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Series expansions for three familiar irrational numbers with free parameters

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Abstract: $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are important irrational numbers, but the series expansions for them are rare in the literature. In order to fill the gap, we established numerous series expansions for these numbers with free parameters in terms of the hypergeometric method.

Key words: series expansions for $\sqrt{2}$; series expansions for $\sqrt{3}$; series expansions for $\sqrt{5}$

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三个常用无理数的带自由参数的级数展开式

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摘要: 虽然$\sqrt{2}$, $\sqrt{3}$ 和$\sqrt{5}$ 都是重要的无理数, 但它们的级数展开式非常少. 为此, 本文用超几何方法建立这三个数学常数的级数展开式, 相应的结论不但数量多而且带有自由参数.

关键词: $\sqrt{2}$ 的级数展开式; $\sqrt{3}$ 的级数展开式; $\sqrt{5}$ 的级数展开式

0 Introduction

For a complex number x and an integer n, define the shifted factorial by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1}(x+k), & \text{when} \quad n>0; \\ 1, & \text{when} \quad n=0; \\ \dfrac{(-1)^n}{\prod_{k=1}^{-n}(k-x)}, & \text{when} \quad n<0. \end{cases}$$

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Recall that the function $\Gamma(x)$ can be given by Euler's integral:

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \quad \text{with} \quad \mathrm{Re}(x)>0.$$

Then we have the following two relations:

$$\Gamma(x+n)=\Gamma(x)(x)_n, \quad \Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)},$$

which will be used frequently without indication in this paper.

Following Bailey[1], define the hypergeometric series by

$${}_{r+1}F_s=\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\,\Big|\, z\right]=\sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k}z^k.$$

Then Dougall's ${}_5F_4$-series identity (cf. [1, p. 27]) can be stated as 工式序号如:19a,19b...

$$\begin{aligned}{}_5F_4&\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d \\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\,\Big|\, 1\right]\\ &=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \qquad (0.1\mathrm{a})\end{aligned}$$

provided that $Re(1+a-b-c-d)>0$.

Recently, Chu[2] and Liu[3,4] deduced many surprising π-formulas from (0.1) and the limiting cases of it. Inspired by these works, we shall explore systematically series expansions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ with free parameters by using (0.1).

1 Series expansions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ with three free parameters

Theorem 1 *For $x,y\in\mathbb{C}$ and $m,n,p\in\mathbb{Z}$ with* $\mathrm{Re}(1-y+m+n-p)>0$, *there holds the summation formula on sine function with five free parameters*:

$$\frac{\sin(\pi x)}{\sin(\pi y)}=\frac{(x)_m(1-x)_n}{(y-x)_{p-m}(y+x)_{p-n-1}(1-y)_{m+n-p}}\sum_{k=0}^{\infty}\frac{(y-x)_{k+p-m}(y+x)_{k+p-n-1}}{k!(y)_{k+p}}.$$

Proof Performing the replacement $d\to 1+a-d$ for (0.1) and then letting $a\to\infty$, we obtain Gauss' ${}_2F_1$-series identity (cf. [1, p.2]):

$${}_2F_1\left[\begin{matrix} b, & c \\ & d \end{matrix}\,\Big|\, 1\right]=\frac{\Gamma(d)\Gamma(d-b-c)}{\Gamma(d-b)\Gamma(d-c)} \quad \text{where} \quad \mathrm{Re}(d-b-c)>0.$$

Employing the substitutions $b\to y-x+p-m$, $c\to y+x+p-n-1$, $d\to y+p$, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(y-x+p-m)_k(y+x+p-n-1)_k}{k!(y+p)_k}&=\frac{\Gamma(y+p)\Gamma(1-y+m+n-p)}{\Gamma(x+m)\Gamma(1-x+n)}\\ &=\frac{(y)_p(1-y)_{m+n-p}}{(x)_m(1-x)_n}\frac{\Gamma(y)\Gamma(1-y)}{\Gamma(x)\Gamma(1-x)}=\frac{(y)_p(1-y)_{m+n-p}}{(x)_m(1-x)_n}\frac{\sin(\pi x)}{\sin(\pi y)}.\end{aligned}$$

Multiplying both sides by $\frac{(x)_m(1-x)_n}{(y)_p(1-y)_{m+n-p}}$, we can proceed as follows.

$$\begin{aligned}\frac{\sin(\pi x)}{\sin(\pi y)} &= \frac{(x)_m(1-x)_n}{(y)_p(1-y)_{m+n-p}}\sum_{k=0}^{\infty}\frac{(y-x+p-m)_k(y+x+p-n-1)_k}{k!(y+p)_k}\\ &= \frac{(x)_m(1-x)_n}{(y-x)_{p-m}(y+x)_{p-n-1}(1-y)_{m+n-p}}\sum_{k=0}^{\infty}\frac{(y-x)_{k+p-m}(y+x)_{k+p-n-1}}{k!(y)_{k+p}}.\end{aligned}$$

This completes the proof of Theorem 1.

1.1 Series expansions for $\sqrt{2}$ with three free parameters

Corollary 1 ($x=1/2$ and $y=3/4$ in Theorem 1)

$$\sqrt{2} = \frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(\frac{1}{4})_{p-m}(\frac{5}{4})_{p-n-1}(\frac{1}{4})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{4})_{k+p-m}(\frac{5}{4})_{k+p-n-1}}{k!(\frac{3}{4})_{k+p}}.$$

Corollary 2 ($x=1/4$ and $y=5/6$ in Theorem 1)

$$\sqrt{2} = \frac{(\frac{1}{4})_m(\frac{3}{4})_n}{(\frac{7}{12})_{p-m}(\frac{13}{12})_{p-n-1}(\frac{1}{6})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{7}{12})_{k+p-m}(\frac{13}{12})_{k+p-n-1}}{k!(\frac{5}{6})_{k+p}}.$$

Corollary 3 ($x=1/4$ and $y=1/2$ in Theorem 1)

$$\frac{\sqrt{2}}{2} = \frac{(\frac{1}{4})_m(\frac{3}{4})_n}{(\frac{1}{4})_{p-m}(\frac{3}{4})_{p-n-1}(\frac{1}{2})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{4})_{k+p-m}(\frac{3}{4})_{k+p-n-1}}{k!(\frac{1}{2})_{k+p}}.$$

Corollary 4 ($x=1/6$ and $y=1/4$ in Theorem 1)

$$\frac{\sqrt{2}}{2} = \frac{(\frac{1}{6})_m(\frac{5}{6})_n}{(\frac{1}{12})_{p-m}(\frac{5}{12})_{p-n-1}(\frac{3}{4})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{12})_{k+p-m}(\frac{5}{12})_{k+p-n-1}}{k!(\frac{1}{4})_{k+p}}.$$

1.2 Series expansions for $\sqrt{3}$ with three free parameters

Corollary 5 ($x=1/3$ and $y=5/6$ in Theorem 1)

$$\sqrt{3} = \frac{(\frac{1}{3})_m(\frac{2}{3})_n}{(\frac{1}{2})_{p-m}(\frac{7}{6})_{p-n-1}(\frac{1}{6})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+p-m}(\frac{7}{6})_{k+p-n-1}}{k!(\frac{5}{6})_{k+p}}.$$

Corollary 6 ($x=1/3$ and $y=1/2$ in Theorem 1)

$$\frac{\sqrt{3}}{2} = \frac{(\frac{1}{3})_m(\frac{2}{3})_n}{(\frac{1}{6})_{p-m}(\frac{5}{6})_{p-n-1}(\frac{1}{2})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+p-m}(\frac{5}{6})_{k+p-n-1}}{k!(\frac{1}{2})_{k+p}}.$$

Corollary 7 ($x=1/6$ and $y=1/3$ in Theorem 1)

$$\frac{\sqrt{3}}{3} = \frac{(\frac{1}{6})_m(\frac{5}{6})_n}{(\frac{1}{6})_{p-m}(\frac{1}{2})_{p-n-1}(\frac{2}{3})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+p-m}(\frac{1}{2})_{k+p-n-1}}{k!(\frac{1}{3})_{k+p}}.$$

Corollary 8 ($x=1/2$ and $y=2/3$ in Theorem 1)

$$\frac{2\sqrt{3}}{3}=\frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(\frac{1}{6})_{p-m}(\frac{7}{6})_{p-n-1}(\frac{1}{3})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+p-m}(\frac{7}{6})_{k+p-n-1}}{k!(\frac{2}{3})_{k+p}}.$$

Corollary 9 ($x=1/4$ and $y=11/12$ in Theorem 1)

$$\sqrt{3}+1=\frac{(\frac{1}{4})_m(\frac{3}{4})_n}{(\frac{2}{3})_{p-m}(\frac{7}{6})_{p-n-1}(\frac{1}{12})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{2}{3})_{k+p-m}(\frac{7}{6})_{k+p-n-1}}{k!(\frac{11}{12})_{k+p}}.$$

Corollary 10 ($x=1/4$ and $y=5/12$ in Theorem 1)

$$\sqrt{3}-1=\frac{(\frac{1}{4})_m(\frac{3}{4})_n}{(\frac{1}{6})_{p-m}(\frac{2}{3})_{p-n-1}(\frac{7}{12})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+p-m}(\frac{2}{3})_{k+p-n-1}}{k!(\frac{5}{12})_{k+p}}.$$

Corollary 11 ($x=5/12$ and $y=3/4$ in Theorem 1)

$$\frac{\sqrt{3}+1}{2}=\frac{(\frac{5}{12})_m(\frac{7}{12})_n}{(\frac{1}{3})_{p-m}(\frac{7}{6})_{p-n-1}(\frac{1}{4})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{3})_{k+p-m}(\frac{7}{6})_{k+p-n-1}}{k!(\frac{3}{4})_{k+p}}.$$

Corollary 12 ($x=1/12$ and $y=1/4$ in Theorem 1)

$$\frac{\sqrt{3}-1}{2}=\frac{(\frac{1}{12})_m(\frac{11}{12})_n}{(\frac{1}{6})_{p-m}(\frac{1}{3})_{p-n-1}(\frac{3}{4})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+p-m}(\frac{1}{3})_{k+p-n-1}}{k!(\frac{1}{4})_{k+p}}.$$

Corollary 13 ($x=5/12$ and $y=11/12$ in Theorem 1)

$$2+\sqrt{3}=\frac{(\frac{5}{12})_m(\frac{7}{12})_n}{(\frac{1}{2})_{p-m}(\frac{4}{3})_{p-n-1}(\frac{1}{12})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+p-m}(\frac{4}{3})_{k+p-n-1}}{k!(\frac{11}{12})_{k+p}}.$$

Corollary 14 ($x=1/12$ and $y=5/12$ in Theorem 1)

$$2-\sqrt{3}=\frac{(\frac{1}{12})_m(\frac{11}{12})_n}{(\frac{1}{3})_{p-m}(\frac{1}{2})_{p-n-1}(\frac{7}{12})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{3})_{k+p-m}(\frac{1}{2})_{k+p-n-1}}{k!(\frac{5}{12})_{k+p}}.$$

1.3 Series expansions for $\sqrt{5}$ with three free parameters

Corollary 15 ($x=1/2$ and $y=9/10$ in Theorem 1)

$$\sqrt{5}+1=\frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(\frac{2}{5})_{p-m}(\frac{7}{5})_{p-n-1}(\frac{1}{10})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{2}{5})_{k+p-m}(\frac{7}{5})_{k+p-n-1}}{k!(\frac{9}{10})_{k+p}}.$$

Corollary 16 ($x=1/2$ and $y=7/10$ in Theorem 1)

$$\sqrt{5}-1=\frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(\frac{1}{5})_{p-m}(\frac{6}{5})_{p-n-1}(\frac{3}{10})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{1}{5})_{k+p-m}(\frac{6}{5})_{k+p-n-1}}{k!(\frac{7}{10})_{k+p}}.$$

Corollary 17 ($x=3/10$ and $y=5/6$ in Theorem 1)

$$\frac{\sqrt{5}+1}{2}=\frac{(\frac{3}{10})_m(\frac{7}{10})_n}{(\frac{8}{15})_{p-m}(\frac{17}{15})_{p-n-1}(\frac{1}{6})_{m+n-p}}\sum_{k=0}^{\infty}\frac{(\frac{8}{15})_{k+p-m}(\frac{17}{15})_{k+p-n-1}}{k!(\frac{5}{6})_{k+p}}.$$

Corollary 18 ($x = 1/10$ and $y = 1/6$ in Theorem 1)

$$\frac{\sqrt{5}-1}{2} = \frac{(\frac{1}{10})_m(\frac{9}{10})_n}{(\frac{1}{15})_{p-m}(\frac{4}{15})_{p-n-1}(\frac{5}{6})_{m+n-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{15})_{k+p-m}(\frac{4}{15})_{k+p-n-1}}{k!(\frac{1}{6})_{k+p}}.$$

Corollary 19 ($x = 3/10$ and $y = 1/2$ in Theorem 1)

$$\frac{\sqrt{5}+1}{4} = \frac{(\frac{3}{10})_m(\frac{7}{10})_n}{(\frac{1}{5})_{p-m}(\frac{4}{5})_{p-n-1}(\frac{1}{2})_{m+n-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{5})_{k+p-m}(\frac{4}{5})_{k+p-n-1}}{k!(\frac{1}{2})_{k+p}}.$$

Corollary 20 ($x = 1/10$ and $y = 1/2$ in Theorem 1)

$$\frac{\sqrt{5}-1}{4} = \frac{(\frac{1}{10})_m(\frac{9}{10})_n}{(\frac{2}{5})_{p-m}(\frac{3}{5})_{p-n-1}(\frac{1}{2})_{m+n-p}} \sum_{k=0}^{\infty} \frac{(\frac{2}{5})_{k+p-m}(\frac{3}{5})_{k+p-n-1}}{k!(\frac{1}{2})_{k+p}}.$$

Corollary 21 ($x = 3/10$ and $y = 1/10$ in Theorem 1)

$$\frac{3+\sqrt{5}}{2} = \frac{(\frac{3}{10})_m(\frac{7}{10})_n}{(-\frac{1}{5})_{p-m}(\frac{2}{5})_{p-n-1}(\frac{9}{10})_{m+n-p}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{5})_{k+p-m}(\frac{2}{5})_{k+p-n-1}}{k!(\frac{1}{10})_{k+p}}.$$

Corollary 22 ($x = 1/10$ and $y = 3/10$ in Theorem 1)

$$\frac{3-\sqrt{5}}{2} = \frac{(\frac{1}{10})_m(\frac{9}{10})_n}{(\frac{1}{5})_{p-m}(\frac{2}{5})_{p-n-1}(\frac{7}{10})_{m+n-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{5})_{k+p-m}(\frac{2}{5})_{k+p-n-1}}{k!(\frac{3}{10})_{k+p}}.$$

2 Further series expansions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ with three free parameters

Theorem 2 *For $x, y \in \mathbb{C}$ and $m, n, p \in \mathbb{Z}$ with* $\mathrm{Re}(2 + 2n + 2p - 3m - 3x) > 0$, *there holds the summation formula on sine function with five free parameters*:

$$\frac{\sin(\pi x)}{\sin(\pi y)} = \frac{(1-x)_{n+p-m}}{(x-y)_{m-n}(x+y)_{m-p}} \sum_{k=0}^{\infty} \frac{(x)_{k+m}(x-y)_{k+m-n}(x+y)_{k+m-p}}{k!\,(y)_{k+n+1}(1-y)_{k+p}} \frac{x+m+2k}{(-1)^k}.$$

Proof Letting $d \to -\infty$ for (0.1), we gain the ${}_4F_3$-series identity (cf. [1, p. 28]):

$${}_4F_3\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c \\ & \frac{a}{2}, & 1+a-b, & 1+a-c \end{matrix} \middle| -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$

where $\mathrm{Re}(2 + a - 2b - 2c) > 0$. Performing the replacements $a \to x + m$, $b \to x - y + m - n$, $c \to x + y + m - p$ for this equation, we attain

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(x+m)_k(x-y+m-n)_k(x+y+m-p)_k}{k!(1+y+n)_k(1-y+p)_k} \frac{x+m+2k}{x+m}(-1)^k \\ &= \frac{\Gamma(1+y+n)\Gamma(1-y+p)}{\Gamma(1+x+m)\Gamma(1-x+n+p-m)} \\ &= \frac{(y)_{1+n}(1-y)_p}{(x)_{1+m}(1-x)_{n+p-m}} \frac{\Gamma(y)\Gamma(1-y)}{\Gamma(x)\Gamma(1-x)} \\ &= \frac{(y)_{1+n}(1-y)_p}{(x)_{1+m}(1-x)_{n+p-m}} \frac{\sin(\pi x)}{\sin(\pi y)}. \end{aligned}$$

Multiplying both sides by $\frac{(x)_{1+m}(1-x)_{n+p-m}}{(y)_{1+n}(1-y)_p}$, the resulting identity reads as

$$\begin{aligned}\frac{\sin(\pi x)}{\sin(\pi y)} =& \frac{(x)_{1+m}(1-x)_{n+p-m}}{(y)_{1+n}(1-y)_p}\\ &\times \sum_{k=0}^{\infty} \frac{(x+m)_k(x-y+m-n)_k(x+y+m-p)_k}{k!(1+y+n)_k(1-y+p)_k} \frac{x+m+2k}{x+m}(-1)^k\\ =& \frac{(1-x)_{n+p-m}}{(x-y)_{m-n}(x+y)_{m-p}}\\ &\times \sum_{k=0}^{\infty} \frac{(x)_{k+m}(x-y)_{k+m-n}(x+y)_{k+m-p}}{k!\,(y)_{k+n+1}(1-y)_{k+p}} \frac{x+m+2k}{(-1)^k}.\end{aligned}$$

This finishes the proof of Theorem 2.

2.1 Series expansions for $\sqrt{2}$ with three free parameters

Corollary 23 ($x=3/4$ and $y=1/2$ in Theorem 2)

$$\sqrt{2} = \frac{(\frac{1}{4})_{n+p-m}}{(\frac{1}{4})_{m-n}(\frac{5}{4})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{3}{4})_{k+m}(\frac{1}{4})_{k+m-n}(\frac{5}{4})_{k+m-p}}{k!(\frac{3}{2})_{k+n}(\frac{1}{2})_{k+p}} \frac{3+4m+8k}{(-1)^k}.$$

Corollary 24 ($x=1/4$ and $y=1/6$ in Theorem 2)

$$\frac{2\sqrt{2}}{3} = \frac{(\frac{3}{4})_{n+p-m}}{(\frac{1}{12})_{m-n}(\frac{5}{12})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{4})_{k+m}(\frac{1}{12})_{k+m-n}(\frac{5}{12})_{k+m-p}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}} \frac{1+4m+8k}{(-1)^k}.$$

Corollary 25 ($x=5/6$ and $y=1/4$ in Theorem 2)

$$\frac{3\sqrt{2}}{4} = \frac{(\frac{1}{6})_{n+p-m}}{(\frac{7}{12})_{m-n}(\frac{13}{12})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{5}{6})_{k+m}(\frac{7}{12})_{k+m-n}(\frac{13}{12})_{k+m-p}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}} \frac{5+6m+12k}{(-1)^k}.$$

2.2 Series expansions for $\sqrt{3}$ with three free parameters

Corollary 26 ($x=1/3$ and $y=1/6$ in Theorem 2)

$$\frac{\sqrt{3}}{2} = \frac{(\frac{2}{3})_{n+p-m}}{(\frac{1}{6})_{m-n}(\frac{1}{2})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{3})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{1}{2})_{k+m-p}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}} \frac{1+3m+6k}{(-1)^k}.$$

Corollary 27 ($x=5/6$ and $y=1/3$ in Theorem 2)

$$\frac{2\sqrt{3}}{3} = \frac{(\frac{1}{6})_{n+p-m}}{(\frac{1}{2})_{m-n}(\frac{7}{6})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{5}{6})_{k+m}(\frac{1}{2})_{k+m-n}(\frac{7}{6})_{k+m-p}}{k!(\frac{4}{3})_{k+n}(\frac{2}{3})_{k+p}} \frac{5+6m+12k}{(-1)^k}.$$

Corollary 28 ($x=2/3$ and $y=1/2$ in Theorem 2)

$$\frac{3\sqrt{3}}{4} = \frac{(\frac{1}{3})_{n+p-m}}{(\frac{1}{6})_{m-n}(\frac{7}{6})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{2}{3})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{7}{6})_{k+m-p}}{k!(\frac{3}{2})_{k+n}(\frac{1}{2})_{k+p}} \frac{2+3m+6k}{(-1)^k}.$$

Corollary 29 ($x = 1/2$ and $y = 1/3$ in Theorem 2)

$$\frac{4\sqrt{3}}{9} = \frac{(\frac{1}{2})_{n+p-m}}{(\frac{1}{6})_{m-n}(\frac{5}{6})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{5}{6})_{k+m-p}}{k!(\frac{4}{3})_{k+n}(\frac{2}{3})_{k+p}} \frac{1+2m+4k}{(-1)^k}.$$

Corollary 30 ($x = 1/4$ and $y = 1/12$ in Theorem 2)

$$\frac{\sqrt{3}+1}{3} = \frac{(\frac{3}{4})_{n+p-m}}{(\frac{1}{6})_{m-n}(\frac{1}{3})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{4})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{1}{3})_{k+m-p}}{k!(\frac{13}{12})_{k+n}(\frac{11}{12})_{k+p}} \frac{1+4m+8k}{(-1)^k}.$$

Corollary 31 ($x = 3/4$ and $y = 5/12$ in Theorem 2)

$$\frac{5(\sqrt{3}-1)}{3} = \frac{(\frac{1}{4})_{n+p-m}}{(\frac{1}{3})_{m-n}(\frac{7}{6})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{3}{4})_{k+m}(\frac{1}{3})_{k+m-n}(\frac{7}{6})_{k+m-p}}{k!(\frac{17}{12})_{k+n}(\frac{7}{12})_{k+p}} \frac{3+4m+8k}{(-1)^k}.$$

Corollary 32 ($x = 5/12$ and $y = 1/4$ in Theorem 2)

$$\frac{3(\sqrt{3}+1)}{2} = \frac{(\frac{7}{12})_{n+p-m}}{(\frac{1}{6})_{m-n}(\frac{2}{3})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{5}{12})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{2}{3})_{k+m-p}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}} \frac{5+12m+24k}{(-1)^k}.$$

Corollary 33 ($x = 11/12$ and $y = 1/4$ in Theorem 2)

$$\frac{3(\sqrt{3}-1)}{2} = \frac{(\frac{1}{12})_{n+p-m}}{(\frac{2}{3})_{m-n}(\frac{7}{6})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{11}{12})_{k+m}(\frac{2}{3})_{k+m-n}(\frac{7}{6})_{k+m-p}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}} \frac{11+12m+24k}{(-1)^k}.$$

2.3 Series expansions for $\sqrt{5}$ with three free parameters

Corollary 34 ($x = 3/10$ and $y = 1/10$ in Theorem 2）

$$\frac{\sqrt{5}+3}{2} = \frac{(\frac{7}{10})_{n+p-m}}{(\frac{1}{5})_{m-n}(\frac{2}{5})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{3}{10})_{k+m}(\frac{1}{5})_{k+m-n}(\frac{2}{5})_{k+m-p}}{k!(\frac{11}{10})_{k+n}(\frac{9}{10})_{k+p}} \frac{3+10m+20k}{(-1)^k}.$$

Corollary 35 ($x = 7/10$ and $y = 1/2$ in Theorem 2）

$$\frac{5(\sqrt{5}+1)}{4} = \frac{(\frac{3}{10})_{n+p-m}}{(\frac{1}{5})_{m-n}(\frac{6}{5})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{7}{10})_{k+m}(\frac{1}{5})_{k+m-n}(\frac{6}{5})_{k+m-p}}{k!(\frac{3}{2})_{k+n}(\frac{1}{2})_{k+p}} \frac{7+10m+20k}{(-1)^k}.$$

Corollary 36 ($x = 9/10$ and $y = 1/2$ in Theorem 2)

$$\frac{5(\sqrt{5}-1)}{4} = \frac{(\frac{1}{10})_{n+p-m}}{(\frac{2}{5})_{m-n}(\frac{7}{5})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{9}{10})_{k+m}(\frac{2}{5})_{k+m-n}(\frac{7}{5})_{k+m-p}}{k!(\frac{3}{2})_{k+n}(\frac{1}{2})_{k+p}} \frac{9+10m+20k}{(-1)^k}.$$

Corollary 37 ($x = 1/2$ and $y = 1/10$ in Theorem 2)

$$\frac{\sqrt{5}+1}{5} = \frac{(\frac{1}{2})_{n+p-m}}{(\frac{2}{5})_{m-n}(\frac{3}{5})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+m}(\frac{2}{5})_{k+m-n}(\frac{3}{5})_{k+m-p}}{k!(\frac{11}{10})_{k+n}(\frac{9}{10})_{k+p}} \frac{1+2m+4k}{(-1)^k}.$$

Corollary 38 ($x = 1/2$ and $y = 3/10$ in Theorem 2)

$$\frac{3(\sqrt{5}-1)}{5} = \frac{(\frac{1}{2})_{n+p-m}}{(\frac{1}{5})_{m-n}(\frac{4}{5})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+m}(\frac{1}{5})_{k+m-n}(\frac{4}{5})_{k+m-p}}{k!(\frac{13}{10})_{k+n}(\frac{7}{10})_{k+p}} \frac{1+2m+4k}{(-1)^k}.$$

Corollary 39 ($x = 3/10$ and $y = 1/6$ in Theorem 2)

$$\frac{5(\sqrt{5}+1)}{6} = \frac{(\frac{7}{10})_{n+p-m}}{(\frac{2}{15})_{m-n}(\frac{7}{15})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{3}{10})_{k+m}(\frac{2}{15})_{k+m-n}(\frac{7}{15})_{k+m-p}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}} \frac{3+10m+20k}{(-1)^k}.$$

Corollary 40 ($x = 9/10$ and $y = 1/6$ in Theorem 2)

$$\frac{5(\sqrt{5}-1)}{6} = \frac{(\frac{1}{10})_{n+p-m}}{(\frac{11}{15})_{m-n}(\frac{16}{15})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{9}{10})_{k+m}(\frac{11}{15})_{k+m-n}(\frac{16}{15})_{k+m-p}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}} \frac{9+10m+20k}{(-1)^k}.$$

Corollary 41 ($x = 1/6$ and $y = 1/10$ in Theorem 2)

$$\frac{3(\sqrt{5}+1)}{10} = \frac{(\frac{5}{6})_{n+p-m}}{(\frac{1}{15})_{m-n}(\frac{4}{15})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{1}{6})_{k+m}(\frac{1}{15})_{k+m-n}(\frac{4}{15})_{k+m-p}}{k!(\frac{11}{10})_{k+n}(\frac{9}{10})_{k+p}} \frac{1+6m+12k}{(-1)^k}.$$

Corollary 42 ($x = 5/6$ and $y = 3/10$ in Theorem 2)

$$\frac{9(\sqrt{5}-1)}{10} = \frac{(\frac{1}{6})_{n+p-m}}{(\frac{8}{15})_{m-n}(\frac{17}{15})_{m-p}} \sum_{k=0}^{\infty} \frac{(\frac{5}{6})_{k+m}(\frac{8}{15})_{k+m-n}(\frac{17}{15})_{k+m-p}}{k!(\frac{13}{10})_{k+n}(\frac{7}{10})_{k+p}} \frac{5+6m+12k}{(-1)^k}.$$

3 Series expansions for $\sqrt{3}$ with four free parameters

Theorem 3 For $x, y \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}$ with $\mathrm{Re}(1/2 - x + n + p - m - q) > 0$, there holds the summation formula on tangent function with six free parameters:

$$\begin{aligned}\frac{\tan(\pi x)}{\tan(\pi y)} = &\frac{(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}}{(x-y)_{m-n}(x+y)_{m-p}(\frac{1}{2}-x)_{n+p-m-q}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(x)_{k+m}(x-y)_{k+m-n}(x+y)_{k+m-p}(\frac{1}{2})_{k+q}}{k!\,(y)_{k+n+1}(1-y)_{k+p}(\frac{1}{2}+x)_{k+m-q}} (x+m+2k).\end{aligned}$$

Proof Employing the substitutions $a \to x+m$, $b \to x-y+m-n$, $c \to x+y+m-p$, $d \to 1/2+q$ for (0.1), we achieve

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(x+m)_k(x-y+m-n)_k(x+y+m-p)_k(\frac{1}{2}+q)_k}{k!(1+y+n)_k(1-y+p)_k(\frac{1}{2}+x+m-q)_k} \frac{x+m+2k}{x+m} \\ &\quad= \frac{\Gamma(1+y+n)\Gamma(1-y+p)\Gamma(\frac{1}{2}+x+m-q)\Gamma(\frac{1}{2}-x+n+p-m-q)}{\Gamma(1+x+m)\Gamma(1-x+n+p-m)\Gamma(\frac{1}{2}+y+n-q)\Gamma(\frac{1}{2}-y+p-q)} \\ &\quad= \frac{(y)_{1+n}(1-y)_p(\frac{1}{2}+x)_{m-q}(\frac{1}{2}-x)_{n+p-m-q}}{(x)_{1+m}(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}} \frac{\Gamma(y)\Gamma(1-y)\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}-x)}{\Gamma(x)\Gamma(1-x)\Gamma(\frac{1}{2}+y)\Gamma(\frac{1}{2}-y)} \\ &\quad= \frac{(y)_{1+n}(1-y)_p(\frac{1}{2}+x)_{m-q}(\frac{1}{2}-x)_{n+p-m-q}}{(x)_{1+m}(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}} \frac{\tan(\pi x)}{\tan(\pi y)}.\end{aligned}$$

Multiplying both sides by $\frac{(x)_{1+m}(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}}{(y)_{1+n}(1-y)_p(\frac{1}{2}+x)_{m-q}(\frac{1}{2}-x)_{n+p-m-q}}$, we can proceed as follows:

$$\begin{aligned}\frac{\tan(\pi x)}{\tan(\pi y)} =& \frac{(x)_{1+m}(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}}{(y)_{1+n}(1-y)_p(\frac{1}{2}+x)_{m-q}(\frac{1}{2}-x)_{n+p-m-q}} \\ &\times \sum_{k=0}^{\infty} \frac{(x+m)_k(x-y+m-n)_k(x+y+m-p)_k(\frac{1}{2}+q)_k}{k!(1+y+n)_k(1-y+p)_k(\frac{1}{2}+x+m-q)_k} \frac{x+m+2k}{x+m} \\ =& \frac{(1-x)_{n+p-m}(\frac{1}{2}+y)_{n-q}(\frac{1}{2}-y)_{p-q}}{(x-y)_{m-n}(x+y)_{m-p}(\frac{1}{2}-x)_{n+p-m-q}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(x)_{k+m}(x-y)_{k+m-n}(x+y)_{k+m-p}(\frac{1}{2})_{k+q}}{k!\,(y)_{k+n+1}(1-y)_{k+p}(\frac{1}{2}+x)_{k+m-q}} (x+m+2k).\end{aligned}$$

This completes the proof of Theorem 3.

Corollary 43 ($x = 1/4$ and $y = 1/6$ in Theorem 3)

$$\begin{aligned}\frac{\sqrt{3}}{6} =& \frac{(\frac{3}{4})_{n+p-m}(\frac{2}{3})_{n-q}(\frac{1}{3})_{p-q}}{(\frac{1}{12})_{m-n}(\frac{5}{12})_{m-p}(\frac{5}{4})_{n+p-m-q-1}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{1}{4})_{k+m}(\frac{1}{12})_{k+m-n}(\frac{5}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}(\frac{3}{4})_{k+m-q}} (1+4m+8k).\end{aligned}$$

Corollary 44 ($x = 5/6$ and $y = 1/4$ in Theorem 3)

$$\begin{aligned}\frac{\sqrt{3}}{6} =& \frac{(\frac{1}{6})_{n+p-m}(\frac{3}{4})_{n-q}(\frac{1}{4})_{p-q}}{(\frac{7}{12})_{m-n}(\frac{13}{12})_{m-p}(\frac{2}{3})_{n+p-m-q-1}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{5}{6})_{k+m}(\frac{7}{12})_{k+m-n}(\frac{13}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}(\frac{4}{3})_{k+m-q}} (5+6m+12k).\end{aligned}$$

Corollary 45 ($x = 1/3$ and $y = 1/4$ in Theorem 3)

$$\begin{aligned}\frac{\sqrt{3}}{8} =& \frac{(\frac{2}{3})_{n+p-m}(\frac{3}{4})_{n-q}(\frac{1}{4})_{p-q}}{(\frac{1}{12})_{m-n}(\frac{7}{12})_{m-p}(\frac{7}{6})_{n+p-m-q-1}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{1}{3})_{k+m}(\frac{1}{12})_{k+m-n}(\frac{7}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}(\frac{5}{6})_{k+m-q}} (1+3m+6k).\end{aligned}$$

Corollary 46 ($x = 3/4$ and $y = 1/3$ in Theorem 3)

$$\begin{aligned}\frac{\sqrt{3}}{9} =& \frac{(\frac{1}{4})_{n+p-m}(\frac{5}{6})_{n-q}(\frac{1}{6})_{p-q}}{(\frac{5}{12})_{m-n}(\frac{13}{12})_{m-p}(\frac{3}{4})_{n+p-m-q-1}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{3}{4})_{k+m}(\frac{5}{12})_{k+m-n}(\frac{13}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{4}{3})_{k+n}(\frac{2}{3})_{k+p}(\frac{5}{4})_{k+m-q}} (3+4m+8k).\end{aligned}$$

Corollary 47 ($x = 5/12$ and $y = 1/12$ in Theorem 3)

$$\begin{aligned}\frac{4\sqrt{3}+7}{12} =& \frac{(\frac{7}{12})_{n+p-m}(\frac{7}{12})_{n-q}(\frac{5}{12})_{p-q}}{(\frac{1}{3})_{m-n}(\frac{1}{2})_{m-p}(\frac{13}{12})_{n+p-m-q-1}(\frac{1}{2})_q} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{5}{12})_{k+m}(\frac{1}{3})_{k+m-n}(\frac{1}{2})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{13}{12})_{k+n}(\frac{11}{12})_{k+p}(\frac{11}{12})_{k+m-q}} (5+12m+24k).\end{aligned}$$

Corollary 48 ($x=5/12$ and $y=1/4$ in Theorem 3)

$$\frac{\sqrt{3}+2}{4}=\frac{(\frac{7}{12})_{n+p-m}(\frac{3}{4})_{n-q}(\frac{1}{4})_{p-q}}{(\frac{1}{6})_{m-n}(\frac{2}{3})_{m-p}(\frac{13}{12})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{5}{12})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{2}{3})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{5}{4})_{k+n}(\frac{3}{4})_{k+p}(\frac{11}{12})_{k+m-q}}(5+12m+24k).$$

Corollary 49 ($x=1/4$ and $y=1/12$ in Theorem 3)

$$\frac{\sqrt{3}+2}{12}=\frac{(\frac{3}{4})_{n+p-m}(\frac{7}{12})_{n-q}(\frac{5}{12})_{p-q}}{(\frac{1}{6})_{m-n}(\frac{1}{3})_{m-p}(\frac{5}{4})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{1}{4})_{k+m}(\frac{1}{6})_{k+m-n}(\frac{1}{3})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{13}{12})_{k+n}(\frac{11}{12})_{k+p}(\frac{3}{4})_{k+m-q}}(1+4m+8k).$$

Corollary 50 ($x=5/12$ and $y=1/6$ in Theorem 3)

$$\frac{2\sqrt{3}+3}{6}=\frac{(\frac{7}{12})_{n+p-m}(\frac{2}{3})_{n-q}(\frac{1}{3})_{p-q}}{(\frac{1}{4})_{m-n}(\frac{7}{12})_{m-p}(\frac{13}{12})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{5}{12})_{k+m}(\frac{1}{4})_{k+m-n}(\frac{7}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}(\frac{11}{12})_{k+m-q}}(5+12m+24k).$$

Corollary 51 ($x=5/12$ and $y=1/3$ in Theorem 3)

$$\frac{2\sqrt{3}+3}{9}=\frac{(\frac{7}{12})_{n+p-m}(\frac{5}{6})_{n-q}(\frac{1}{6})_{p-q}}{(\frac{1}{12})_{m-n}(\frac{3}{4})_{m-p}(\frac{13}{12})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{5}{12})_{k+m}(\frac{1}{12})_{k+m-n}(\frac{3}{4})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{4}{3})_{k+n}(\frac{2}{3})_{k+p}(\frac{11}{12})_{k+m-q}}(5+12m+24k).$$

Corollary 52 ($x=1/6$ and $y=1/12$ in Theorem 3)

$$\frac{2\sqrt{3}+3}{18}=\frac{(\frac{5}{6})_{n+p-m}(\frac{7}{12})_{n-q}(\frac{5}{12})_{p-q}}{(\frac{1}{12})_{m-n}(\frac{1}{4})_{m-p}(\frac{4}{3})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+m}(\frac{1}{12})_{k+m-n}(\frac{1}{4})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{13}{12})_{k+n}(\frac{11}{12})_{k+p}(\frac{2}{3})_{k+m-q}}(1+6m+12k).$$

Corollary 53 ($x=1/3$ and $y=1/12$ in Theorem 3)

$$\frac{2\sqrt{3}+3}{24}=\frac{(\frac{2}{3})_{n+p-m}(\frac{7}{12})_{n-q}(\frac{5}{12})_{p-q}}{(\frac{1}{4})_{m-n}(\frac{5}{12})_{m-p}(\frac{7}{6})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{1}{3})_{k+m}(\frac{1}{4})_{k+m-n}(\frac{5}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{13}{12})_{k+n}(\frac{11}{12})_{k+p}(\frac{5}{6})_{k+m-q}}(1+3m+6k).$$

Corollary 54 ($x=11/12$ and $y=1/6$ in Theorem 3)

$$\frac{10\sqrt{3}-15}{6}=\frac{(\frac{1}{12})_{n+p-m}(\frac{2}{3})_{n-q}(\frac{1}{3})_{p-q}}{(\frac{3}{4})_{m-n}(\frac{13}{12})_{m-p}(\frac{7}{12})_{n+p-m-q-1}(\frac{1}{2})_q}\times\sum_{k=0}^{\infty}\frac{(\frac{11}{12})_{k+m}(\frac{3}{4})_{k+m-n}(\frac{13}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{7}{6})_{k+n}(\frac{5}{6})_{k+p}(\frac{17}{12})_{k+m-q}}(11+12m+24k).$$

Corollary 55 ($x = 11/12$ and $y = 1/3$ in Theorem 3)

$$\frac{10\sqrt{3}-15}{9} = \frac{(\frac{1}{12})_{n+p-m}(\frac{5}{6})_{n-q}(\frac{1}{6})_{p-q}}{(\frac{7}{12})_{m-n}(\frac{5}{4})_{m-p}(\frac{7}{12})_{n+p-m-q-1}(\frac{1}{2})_q} \times \sum_{k=0}^{\infty} \frac{(\frac{11}{12})_{k+m}(\frac{7}{12})_{k+m-n}(\frac{5}{4})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{4}{3})_{k+n}(\frac{2}{3})_{k+p}(\frac{17}{12})_{k+m-q}}(11+12m+24k).$$

Corollary 56 ($x = 5/6$ and $y = 5/12$ in Theorem 3)

$$\frac{10\sqrt{3}-15}{18} = \frac{(\frac{1}{6})_{n+p-m}(\frac{11}{12})_{n-q}(\frac{1}{12})_{p-q}}{(\frac{5}{12})_{m-n}(\frac{5}{4})_{m-p}(\frac{2}{3})_{n+p-m-q-1}(\frac{1}{2})_q} \times \sum_{k=0}^{\infty} \frac{(\frac{5}{6})_{k+m}(\frac{5}{12})_{k+m-n}(\frac{5}{4})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{17}{12})_{k+n}(\frac{7}{12})_{k+p}(\frac{4}{3})_{k+m-q}}(5+6m+12k).$$

Corollary 57 ($x = 2/3$ and $y = 5/12$ in Theorem 3)

$$\frac{10\sqrt{3}-15}{24} = \frac{(\frac{1}{3})_{n+p-m}(\frac{11}{12})_{n-q}(\frac{1}{12})_{p-q}}{(\frac{1}{4})_{m-n}(\frac{13}{12})_{m-p}(\frac{5}{6})_{n+p-m-q-1}(\frac{1}{2})_q} \times \sum_{k=0}^{\infty} \frac{(\frac{2}{3})_{k+m}(\frac{1}{4})_{k+m-n}(\frac{13}{12})_{k+m-p}(\frac{1}{2})_{k+q}}{k!(\frac{17}{12})_{k+n}(\frac{7}{12})_{k+p}(\frac{7}{6})_{k+m-q}}(2+3m+6k).$$

Remark The convergent conditions for all the corollaries can be derived from the corresponding theorems without difficulty. By specifying the values of x and y, Theorems 1–3 can create more series expansions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ with free parameters. Setting $m = n = p = 0$ in Corollary 2, we have

$$\sqrt{2} = \sum_{k=0}^{\infty} \frac{(\frac{1}{4})_k^2}{k!(\frac{3}{4})_k}.$$

This formula gives the interesting inequality:

$$\sqrt{2} \geqslant \sum_{k=0}^{\varepsilon} \frac{(\frac{1}{4})_k^2}{k!(\frac{3}{4})_k},$$

where ε is an arbitrary nonnegative integer. Of course, countless inequalities of this type can be deduced in the same method. We shall not display them due to the limitation of space.

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