \widetilde{B}_3$ which are in the same right cell of $\widetilde{B}_3$ and also in the same right-connected component of σ_6; the second kind of elements are the form $\mathbf{234234} \cdot z'$, $z' \in \widetilde{B}_3$ all of which are in another right cell of $\widetilde{B}_3$ and also in the same right-connected component of σ_6. Since $\mathbf{012012} \underset{L}{\sim} \mathbf{34012012} \underset{L}{\sim} \mathbf{234012012} \underset{L}{\sim} \mathbf{1234012012} \underset{L}{\sim} \mathbf{01234012012} \underset{L}{\sim} \mathbf{201234012012} \underset{R}{\sim} \mathbf{20123401201234} \underset{R}{\sim} \mathbf{201234012012342} \underset{R}{\sim} \mathbf{2012340120123423 4} \underset{L}{\sim} \mathbf{0123401201234234} \underset{L}{\sim} \mathbf{123401201234234} \underset{L}{\sim} \mathbf{23401201234234} \underset{L}{\sim} \mathbf{3401201234234} \underset{L}{\sim} \mathbf{01201234234} \underset{L}{\sim} \mathbf{1201234234} \underset{L}{\sim} \mathbf{201234234} \underset{L}{\sim} \mathbf{01234234} \underset{L}{\sim} 1234234 \underset{L}{\sim} \mathbf{234234}$, this implies that the set $\mathfrak{g}_6$ is contained in some two-sided cell of $\widetilde{B}_3$ and also in some two-sided-connected component of σ_6.

By Tab. 13, we see that the elements in $\mathfrak{g}_7$ can be put into four classes according to their reduced expressions: $\mathbf{2342340234} \cdot z_1$, $\mathbf{2342341234} \cdot z_2$, $\mathbf{0342340234} \cdot z_3$, $\mathbf{1342341234} \cdot z_4$, $z_j \in \widetilde{B}_3$ for $j = 1, 2, 3, 4$. It is evident that the members of the same class are in the same right cell of $\widetilde{B}_3$ and also in the same right-connected component of σ_6. Besides, we find $\mathbf{2342340234} \underset{L}{\sim} \mathbf{02342340234} = \mathbf{03423402340} \underset{R}{\sim} \mathbf{0342340234}$, $\mathbf{2342341234} \underset{L}{\sim} \mathbf{12342341234} = \mathbf{13423412341} \underset{R}{\sim} \mathbf{1342341234}$, $\mathbf{1342341234} \underset{R}{\sim} \mathbf{13423412340} \underset{R}{\sim} \mathbf{134234123402} \underset{R}{\sim} \mathbf{1342341234021} \underset{R}{\sim} \mathbf{13423412340212} \underset{R}{\sim} \mathbf{134234123402123 4} \underset{R}{\sim} \mathbf{1342341234021234 2} \underset{R}{\sim} \mathbf{13423412340212342 0} \underset{R}{\sim} \mathbf{134234123402123420 1} = \mathbf{1234210342340234120} \underset{L}{\sim} \mathbf{234210342340234120} \underset{L}{\sim} \mathbf{34210342340234120} \underset{L}{\sim} \mathbf{210342340234120} \underset{L}{\sim} \mathbf{10342340234120} \underset{L}{\sim} \mathbf{0342340234120} \underset{R}{\sim} \mathbf{034234023412} \underset{R}{\sim} \mathbf{03423402341} \underset{R}{\sim} \mathbf{0342340234}$, hence $\mathfrak{g}_7$ is contained in some two-sided cell of $\widetilde{B}_3$ and also in some two-sided-connected component of σ_7.

By Tab. 14, we know that there are totally two kinds of elements in $\mathfrak{g}_{12}$. One kind of elements which can be written in the form $\mathbf{023402340234} \cdot z$ where $z \in \widetilde{B}_3$ are in a right cell and in a right-connected component of σ_{12}, while the other kind of elements which can be written in the form $\mathbf{123412341234} \cdot z'$ where $z' \in \widetilde{B}_3$ are in another right cell and in a right-connected component of σ_{12}. Since $\mathbf{123412341234} \underset{R}{\sim} \mathbf{1234123412340} \underset{R}{\sim} \mathbf{12341234123402} \underset{R}{\sim} \mathbf{123412341234023 4} \underset{R}{\sim} \mathbf{1234123412340234 2} \underset{R}{\sim} \mathbf{12341234123402342 0} = \mathbf{12342102340234023 4} \underset{L}{\sim} \mathbf{2342102340234023 4} \underset{L}{\sim} \mathbf{342102340234023 4} \underset{L}{\sim} \mathbf{21023402340234} \underset{L}{\sim} \mathbf{1023402340234} \underset{L}{\sim} \mathbf{023402340234}$, we see that $\mathfrak{g}_{12}$ is contained in some two-sided cell of $\widetilde{B}_3$ and in some two-sided-connected component of σ_{12}.

The proof is completed.

Theorem 4.7 *For $i\in I$, the set σ_i forms a single two-sided cell of $(\widetilde{B}_3,\widetilde{\ell})$. Furthermore, σ_i is two-sided-connected.*

Proof By 1.6(1), we see that for any $i\in I$, the set σ_i is a union of some two-sided cells of $(\widetilde{B}_3,\widetilde{\ell})$. Then Lemma 4.6 tells us that $x \underset{\sigma_i LR}{\text{---}} y$ for any $x,y\in\mathfrak{g}_i$. Since each left cell of $(\widetilde{B}_3,\widetilde{\ell})$ is proved to be left-connected and contains an element of $\mathfrak{g}_i$, this implies that σ_i is two-sided-connected. Hence σ_i is a single two-sided cell of $\widetilde{B}_3$ by Lemma 4.6.

By Theorem 4.7, Lemma 2.3 and 3.5, we see that there are totally eight two-sided cells in $(\widetilde{B}_3,\widetilde{\ell})$.

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