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# Efficient characterization for $I\{2\}$ and $M\{2\}$

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**Abstract:** An important characterization formula for  $M\{2\}$  was given by Stewart where  $M \in \mathbb{C}^{m \times n}$ . But this formula contains redundant arbitrary parameters, and therefore is nonefficient. This paper, by using the matrix full rank decomposition, showed that for a proper subset of  $I\{2\}_s$ , which is denoted as  $\mathbb{B}_1$ , the redundant arbitrary parameters in Stewart's formula can be eliminated, and  $I\{2\}_s$  is a union set of its certain subsets, and each of the subsets is 2-norm isometry with  $\mathbb{B}_1$ . Finally, the efficient characterization formulas for  $I\{2\}_s$ ,  $I\{2\}$  and  $M\{2\}$  are obtained respectively. An algorithm was provided that can be used to compute any element of  $I\{2\}_s$ , and avoid the repeat computation work for each element of  $I\{2\}_s$ .

**Key words:** efficient characterization for a generalized inverse matrix set; 2-norm isometry of two sets; matrix full-rank factorization; solution set of a matrix equation

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## $I\{2\}$ 和 $M\{2\}$ 的有效刻画

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**摘要:** Stewart 给出了一个矩阵 2-逆集合  $M\{2\}$  的刻画公式. 但其中含有多余的任意参数, 因而不是一个有效刻画. 本文利用方阵的满秩分解, 为  $I\{2\}_s$  的一个真子集  $\mathbb{B}_1$  剔除了 Stewart 公式中的多余任意参数, 得到了  $\mathbb{B}_1$  的有效刻画公式; 还证明了  $I\{2\}$  是其有限个子集的并集, 其中每个子集与  $\mathbb{B}_1$  等距同构. 由此可分别建立  $I\{2\}$ ,  $I\{2\}$ ,  $M\{2\}$  和  $M\{2\}$  的有效刻画公式. 算法 2.1 则可用于无重复地计算  $I\{2\}_s$  的每个元素.

**关键词:** 广义逆矩阵类的有效刻画; 两个集合的 2-范等距同构; 矩阵的满秩分解; 矩阵方程的解

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## 1 Introduction

**Definition 1.1** Assume  $f$  is a mapping from set  $\mathbb{A}$  onto set  $\mathbb{B}$ , (for example,  $\mathbb{A} = \{\alpha : \alpha = (Y, Z) : Y \in \mathbb{C}_s^{n \times s}, Z \in \mathbb{C}_s^{m \times s}, Z^*MY = I_s\}$ ,  $\mathbb{B} = \{\beta : \beta = f(\alpha) = f(Y, Z) = YZ^*, \alpha \in \mathbb{A}\}$ ). Then  $f$  is called a characterization for  $\mathbb{B}$ . And it is called efficient  $\Leftrightarrow$  for each  $\beta \in \mathbb{B}$ , if the inverse image set  $f^{-1}(\beta) = \{\alpha : f(\alpha) = \beta, \alpha \in \mathbb{A}\}$  has one and only one element.

In Ben-Israel's book<sup>[1]</sup>, certain efficient characterizations for some  $\{1\}$ -generalized inverse classes was given. According to Ben-Israel's term, in an efficient characterization, there is no redundant arbitrary parameter.

The main aim of this paper is to establish efficient characterization formulas for  $M\{2\}_s$  and  $M\{2\}$ . An important fact is that the  $\{2\}$ -generalized inverse can be used in the iterative scheme of a nonlinear systems<sup>[1]</sup>. A formula in [5] is quoted as a proposition here.

**Proposition 1.1** ([1, Th. 2.5]) Let  $M \in \mathbb{C}_r^{m \times n}$ ,  $0 < s \leq r$ . Then

$$M\{2\} = \{0\} \bigcup \bigcup_{1 \leq s \leq r} M\{2\}_s \quad (1.1)$$

$$\text{with } M\{2\}_s = \{YZ^* : Y \in \mathbb{C}_s^{n \times s}, Z \in \mathbb{C}_s^{m \times s}, Z^*MY = I_s\}. \quad (1.2)$$

(1.2) is a good general solution formula for the Penrose second matrix equation  $XXM = X$ , and it means the inverse image set  $\mathbb{A} = \{\alpha : \alpha = (Y, Z), Y \in \mathbb{C}_s^{n \times s}, Z \in \mathbb{C}_s^{m \times s}, Z^*MY = I_s\}$  has  $(m+n)s - s^2$  arbitrary parameters. But (1.2) is nonefficient. In fact, if  $\alpha_0 = (Y_0, Z_0) \in \mathbb{A}$ , set  $Y = cY_0, Z = 1/cZ_0, 1 \neq c \neq 0$ , then  $\alpha_0 \neq \alpha = (Y, Z) \in \mathbb{A}$  and  $YZ^* = Y_0Z_0^*$ . It means (1.2) has redundant arbitrary parameter, so according to Ben-Israel's term or Definition 1.1, it is nonefficient. Furthermore, if  $\beta = L_1R_1^* \in \mathbb{C}_r^{m \times n}, L_1 \in \mathbb{C}_s^{n \times s}, R_1 \in \mathbb{C}_s^{m \times s}, T \in \mathbb{C}_s^{s \times s}$ , then  $(L_1T)(T^{-1}R_1^*) = L_1R_1^*$ . So (1.2) has at least  $s^2$  redundant arbitrary parameters.

In Dong's paper<sup>[2]</sup>, denote  $D = \text{diag}(I_r, 0)$ , a formula is

$$D\{2\} = \left\{ \begin{pmatrix} S^{-1} \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} S & S^{-1} \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} C_1 & 0 \\ C_2 & 0 \end{pmatrix} S & \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (B_1, B_2) \end{pmatrix} : S \in \mathbb{C}_r^{r \times r}, 0 \leq t \leq r \right\}. \quad (1.3)$$

We can show (1.3) contains redundant arbitrary parameters when  $r > 0$ . So (1.3) is also nonefficient.

In this paper, based on (1.2) and using full-rank factorizations of  $M$ , an efficient characterization formula for  $M\{2\}$  is obtained in §3. In the process to deduce the formula, the most difficult task is to seek the efficient characterization for  $I\{2\}_s$ . This task is completed by (2.7(i)) and (2.7(ii)). The Algorithm 2.1 is designed to avoid the repeat computation for any element of  $I\{2\}_s$ .

## 2 Efficient characterization for $I\{2\}$

The  $n \times n$  identity matrix is denoted as  $I$  or  $I_n$ . Given  $i, j \in \{1, 2, \dots, n\}$ , to exchange the  $i, j$ -th rows of  $I_n$ , we obtain the exchange (unitary permutation) matrix  $I_{i,j}$ . From (1.2),

we obtain

$$I\{2\} = \{0\} \bigcup \bigcup_{1 \leq s \leq n}^{\oplus} I\{2\}_s, \quad (2.1)$$

$$I\{2\}_s = \{YZ^* : Y, Z \in \mathbb{C}_s^{n \times s}, Z^*Y = I_s, 1 \leq s \leq n\}. \quad (2.2)$$

And (2.2) is not an efficient characterization for  $I\{2\}_s$ .

In this section, our main task is for any given  $s \in \{1, \dots, n\}$ , to establish an efficient and 'feasible' characterization formula for  $I\{2\}_s$ .

**Lemma 2.1** (1)  $I\{2\} = \{X : X^2 = X, X \in \mathbb{C}^{n \times n}\}$ .

(2)  $I\{2\}_n = \{I\}$ , and  $I\{2\}_0 = \{0\}$ .

(3) Let  $T \in \mathbb{C}_n^{n \times n}$ ,  $A = TBT^{-1}$ , Then  $A^2 = A \Leftrightarrow B^2 = B$ . Especially,  $T$  may be a permutation matrix or a unitary matrix.

**Proof** According to the definition of  $I\{2\}$ , we obtain (1). If  $X$  is nonsingular satisfies  $XIX = X$ , we directly obtain  $X = I$ . If  $\text{rank}(X) = 0$ , then we know  $X = 0$ . So (2) holds. (3) is obviously true.

In (1.1) or (1.2), if for each given  $f^{-1}(\beta) \neq \emptyset$ , one fixed element  $\alpha_\beta \in f^{-1}(\beta)$  is chosen (we can say the  $\alpha_\beta$  is an index element of  $f^{-1}(\beta)$ ), then the characterization  $\beta = f(\alpha_\beta)$  is an efficient characterization. The most important problem is how can we choose the index element  $\alpha_\beta$  for each  $\beta \in \mathbb{B} = I\{2\}_s$  practically.

Lemma 2.1(3) means for each permutation matrix  $P$  and  $\beta = YZ^*$ ,  $Y, Z \in \mathbb{C}_s^{n \times s}$ .  $\beta \in I_s\{2\} \Leftrightarrow P\beta P^T \in I_s\{2\}$ .

**Definition 2.1** For given  $n, s$  satisfy  $0 < s \leq n$ , denote  $\mathbb{A}_1 = \{\alpha : \alpha = (\alpha_y, \alpha_z) \triangleq \begin{pmatrix} I_s & I_s - \eta^* \xi \\ \eta & \xi \end{pmatrix} \in \mathbb{C}^{n \times (2s)}, \eta, \xi \in \mathbb{C}^{(n-s) \times s}, \alpha_z^* \alpha_y = I_s\}$ ,  $\mathbb{B}_1 = \{\beta : \beta = f(\alpha) = \alpha_y \alpha_z^* = \begin{pmatrix} I_s \\ \eta \end{pmatrix} (I_s - \xi^* \eta, \xi^*), \alpha \in \mathbb{A}_1\}$ .

**Theorem 2.1** Let  $\beta = YZ^*$ ,  $Y = \begin{pmatrix} Y_{11} \\ Y_{21} \end{pmatrix}$ ,  $Z = \begin{pmatrix} Z_{11} \\ Z_{21} \end{pmatrix} \in \mathbb{C}_s^{n \times s}$ ,  $Y_{11}, Z_{11} \in \mathbb{C}^{s \times s}$ .

We have

$$(1) \beta^2 = \beta \Leftrightarrow Z_{11}^* Y_{11} = I_s - Z_{21}^* Y_{21}. \quad (2.3)$$

$$(2) \beta^2 = \beta \in \mathbb{B}_1 \Leftrightarrow Y_{11} \in \mathbb{C}_s^{s \times s}. \quad (2.4)$$

$$(3) \text{ Assume } (\eta_1, \xi_1) \neq (\eta_2, \xi_2), \text{ and } \beta_i = \begin{pmatrix} I_s - \xi_i^* \eta_i & \xi_i^* \\ \eta_i - \eta_i \xi_i^* \eta_i & \eta_i \xi_i^* \end{pmatrix} \in \mathbb{B}_1, i = 1, 2, \text{ then we}$$

have

$$\beta_1 \neq \beta_2. \quad (2.5)$$

It also means if  $\mathbb{A}_1 \ni \alpha_i, \beta_i = f(\alpha_i), i=1,2$ , then  $\beta_1 \neq \beta_2 \Leftrightarrow \alpha_1 \neq \alpha_2$ .

**Proof** (1) (2.3) is equivalent to (2.2). (2) When  $Y_{11} \in \mathbb{C}_s^{s \times s}$ , set  $\xi^* = Y_{11} Z_{21}^*$ ,  $\eta = Y_{21} Y_{11}^{-1}$ , we obtain  $\beta = YZ^* = (I_s \eta^T)^T (I_s - \xi^* \eta, \xi^*) \in \mathbb{B}_1$ . Conversely, if  $\beta = YZ^* \in \mathbb{B}_1$  with  $Y_{11} \in \mathbb{C}_t^{s \times s}, t < s$ . Then we obtain  $\xi^* = Y_{11} Z_{21}^*$  and  $I_s - \xi^* \eta = Y_{11} Z_{11}^*$ . Thus  $I_s = Y_{11} (Z_{11}^* + Z_{21}^* \eta)$ . This is a contradiction. So (2) is valid.

(3-i) If  $\xi_1 \neq \xi_2$ , then the (1,2) blocks of  $\beta_1$  and  $\beta_2$  are different.

(3-ii) When  $\xi_1 = \xi_2$ , and  $\eta_1 \xi_1^* \neq \eta_2 \xi_2^*$ , we also obtain  $\beta_1 \neq \beta_2$ .

(3-iii) When  $\xi_1 = \xi_2$ , and  $\xi_1^* \eta_1 \neq \xi_2^* \eta_2$ , then  $I - \xi_1^* \eta_1 \neq I - \xi_2^* \eta_2$ .

(3-iv) If  $\xi_1 = \xi_2, \eta_1 \neq \eta_2, \xi_1^* \eta_1 = \xi_2^* \eta_2$  and  $\eta_1 \xi_1^* = \eta_2 \xi_2^*$ . Then we obtain  $\eta_1 - \eta_1 \xi_1^* \eta_1 - (\eta_2 - \eta_2 \xi_2^* \eta_2) = \eta_1 - \eta_2 + \eta_2(\xi_2^* \eta_2 - \xi_2^* \eta_1) + (\eta_2 \xi_2^* - \eta_1 \xi_1^*) \eta_1 = \eta_1 - \eta_2 \neq 0$ . So (3) is valid.

**Remark 2.1** Theorem 2.1 Means an efficient characterization from  $\mathbb{A}_1$  onto  $\mathbb{B}_1$  is obtained. Later, this result will be enlarged to the  $I\{2\}_s$  and  $I\{2\}$  cases. An important fact is that  $\mathbb{B}_1$  is completely dependent on  $2(n-s)s$  arbitrary parameters.

**Definition 2.2** Assume  $0 < s \leq n$ .

(1) The set  $\mathbb{O}_{(n,s)} = \{(i_1, \dots, i_s) : 1 \leq i_1 < \dots < i_s \leq n\}$  has  $\frac{n!}{s!(n-s)!} (\triangleq g \text{ or } g_{ns})$  elements. An appoint is if  $\alpha_t = (i_1^{[t]}, \dots, i_s^{[t]})$ ,  $\alpha_k = (i_1^{[k]}, \dots, i_s^{[k]}) \in \mathbb{O}_{(n,s)}$ , then  $\alpha_k \prec \alpha_t$  implies  $i_s^{[k]} < i_s^{[t]}$  or  $\exists p$  satisfies  $1 < p \leq s, i_{p-1}^{[k]} < i_{p-1}^{[t]}$  and  $i_h^{[k]} = i_h^{[t]}$  when  $h \geq p$ . Then we can denote  $\mathbb{O}_{(n,s)} = \{\alpha_1, \alpha_2, \dots, \alpha_g\}$  with  $\alpha_1 = (1, 2, \dots, s) \prec \dots \prec \alpha_g = (n-s+1, \dots, n)$ , and the  $\alpha_l$  is called the ' $l$ -th smallest' element of  $\mathbb{O}_{(n,s)}$ . Thus an 'ordered monotone increase' function can be defined as  $\phi(\alpha_l) = \phi_{n,s}(\alpha_l) = l, l = 1, \dots, g$ . Later, we will show for a given  $\alpha = (i_1, \dots, i_s) \in \mathbb{O}_{(n,s)}$ , how can we recursively obtain the value of  $\phi_{n,s}(\alpha)$ .

(2) Define  $\mathbb{P}_{(n,s)} = \{P_l = P(i_1, \dots, i_s) : (i_1, \dots, i_s) \in \mathbb{O}_{(n,s)}, \phi(i_1, \dots, i_s) = l, P(i_1, \dots, i_s) \triangleq \prod_{1 \leq k \leq s} I_{k, i_k}\} = \{I = P_1, \dots, P_g\}$

(3) Define  $\mathbb{A}_l = P_l \mathbb{A}_1, \mathbb{B}_l = \mathbb{B}(i_1, \dots, i_s) = P_l \mathbb{B}_1 P_l^T, 1 \leq l \leq g. \mathbb{A} = \bigcup_{l=1, \dots, g} \mathbb{A}_l, \mathbb{B} = \bigcup_{l=1, \dots, g} \mathbb{B}_l$ . So  $\mathbb{A}(1, \dots, s) = \mathbb{A}_1, \mathbb{B}(1, \dots, s) = \mathbb{B}_1$ . Define

$$\mathbb{D}_1 = \mathbb{A}_1, \mathbb{D}_l = \mathbb{A}_l \setminus \bigcup_{k < l} \mathbb{A}_k; \mathbb{C}_1 = \mathbb{B}_1, \mathbb{C}_l = \mathbb{B}_1 \setminus \bigcup_{k < l} \mathbb{B}_k, l = 2, \dots, g. \quad (2.6)$$

Thus, we have

$$\mathbb{A} = \bigcup_{l=1, \dots, g}^{\oplus} \mathbb{D}_l, \quad \mathbb{B} = \bigcup_{l=1, \dots, g}^{\oplus} \mathbb{C}_l.$$

(4) Let  $L = (l_1^T, \dots)^T \in \mathbb{C}^{n \times s}$ . If  $t \leq n$ , denote  $L(j_1, \dots, j_t) = (l_{j_1}^T, \dots, l_{j_t}^T)^T$ . The first nonsingular  $s \times s$  submatrix of  $L$  is  $L(i_1, \dots, i_s) (\triangleq L_{fns})$  means  $L(i_1, \dots, i_s)$  is nonsingular, and for each  $(j_1, \dots, j_s) \prec (i_1, \dots, i_s), L(j_1, \dots, j_s)$  is singular.

**Notice 2.1** In Definition 2.2(1), for each  $\alpha = (i_1, \dots, i_s) \in \mathbb{O}_{(n,s)}$ , the function value  $l = \phi(\alpha)$  can be recursively obtained as follows: **(r-i)**  $s = 1$  means  $\phi(i_1) = i_1, 1 \leq i_1 \leq n$ . **(r-ii)** Denote  $i_0 = 0, i_{s+1} = +\infty$ . If  $1 < s, 1 \leq p \leq s \leq n$ , satisfy  $i_{k-1} + 1 = i_k$  when  $k \leq p$ , and  $i_p + 1 < i_{p+1}$ , then  $\phi(i_1, \dots, i_s) + 1 = \phi(1, \dots, p-1, i_p + 1, i_{p+1}, \dots, i_s)$ . Especially we have  $\phi(1, \dots, s) = 1, \phi(i_s - s + 1, i_s - s + 2, \dots, i_s) = g_{i_s, s}, i_s \leq n$ . **(r-iii)** If  $s > p \geq 1$  and  $(i_1, \dots, i_p, i_{p+1}, \dots, i_s) \in \mathbb{O}_{(n,s)}$  with  $i_p + 1 < i_{p+1}, i_k + 1 = i_{k+1}$  when  $s \geq k \geq p+1$ . Then  $\phi(i_1, \dots, i_s) = \phi(i_s - s + 1, \dots, i_s) - (\phi(i_s - s + 1, \dots, i_s - s + p) - \phi(i_1, \dots, i_p)) = g_{i_s, s} - g_{i_s - s + p, p} + \phi(i_1, \dots, i_p)$ . For example:  $\phi(2, 3, 5, 6) = \phi(3, 4, 5, 6) - (\phi(3, 4) - \phi(2, 3)) = 15 - (6 - 3) = 12$ .  $\phi(2, 4, 6) = \phi(4, 5, 6) - (\phi(4, 5) - \phi(2, 4))$ ,  $\phi(2, 4) = \phi(3, 4) - (\phi(3) - \phi(2)) = 6 - (3 - 2) = 5, \phi(4, 5) = 10, \phi(4, 5, 6) = 20$ . So  $\phi(2, 4, 6) = 15$ . Indeed, we have  $\phi(2, 3, 5, 6) = 12$ ; and  $\phi(2, 4, 6) = 15$ .

**Definition 2.2(1)** Also means for given  $s, s \leq n$ , and independent value  $\alpha_l^{[n,s]} = (i_1^{[l]}, \dots, i_s^{[l]}) \in \mathbb{O}_{n,s}$ , the function  $\phi_{n,s}^{-1}$  satisfies  $\alpha_l^{[n,s]} = \phi_{n,s}^{-1}(l)$  is an inverse function of  $\phi \triangleq \phi_{n,s}$ .

if  $n < m$ , then  $\alpha_l^{[n,s]} = \alpha_l^{[m,s]}, l=1, \dots, g_{ns}$ . So for given  $s < +\infty$ , we can assume an increase function  $\phi_s$  is defined on  $\bigcup_{s \leq n < +\infty} \mathbb{O}_{(n,s)} \triangleq \mathbb{O}_s$ . Thus, a table  $\triangleq$  Tab. 1 can be designed as follows: when  $t \geq 1$ , the  $(t, 1)$  entry is the inverse function value  $t = \phi_s^{-1}(\alpha_t^{[s]})$ , where  $\alpha_t^{[s]} = (i_1, \dots, i_s)$  is the  $t$ -th ‘smallest’ element of  $\mathbb{O}_s$ . The  $(t, s)$  entry equals  $(i_1, \dots, i_s)$ .

Tab.1 Function $\phi_s$ and $\phi_s^{-1}$					
$t \setminus s$	1	2	3	4	...
1	(1)	(1,2)	(1,2,3)	(1,2,3,4)	...
2	(2)	(1,3)	(1,2,4)	(1,2,3,5)	...
3	(3)	(2,3)	(1,3,4)	(1,2,4,5)	...
4	(4)	(1,4)	(2,3,4)	(1,3,4,5)	...
5	(5)	(2,4)	(1,2,5)	(2,3,4,5)	...
6	(6)	(3,4)	(1,3,5)	(1,2,3,6)	...
7	(7)	(1,5)	(2,3,5)	(1,2,4,6)	...
8	(8)	(2,5)	(1,4,5)	(1,3,4,6)	...
9	(9)	(3,5)	(2,4,5)	(2,3,4,6)	...
10	(10)	(4,5)	(3,4,5)	(1,2,5,6)	...
11	(11)	(1,6)	(1,2,6)	(1,3,5,6)	...
12	(12)	(2,6)	(1,3,6)	(2,3,5,6)	...
13	(13)	(3,6)	(2,3,6)	(1,4,5,6)	...
14	(14)	(4,6)	(1,4,6)	(2,4,5,6)	...
15	(15)	(5,6)	(2,4,6)	(3,4,5,6)	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

**Lemma 2.2** Let  $L \in \mathbb{C}_s^{n \times s}$ . (1) Denote  $G_t = L(1, \dots, i_t - 1) \in \mathbb{C}^{(i_t-1) \times s}$ , ( $G_1 = \emptyset$  when  $i_1 = 1$ ),  $G^{[t]} = (G_t^T, l_{i_{t+1}}^T, l_{i_{t+2}}^T, \dots, l_{i_s}^T)^T \in \mathbb{C}^{(s-t+i_t-1) \times s}$ ,  $t=1, 2, \dots, s$ . (e.g., assume  $n=8, s=3, i_1=2, i_2=4, i_3=7$ . So  $G_1 = (l_1)$ ,  $G^{[1]} = L(1, 4, 7)$ ,  $G_2 = L(1, 2, 3)$ ,  $G^{[2]} = L(1, 2, 3, 7)$ ,  $G_3 = L(1, 2, 3, 4, 5, 6)$ ,  $G^{[3]} = G_3$ .) Then  $L_{fns} = L(i_1, \dots, i_s) \Leftrightarrow \text{rank}(L(i_1, \dots, i_s)) = s$ , and  $\text{rank}(G^{(t)}) = s-1, t=1, \dots, s$ . Especially, if  $L(i_1, \dots, i_s) = I_s$ , then  $L_{fns} = L(i_1, \dots, i_s) \Leftrightarrow$  the  $t$ -th column of  $G^{(t)}$ , denoted as  $g^{(t)}$ , is a 0-vector,  $t=1, 2, \dots, s$ .

$$(2) (P(i_1, \dots, i_s)L)(i_1, \dots, i_s) = (l_1^T, l_2^T, \dots, l_s^T)^T = L(1, 2, \dots, s).$$

$$(P^T(i_1, \dots, i_s)L)(1, 2, \dots, s) = (l_{i_1}^T, l_{i_2}^T, \dots, l_{i_s}^T)^T = L(i_1, \dots, i_s).$$

**Proof** (1)  $\Rightarrow \text{rank}(L(i_1, \dots, i_s)) = s$  means  $L(i_1, \dots, i_{t-1}, i_{t+1}, \dots, i_s) \in \mathbb{C}_{s-1}^{(s-1) \times s}$  is a submatrix of  $G^{(t)}$ . So  $\text{rank}(G^{(t)}) \geq s-1, t=1, 2, \dots, s$ . According to Definition 2.2 (4), if  $\text{rank}(G^{(t)}) = s$ , then there exists a nonsingular submatrix  $L(j_1, \dots, j_s)$  of  $G^{(t)}$  satisfies  $(j_1, \dots, j_s) \prec (i_1, \dots, i_s)$ , So  $L(i_1, \dots, i_s) \neq L_{fns}$ . This contradiction implies  $\text{rank}(G^{(t)}) = s-1, t=1, 2, \dots, s$ .

$\Leftarrow$  we want to show if  $(j_1, \dots, j_s) \prec (i_1, \dots, i_s)$ , then  $L(j_1, \dots, j_s)$  is singular. For example, assume  $j_s = i_s, j_{s-1} < i_{s-1}$ . Then  $L(j_1, \dots, j_s)$  is a  $s \times s$  submatrix of  $G^{(s-1)}$ . Hence  $\text{rank}(L(j_1, \dots, j_s)) = s-1$ . So  $L_{fns} = L(i_1, \dots, i_s)$ .

When  $L(i_1, \dots, i_s) = I_s$ , we can show  $\text{rank}(G^{(t)}) = s-1 \Leftrightarrow g^{(t)} = 0, 1 \leq t \leq s$ .

Notice that when  $L(i_1, \dots, i_s) = I_s$ , then  $g^{(t)} = 0 \Leftrightarrow g_t = 0$  or  $\emptyset$ , where  $g_t$  is the  $t$ -th column of  $G_t, t=1, \dots, s$ .

(2) An example is used to explain our conclusion. The  $j$ th column of  $I$  is denoted as  $e_j$ .

If  $n = 4, s = 2$ , we obtain  $P(2, 4) = I_{1,2}I_{2,4} = (e_2, e_4, e_3, e_1)$ . Then

$$(P(2, 4)L)(2, 4) = L(4, 1, 3, 2)(2, 4) = L(1, 2) \cdot (P(2, 4)^T L)(1, 2) = L(2, 4).$$

**Theorem 2.2** Let  $1 \leq s \leq n$ ,  $L, R \in \mathbb{C}_s^{n \times s}, R^*L = I_s, \beta = LR^*$ . Then

- (1)  $\beta \in \mathbb{B}_1 \Leftrightarrow L(1, \dots, s) \in \mathbb{C}_s^{s \times s}$ .
- (2)  $\beta \in \mathbb{B}(i_1, \dots, i_s) = \mathbb{B}_l \Leftrightarrow L(i_1, \dots, i_s) \in \mathbb{C}_s^{s \times s}$ .
- (3)  $\mathbb{B}(i_1, \dots, i_s) \subset I\{2\}_s$ , and  $\mathbb{B}_1 = \mathbb{B}(1, \dots, s)$  is a proper subset of  $I\{2\}_s$ .
- (4)  $I\{2\}_s = \bigcup_{1 \leq l \leq g_{ns}} \mathbb{B}_l$ . (2.7(ii))

$$I\{2\}_s = \bigoplus_{1 \leq l \leq g_{ns}} \mathbb{C}_l. \quad (2.7(ii))$$

**Proof** (1) Theorem 2.1(2) means (1) is valid.

(2) Lemma 2.2 and (1) tell us that  $L(i_1, \dots, i_s) = (P(i_1, \dots, i_s)^T L)(1, \dots, s) \in \mathbb{C}_s^{s \times s} \Leftrightarrow P(i_1, \dots, i_s)^T \beta P(i_1, \dots, i_s) \in \mathbb{B}_1 \Leftrightarrow \beta \in \mathbb{B}(i_1, \dots, i_s)$ .

(3) Obviously,  $\mathbb{B}_1 \subset I\{2\}_s$ . Conversely, take  $R=L=\begin{pmatrix} 0 \\ I_s \end{pmatrix}$ , set  $\beta = LR^*$ . Then  $\beta = \beta^2 \in I\{2\}_s$  and the first row of  $L(1, \dots, s)$  is 0. Hence  $\beta \notin \mathbb{B}_1 = \mathbb{B}(1, 2, \dots, s)$ . So  $\mathbb{B}_1$  is a proper subset of  $I\{2\}_s$ .

(4) For each  $\beta = LR^* \in I\{2\}_s$  with  $R^*L = I_s$ ,  $L, R \in \mathbb{C}_s^{n \times s}$ ,  $L(1, \dots, s) \in \mathbb{C}_t^{s \times s}$ . When  $t = s$ , (1) means  $\beta \in \mathbb{B}_1$ . When  $t < s$ , one can find a nonsingular submatrix  $L(i_1, \dots, i_s)$ . In this case (2) means  $\beta \in \mathbb{B}(i_1, \dots, i_s)$ . Thus (2.7(i)) holds. (2.7(ii)) is equivalent to (2.7(i)).

**Remark 2.2** (1) (2.7(ii)) has  $2(ns - s^2)$  arbitrary parameters. So we can show (2.2) has at least  $s^2$  redundant arbitrary parameters.

(2) Since for any permutation matrix  $P$  and matrix  $X$ ,  $\|PXP^T\|_2 = \|X\|_2$ , so for each  $t, 1 < t \leq g_{ns}$ ,  $\mathbb{B}_t$  is isometry with  $\mathbb{B}_1$ . Although (2.7(i)) has no redundant arbitrary parameters, but if  $n > s \geq 1$ , from Theorem 2.2, we can show  $\mathbb{B}_1 \cap \mathbb{B}_l \neq \emptyset$  when  $l > 1$ . So a shortcoming of (2.7(i)) is it may leads to a finite times repeat computation work when an element of  $I\{2\}_s$  is computed. For example,  $\beta = (1, 1, 1)^T (1/\sqrt{3}(1, 1, 1)) \in \mathbb{B}_1 \cap \mathbb{B}_2 \cap \mathbb{B}_3$ . So in (2.7(i)),  $\beta$  may be computed three times. On the other hand, (2.7(ii)) can be used to design an algorithm to avoid any repeat computation work for each element of  $I\{2\}_s$ .

**Lemma 2.3** Let  $\alpha = \begin{pmatrix} I_s & I_s - \eta^* \xi \\ \eta & \xi \end{pmatrix} = (Y, Z) \in \mathbb{A}_1$ ,  $Y_l = P_l Y = P(i_1^{[l]}, \dots, i_s^{[l]})$

$Y, 1 \leq l \leq g_{ns}$ ,  $\beta_l(\alpha) = f(\alpha) = f(Y, Z) = YZ^* \in \mathbb{B}_1$ .  $\beta_l(\alpha) = P_l f(\alpha) P_l^T$ . Then (1)  $f(P_l(\alpha)) = P_l f(\alpha) P_l^T \in \mathbb{B}_l, l = 1, \dots, g_{ns}$ . And generally,  $\mathbb{B}_l = f(\mathbb{A}_l) = P_l f(\mathbb{A}_1) P_l^T, f(\mathbb{D}_l) = P_l f(\mathbb{D}_1) P_l^T = \mathbb{C}_l = f(P_l \mathbb{D}_1), l = 1, \dots, g_{ns}$ . And  $f(\mathbb{A}_l), f(\mathbb{D}_l), l = 1, \dots, g_{ns}$  are all efficient.

(2)  $\beta_l(\alpha) \in \mathbb{C}_l \Leftrightarrow Y_l \in \mathbb{D}_l \Leftrightarrow (Y_l)_{fns} = I_s$ .

**Proof** The proof of (1) is omitted.

(2)  $f(\mathbb{D}_l) = \mathbb{C}_l$  is efficient means ' $\beta_l(\alpha) \in \mathbb{C}_l \Leftrightarrow Y_l \in \mathbb{D}_l$ '.

$\Rightarrow$  From Theorem 2.2(2),  $\beta_l \in \mathbb{C}_l$  means  $Y_l(i_1^{[l]}, \dots, i_s^{[l]})$  is nonsingular and for any  $(j_1^{[k]}, \dots, j_s^{[k]}) \prec (i_1^{[l]}, \dots, i_s^{[l]})$ ,  $\beta_l(\alpha) \notin \mathbb{B}_k$  and  $Y_l(j_1^{[k]}, \dots, j_s^{[k]})$  is singular, Hence  $(Y_l)_{fns} = Y_l(i_1^{[l]}, \dots, i_s^{[l]}) = I_s$ .

$\Leftarrow (Y_l)_{fns} = Y_l(i_1^{[l]}, \dots, i_s^{[l]}) = I_s$  means  $Y_l(i_1^{[l]}, \dots, i_s^{[l]})$  is nonsingular. Theorem 2.2(2) means  $\beta_l \in \mathbb{B}_l$ . For each  $(j_1^{[k]}, \dots, j_s^{[k]}) \prec (i_1^{[l]}, \dots, i_s^{[l]})$ , we have  $Y_l(j_1^{[k]}, \dots, j_s^{[k]})$  is singular. So  $\beta_l(\alpha) \notin \mathbb{B}_k$ . Thus  $\beta_l(\alpha) \in \mathbb{C}_l$ .

**Algorithm 2.1** Given  $n, s$  satisfy  $1 \leq s < n$ , we can compute the elements of  $I\{2\}_s$  nonrepeatedly.

**Step 1** If  $1 \leq l \leq g_{ns}$ , compute  $P_l$ . As the foremost computed point, take an element  $\alpha = (Y, Z) = \begin{pmatrix} I_s & I_s - \eta^* \xi \\ \eta & \xi \end{pmatrix} \in \mathbb{A}_1$ .

**Step 2**  $\beta_1(\alpha) = YZ^*$  is computed and stored.

**Step 3** If  $2 \leq l \leq g_{ns}$ , compute  $Y_l = P_l Y$ . Assume  $P_l = P(i_1, \dots, i_s)$ . From Lemma 2.2(1), if the  $t$ -th column of  $G^{(t)}$  of  $Y_l$  equals 0,  $t = 1, \dots, s$ , then compute  $\beta_l(\alpha) = Y_l Z^* P_l^T$ .

**Step 4** If the stop condition is not satisfied, take next element of  $\mathbb{A}_1$ . Go to Step 2. End.

**Theorem 2.3** If Algorithm 2.1 is used, then the repeated computation may be avoided completely.

**Proof** It is obviously, Step 2 of Algorithm 2.1 is an efficient characterization from  $\mathbb{A}_1$  onto  $\mathbb{C}_1 = \mathbb{B}_1$ .

In Step 3, when  $l = 2$ ,  $P_2 \alpha = (P_2 Y, P_2 Z) = (Y_2, Z_2)$  is considered.

According to Lemma 2.2(1), if for  $Y_2$ , we have  $g^{(t)} = 0, t = 1, \dots, s$ , then  $Y_2 Z^* P_2^T \in \mathbb{C}_2$  is computed and stored, else  $P_2 \alpha$  is not used. It means each element of  $\mathbb{B}_2$  is served at most a time, only when this observed element is an element of  $\mathbb{C}_2$ , it then is computed. So each element of  $\mathbb{C}_2$  is computed at most one time. This fact means Step 3 of Algorithm 2.1 is an efficient characterization from  $\mathbb{D}_2$  onto  $\mathbb{C}_2$ . When  $2 < l \leq s$ , one can also show that each element of  $\mathbb{C}_l$  is computed at most one time. In a word, by Lemma 2.2, 2.3, Algorithm 2.1 is an 'efficient' characterization from

$$\mathbb{A} = \bigcup_{1 \leq l \leq g_{ns}}^{\oplus} \mathbb{D}_l \text{ onto } I\{2\}_s = \bigcup_{1 \leq l \leq g_{ns}}^{\oplus} \mathbb{C}_l.$$

The next theorem tells us in Algorithm 2.1, the number of the judgement times for  $g^{(t)} = 0$  may be decreased greatly when a large number of elements of  $I\{2\}_s$  are computed.

**Theorem 2.4** Let  $\alpha = (Y, Z), Y, Z \in \mathbb{C}_s^{n \times s}, Y(1, \dots, s) = I_s$  is given,  $\beta_1(\alpha) = YZ^*, Z^* Y = I_s, Y = (y_{i,j}), y_{s+1,s} \neq 0$ . Then in Algorithm 2.1, only  $\beta_1(\alpha)$  is computed.

**Proof** Denote  $Y_l = P_l Y, l = 1, \dots, g_{ns}$ . For given  $l, l > 1$ , if  $\phi(i_1, \dots, i_s) = l, (P_l Y)(i_1, \dots, i_s) = Y(1, \dots, s) = I_s$ , from Lema 2.2,  $y_{s+1,s} \neq 0$  means the  $s$ -th column  $g^{(s)}$  of  $G^{(s)}$  for  $Y_l$  is not a 0-vector, and  $Y_l(i_1, \dots, i_s) \neq (Y_l)_{fns}$ . So in Algorithm 2.1  $\beta_l(\alpha) \notin \mathbb{C}_l$  and it is not computed.

Notice that generally, the probability of  $y_{s+1,s} \neq 0$  is 1. So this theorem is important.

**Example 2.1** Assume  $n = 4, s = 2$ . Then  $g_{ns} = 6, \phi(1, 2) = 1, \phi(1, 3) = 2, \phi(2, 3) = 3, \phi(1, 4) = 4, \phi(2, 4) = 5, \phi(3, 4) = 6$ .  $P_1 = I_4, P_2 = I_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (e_1, e_3, e_2, e_4),$   
 $P_3 = I_{1,2} I_{2,3} = (e_2, e_3, e_1, e_4), P_4 = I_{2,4} = (e_1, e_4, e_3, e_2), P_5 = I_{1,2} I_{2,4} = (e_2, e_4, e_3, e_1),$   
 $P_6 = P(3, 4) = I_{1,3} I_{2,4} = (e_3, e_4, e_1, e_2)$ . Denote  $\alpha(k)(s+1, \dots, n) = (\eta(k), \xi(k)), \alpha(k) =$

$$(Y(k), Z(k)) = \begin{pmatrix} I & I - \eta(k)^* \xi(k) \\ \eta(k) & \xi(k) \end{pmatrix}. \text{ Take } \alpha(1)(3, 4)=0, \alpha(2)(3, 4)=\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha(3)(3, 4)=\begin{pmatrix} 3 & \sqrt{-1} & \sqrt{-1} & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \alpha(4)(3, 4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & -4 & -6 \end{pmatrix}, \alpha(5)(3, 4) =$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$k = 1$  Lemma 2.2(1) implies  $Y_l(1)_{fns} = (P_l Y(1))_{fns} = I_2$  and  $P_l Y(1)$  has only one  $2 \times 2$  nonsingular submatrix, so  $\beta_l(\alpha(1)) \in \mathbb{C}_l, 1 \leq l \leq 6$ . Hence in Algorithm 2.1, they are all computed and stored.

$$k = 2 \text{ We have } \beta_1(\alpha(2)) = \begin{pmatrix} I_2 \\ \eta(2) \end{pmatrix} \begin{pmatrix} -1 & -4 & 2 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix} = Y(2)Z(2)^* \in \mathbb{C}_1, \beta_2(\alpha(2)) = P_2 Y(2)Z^*(2)P_2^T \in \mathbb{C}_1, \beta_3(\alpha(2)) = P_3 Y(2)Z^*(2)P_3^T \in \mathbb{C}_1, \beta_4(\alpha(2)) = P_4 Y(2)Z^*(2)P_4^T \in \mathbb{C}_2, \beta_5(\alpha(2)) =$$

$$P_5 Y(2)Z^*(2)P_5^T \in \mathbb{C}_3, \beta_6(\alpha(2)) = \begin{pmatrix} I \\ \eta(2) \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & -4 \\ 1 & 0 & -1 & -1 \end{pmatrix} = P_6 Y(2)Z^*(2)P_6^T \in \mathbb{C}_2.$$

Except  $\beta_1(\alpha(2)) \in \mathbb{C}_1$ , we have  $\beta_t(\alpha(2)) \notin \mathbb{C}_t, 1 < t \leq 6$ . So only  $\beta_1(\alpha(2))$  is computed in Algorithm 2.1. This situation is coincide with Theorem 2.4, since the  $(3, 2)$  component of  $Y(2)$  is  $2 \neq 0$ . A fact is

$$\beta_5(\alpha(2)) = (e_2 - 1/2e_4, e_3 + 1/2e_4) \begin{pmatrix} 0 & -1 & 2 & -4 \\ 0 & -3 & 4 & -6 \end{pmatrix} = \beta_3(\alpha(4)) = P_3 Y(4)Z^*(4)P_3^T.$$

$k = 3$  Since all six  $2 \times 2$  submatrices of  $Y(3)$  are nonsingular, so  $\beta_l(\alpha(3)) \in \mathbb{B}_1, 1 \leq l \leq 6$ . In this case only the  $\beta_1(\alpha(3))$  is computed.

$$k = 5 \text{ We have } Y(5) = (e_1 + e_3, e_2 + e_3 + e_4).$$

(5-i) When  $l = 2$ , by Lemma 2.2, we have  $Y_2(5) = P_2 Y(5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}^T, i_1 = 1, i_2 = 3, G_2^{(1)} = (0, 1), g_2^{(1)} = 0, G_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, g_2^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0$ . So  $(Y_2(5))_{fns} \neq (Y_2(5))(1, 3)$  and  $\beta_2(\alpha(5)) \notin \mathbb{C}_2$ . In fact  $\beta_2(\alpha(5)) \in \mathbb{C}_1$ .

(5-ii)  $Y_3(5) = I_{1,2} I_{2,3} Y(5) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}^T$ , we obtain  $i_1 = 2, i_2 = 3, G_3^{(1)} = (0, 1), g_3^{(1)} = 0, G_3^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, g_3^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . So  $\mathbb{C}_1 \ni \beta_3(\alpha(5)) \notin \mathbb{C}_3$ . We can further show  $\beta_4(\alpha(5)) \in \mathbb{C}_1, \beta_5(\alpha(5)) \in \mathbb{C}_1$  and  $\beta_6(\alpha(5)) \in \mathbb{C}_1$ . So only  $\beta_1(\alpha(5))$  is computed. Obviously, this result is coincide with **Theorem 2.4**.

### 3 Efficient characterization for $M\{2\}$

**Lemma 3.1** Let  $L_1, R_1 \in \mathbb{C}_s^{r \times s}, R_1^* L_1 = I_s, L = (L_1, L_2), R = (R_1, R_2) \in \mathbb{C}^{r \times r}, \text{rank}(L_2) = \text{rank}(R_2) = r - s, R_1^* L_2 = 0, L_1^* R_2 = 0$ . Then  $L, R \in \mathbb{C}_r^{r \times r}$ .

**Proof** Lemma 3.1 is valid when  $s = r$ . When  $0 < s < r$ , the  $i$ -th column of  $L$  is denoted as  $l_i, 1 \leq i \leq r$ . If  $\sum_{l=1}^s c_l l_l + \sum_{j=s+1}^r c_j l_j = 0$ . Using  $R_1^* L_1 = I_s$  and  $R_1^* L_2 = 0$ , we obtain



$c_i = 0, 1 \leq i \leq s$ , and  $\sum_{j=s+1}^r c_j l_j = 0$ . Since  $L_2 \in \mathbb{C}_{r-s}^{r \times (r-s)}$ , we obtain  $c_j = 0, s+1 \leq j \leq r$ . Thus  $L$  is nonsingular. Analogically, we can also show  $R$  is nonsingular.

**Lemma 3.2** Let  $0 < s < r, L_1 \in \mathbb{C}_s^{r \times s}$ . The unitary matrix  $Q = (Q_1, Q_2)$  satisfies  $Q_1 \in \mathbb{C}^{r \times s}, Q^* L_1 = \begin{pmatrix} L_{11} \\ 0 \end{pmatrix}^{[3,4]}$ , then  $L_1^* Q_2 = 0$ .

**Proof** We have  $Q^*(L_1, Q_2) = \begin{pmatrix} L_{11} & 0 \\ 0 & I_{r-s} \end{pmatrix}$ . Hence  $L_1^* Q_2 = 0$ .

If  $M = 0$ , then  $M\{2\} = \{0\}$ . Next we always assume  $M \neq 0$ .

**Theorem 3.1** Let  $r > 0, \mathbb{C}_r^{m \times n} \ni M = L \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} R^*$ , where  $L = (L_1, L_2) \in \mathbb{C}_m^{m \times m}$ ,  $L_1 \in \mathbb{C}_r^{m \times r}, R = (R_1, R_2) \in \mathbb{C}_n^{n \times n}, R_1 \in \mathbb{C}_r^{n \times r}, 0 < s \leq r$ . Then

(1)  $M\{2\} = \{0\} \cup \bigcup_{s=1, \dots, r} M\{2\}_s =$

$$\left\{ R^{-*} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} L^{-1} : I_r\{2\} \ni X_{11}^2 = X_{11} \in \mathbb{C}^{r \times r}, \right. \\ \left. X_{12} = X_{11} X_{12}, X_{21} = X_{21} X_{11}, X_{22} = X_{21} X_{12}. \right\} \quad (3.1)$$

with  $\text{rank}(X_{11}) = \text{rank} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{21} X_{12} \end{pmatrix}$ .

(2)

$$M\{2\}_s = \left\{ R^{-*} \begin{pmatrix} Y_1 Z_1^* & Y_1 H_1 \\ G_1 Z_1^* & G_1 H_1 \end{pmatrix} L^{-1} : Y_1, Z_1 \in \mathbb{C}_s^{r \times s}, Z_1^* Y_1 = I_s \right\} \quad (3.2)$$

with  $H_1 \in \mathbb{C}^{s \times (n-r)}, G_1 \in \mathbb{C}^{(m-r) \times s}, Y_1 = P_l(I_s, \eta^T)^T, Z_1^* = (I_s - \xi^* \eta, \xi^*) P_l^T, 1 \leq l \leq g_{rs}$ .

**Proof** (1) Denote

$$X = R^{-*} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} L^{-1} \triangleq R^{-*} S L^{-1}, X_{11} \in \mathbb{C}^{r \times r}. \quad (3.3)$$

Then  $X \in M\{2\} \Leftrightarrow X = X M X \Leftrightarrow \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{11}^2 & X_{11} X_{12} \\ X_{21} X_{11} & X_{21} X_{12} \end{pmatrix}$ . So (3.1) holds.

From  $\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}^2 & X_{11} X_{12} \\ X_{21} X_{11} & X_{21} X_{12} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{21} X_{12} \end{pmatrix}$ , we obtain

$$\text{rank}(X_{11}) \leq \text{rank}(S) \leq \min \left\{ \text{rank} \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}, \text{rank} \begin{pmatrix} X_{11}^T \\ X_{12}^T \end{pmatrix}^T \right\} = \min \left\{ \text{rank} \begin{pmatrix} X_{11} \\ X_{21} X_{11} \end{pmatrix}, \text{rank} \right. \\ \left. ((X_{11}, X_{11} X_{12})) \right\} \leq \text{rank}(X_{11}).$$

(2) From (1) and §2, we have  $X_{11} = Y_1 Z_1^*$  with  $Y_1, Z_1 \in \mathbb{C}_s^{r \times s}, Z_1^* Y_1 = I_s$ , and for given  $l, 1 \leq l \leq g_{rs}, Y_1 = P_l \begin{pmatrix} I_s \\ \eta \end{pmatrix}$ . (3.1) means  $(I - Y_1 Z_1^*) X_{12} = 0$  and  $X_{21} (I - Y_1 Z_1^*) = 0$ . Using

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