Article ID: 1000-5641(2016)01-0027-12

Cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{l}_6)$

YUE Ming-shi

(*School of Logistics, Linyi University, Linyi Shandong* 276000, *China*)

Abstract: Let α be a group automorphism of the affine Weyl group $(\widetilde{A}_{2n}, \widetilde{S})$ with $\alpha(\widetilde{S}) = \widetilde{S}$. Affine Weyl group $(\widetilde{C}_n, S)$ can be seen as the fixed point set of the affine Weyl group $(\widetilde{A}_{2n}, \widetilde{S})$ under its group automorphism α. The restriction to $\widetilde{C}_n$ of the length function $\widetilde{l}_{2n}$ on $\widetilde{A}_{2n}$ can be seen as a weight function on $\widetilde{C}_n$. In this paper, we give the description for all the left and two-sided cells of the specific weighted Coxeter group $(\widetilde{C}_3, \widetilde{l}_6)$ and prove that each left cell in $(\widetilde{C}_3, \widetilde{l}_6)$ is left-connected.

Key words: affine Weyl group; weighted Coxeter group; quasi-split case; partitions of n; left cells

CLC number: O152 **Document code:** A

DOI: 10.3969/j.issn.1000-5641.2016.01.004

加权 Coxeter 群$(\widetilde{C}_3, \widetilde{l}_6)$的胞腔

岳明仕

(临沂大学物流学院, 山东 临沂 276000)

摘要: 取 α 是仿射 Weyl 群 $(\widetilde{A}_{2n}, \widetilde{S})$ 上某个满足 $\alpha(\widetilde{S}) = \widetilde{S}$ 的群自同构. 仿射 Weyl 群 $(\widetilde{C}_n, S)$ 可以看做仿射 Weyl 群 $(\widetilde{A}_{2n}, \widetilde{S})$ 在其群自同构 α 下的固定点集合. $\widetilde{A}_{2n}$ 上的长度函数 $\widetilde{l}_{2n}$ 在 $\widetilde{C}_n$ 上的限制可以看做 $\widetilde{C}_n$ 上的某个权函数. 本文给出了加权的 Coxeter 群 $(\widetilde{C}_3, \widetilde{l}_6)$ 中所有左胞腔以及双边胞腔的清晰刻画并且证明 $(\widetilde{C}_3, \widetilde{l}_6)$ 中的每个左胞腔都是左连通的.

关键词: 仿射 Weyl 群; 加权 Coxeter 群; 拟分裂情形; 整数 n 的划分; 左胞腔

0 Introduction

Denote by $\mathbb{Z}$ (resp., $\mathbb{N}$, $\mathbb{N}^*$) the set of all integers (resp., non-negative integers, positive integers). For any $i \leqslant j$ in $\mathbb{Z}$, denote by $[i, j]$ the set $\{i, i+1, \cdots, j\}$ and $[j]$ the set $[1, j]$. For any $a \in \mathbb{Q}$, denote by $\lfloor a \rfloor$ the largest integer with $\lfloor a \rfloor \leqslant a$ and $|a|$ the absolute value of a. For any $k \in \mathbb{Z}$, denote by $\langle k \rangle$ the unique integer in $[2n+1]$ with $k \equiv \langle k \rangle \pmod{2n+1}$. Denote by $|S|$ the cardinal of the set S.

A weighted Coxeter group (W, L) is, by definition, a Coxeter group W together with a weight function $L : W \longrightarrow \mathbb{N}$ on it. Suppose that (W, L) is in the quasi-split case, W can be

收稿日期: 2014-12

基金项目: 国家自然科学基金(11071073)

作者简介: 岳明仕, 男，讲师, 研究方向为 Heck 代数及表示理论. E-mail: lymsyue@gmail.com.

seen as a fixed point set of some Coxeter system $(\widetilde{W}, \widetilde{S})$ under its group automorphism (denote by α) with $\alpha(\widetilde{S}) = \widetilde{S}$ (see [1]). Let $\widetilde{S} = \{s_i | 0 \leqslant i \leqslant 2n\}$ be the Coxeter generator set of the affine Weyl group $\widetilde{A}_{2n}$ with $s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1$ for any i, j in $[0, 2n]$ (write $s_{2n+1} = s_0$) with $\langle j \rangle \neq \langle i \pm 1 \rangle$. Let α be the group automorphism of $(\widetilde{A}_{2n}, \widetilde{S})$ with $\alpha(s_i) = s_{2n-i}$ for any $i \in [0, 2n]$. Affine Weyl group $\widetilde{C}_n$ can be seen as a fixed point set of $(\widetilde{A}_{2n}, \widetilde{S})$ under α. We can see the restriction to $\widetilde{C}_n$ of the length function $\widetilde{l}_{2n}$ (or $\widetilde{l}$ in short) on $\widetilde{A}_{2n}$ as a weight function on $\widetilde{C}_n$. The cells of the weighted Coxeter groups $(\widetilde{C}_3, \widetilde{l}_5)$, $(\widetilde{C}_3, \widetilde{l}_7)$ and $(\widetilde{C}_4, \widetilde{l}_7)$ have been described in [2], [3], [4]. In this paper, we give the description for all the cells of the specific weighted Coxeter group $(\widetilde{C}_3, \widetilde{l}_6)$.

1 Cell theories of a weighted Coxeter group

Some concepts and results concerning a weighted Coxeter group will be introduced in this section. All follow from Lusztig in [1] but Lemma 1.1 follows from Shi in [2].

Let (W, S) be a Coxeter System and l be the length function on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $l(w) = r$. A *weight function* on W is, by definition, a map $L : W \longrightarrow \mathbb{N}$ with $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r \in W$ and $L(s) = L(t)$ for any conjugated s, t in S. A *weighted Coxeter group* (W, L) is a Coxeter group W together with a weight function L on it. In the case $L = l$, we call (W, L) in the *split case*.

Suppose that α is a group automorphism of W with $\alpha(S) = S$. Let W^α be the set of all $w \in W$ satisfying $\alpha(w) = w$. For any α-orbit J on S, let w_J be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of all elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) can be seen as a Coxeter System and the restriction of l to W^α can be seen as a weight function on W^α. The weighted Coxeter group (W^α, l) is called in the *quasi-split case*.

Let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation associated to the preorder $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) defined by Lusztig. The corresponding equivalence classes in W are called *left cells* (resp., *right cells*, *two-sided cells*) of W.

Denote by a the a-function on W defined by Lusztig. We have the following result.

Lemma 1.1(see [2, Lemma 1.7]) *Suppose that W is either a finite or an affine Coxeter group and that (W, L) is either in the split case or in the quasi-split case. A non-empty subset E of W is a union of some two-sided cells of W if the following conditions (a)—(b) hold:*

(a) *There exists some $k \in \mathbb{N}$ with $a(x) = k$ for all $x \in E$;*

(b) *The set E with $E = \{x | x^{-1} \in E\}$ is a union of some left cells of W.*

2 Specific weighted Coxeter groups $(\widetilde{A}_{2n}, \widetilde{l}_{2n})$ and $(\widetilde{C}_n, \widetilde{l}_{2n})$

Affine Weyl group $\widetilde{A}_{2n}$ can be seen as the following permutation group on $\mathbb{Z}$ (see [1]):

$$\widetilde{A}_{2n} = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \middle| (i + 2n + 1)w = (i)w + 2n + 1, \sum_{i=1}^{2n+1} (i)w = \sum_{i=1}^{2n+1} i \right\}.$$

Let $\widetilde{S}=\{s_i|0\leqslant i\leqslant 2n\}$ be the Coxeter generator set of $\widetilde{A}_{2n}$ satisfying $(t)s_i=t+1$ if $\langle t\rangle=\langle i\rangle$, $(t)s_i=t-1$ if $\langle t\rangle=\langle i+1\rangle$ and $(t)s_i=t$ otherwise, for any $t\in\mathbb{Z}$. Let α be the group automorphism of $(\widetilde{A}_{2n},\widetilde{S})$ determined by $\alpha(s_i)=s_{2n-i}$ for any $i\in[0,2n]$. Any $w\in\widetilde{A}_{2n}$ can be seen as a $\mathbb{Z}\times\mathbb{Z}$ monomial matrix $A_w=(\delta_{(i)w,j})_{i,j\in\mathbb{Z}}$. The row (resp., column) indices of A_w are increasing from top to bottom (resp., from left to right).

Let $S=\{t_i|0\leqslant i\leqslant n\}$ be the Coxeter generator set of $\widetilde{C}_n$ determined by $t_0=s_0s_{2n}s_0$, $t_n=s_n$ and $t_i=s_is_{2n-i}$ for any $i\in[n-1]$. Under the group automorphism α of $(\widetilde{A}_{2n},\widetilde{S})$, the affine Weyl group $(\widetilde{C}_n,S)$ can be seen as a fixed point set of $(\widetilde{A}_{2n},\widetilde{S})$ and can also be seen as a permutation group on $\mathbb{Z}$:

$$\widetilde{C}_n=\Big\{w:\mathbb{Z}\longrightarrow\mathbb{Z}\Big|(i+2n+1)w=(i)w+2n+1,(-i)w=-(i)w,\forall i\in\mathbb{Z}\Big\}.$$

A *partition* of positive integer n is, by definition, an r-tuple $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_r)$ with $\lambda_1+\lambda_2+\cdots+\lambda_r=n$ and $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_r$. λ_i with $i\in[r]$ is called a *part* of λ. Let Λ_n be the set of all partitions of n. We usually denote λ by the form $\boldsymbol{j_1^{k_1}j_2^{k_2}\cdots j_m^{k_m}}$ (boldfaced), where j_i is a part of λ with multiplicity $k_i\geqslant 1$ for any $i\in[m]$ with $j_1>j_2>\cdots>j_m\geqslant 1$.

Fix $w\in\widetilde{A}_{2n}$. Let $i\neq j$ in $[2n+1]$. We write $i\prec_w j$ if there are some p,q in $\mathbb{Z}$ with $2pn+p+i>2qn+q+j$ and $(2pn+p+i)w<(2qn+q+j)w$. In terms of the entries of A_w, this means the entry 1 at the position $(2pn+p+i,(2pn+p+i)w)$ is located to the southwestern of the entry 1 at the position $(2qn+q+j,(2qn+q+j)w)$. Then we can define a partial order $\preceq_w$ on $[2n+1]$ (see [2, 2.2]).

By a *w-chain*, we mean an integer sequence $a_1,a_2,\cdots,a_r$ in $[2n+1]$ satisfying $a_1\prec_w a_2\prec_w\cdots\prec_w a_r$. A *k-w-chain-family* with $k\geqslant 1$ is, by definition, a disjoint union of some k w-chains. Denote by d_k the maximally possible cardinal of all k-w-chain-families. Then $d_1<d_2<\cdots<d_r=2n$ with some $r\geqslant 1$. Let $\lambda_1=d_1$ and $\lambda_i=d_i-d_{i-1}$ for any $i\in[2,r]$. By a result in [6] we have $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_r$. Then a map ψ from $\widetilde{A}_{2n}$ to the set Λ_{2n+1} can be defined by setting $\psi(w)=(\lambda_1,\lambda_2,\cdots,\lambda_r)$ for any $w\in\widetilde{A}_{2n}$. For any $\lambda\in\Lambda_{2n+1}$, let $E_\lambda=\psi^{-1}(\lambda)\cap\widetilde{C}_n$.

Let $i\neq j$ in $[2n]$. Two integers i,j are called *2n-dual* if $i+j=2n+1$ (we denote $\bar{i}=j$). i,j are called *w-comparable* if $i\prec_w j$ or $j\prec_w i$ and *w-uncomparable* if otherwise. The integer i is called *w-wild* (resp., *w-tame*) if $i,\bar{i}$ are w-comparable (resp., w-uncomparable) and *a w-wild head* (resp., *a w-tame head*) if i with $(\bar{i})w<(i)w$ is *w-wild* (resp., *w-tame*).

Denote by $\widetilde{l}_{2n}$ (or $\widetilde{l}$ in short) the length function on $\widetilde{A}_{2n}$ and l the length function on $\widetilde{C}_n$. Then the affine Weyl group $(\widetilde{A}_{2n},\widetilde{l}_{2n})$ is in the split case and the weighted Coxeter group $(\widetilde{C}_n,\widetilde{l}_{2n})$ is in the quasi-split case. For any $x\in(\widetilde{A}_{2n},\widetilde{l}_{2n})$. Let $\widetilde{\mathcal{L}}(x)=\{s\in\widetilde{S}|sx<x\}$ and $\widetilde{\mathcal{R}}(x)=\{s\in\widetilde{S}|xs<x\}$. For any $y\in\widetilde{C}_n$. Let $\widetilde{L}(y)=\{t\in S|ty<y\}$ and $\widetilde{R}(y)=\{t\in S|yt<y\}$. Let $m_k(x)=|\{i\in\mathbb{Z}|i<k,(i)x>(k)x\}|$ for any $x\in\widetilde{A}_{2n}$ and $k\in\mathbb{Z}$.

Lemma 2.1(see [2, Proposition 2.4]) *Let $w\in\widetilde{A}_{2n}$ and $x\in\widetilde{C}_n$, we have*

(1) $\widetilde{l}(w)=\sum\limits_{1\leqslant i<j\leqslant 2n+1}|\lfloor\frac{(j)w-(i)w}{2n+1}\rfloor|=\sum_{k=1}^{2n+1}m_k(w)$;

(2) $l(x)=\frac{1}{2}(\widetilde{l}(x)-m_0(x)+m_n(x))$.

Let $w \in \widetilde{C}_n$. We see that w is uniquely determined by the n-tuple $((1)w, (2)w, \cdots, (n)w)$. Then w can be identified with the n-tuple $((1)w, (2)w, \cdots, (n)w)$ and we can denote the latter simple by $[(1)w, (2)w, \cdots, (n)w]$.

Lemma 2.2 *Let $w = [a_1, a_2, \cdots, a_n]$ and $w' = t_i w = [a'_1, a'_2, \cdots, a'_n]$ in $\widetilde{C}_n$ with $i \in [0, n]$. For any $j \in [n]$*

(1) *In the case $i = 0$,* $$a'_j = \begin{cases} -a_j\ , & \text{if } j = 1, \\ a_j, & \text{if } j \in [2, n]. \end{cases}$$

(2) *In the case $i \in [n-1]$,* $$a'_j = \begin{cases} a_{j+1}, & \text{if } j = i, \\ a_{j-1}, & \text{if } j = i+1, \\ a_j, & \text{otherwise}. \end{cases}$$

(3) *In the case $i = n$,* $$a'_j = \begin{cases} \bar{a}_j, & \text{if } j = n, \\ a_j, & \text{if } j \in [n-1]. \end{cases}$$

Lemma 2.3 *Let $w = [a_1, a_2, \cdots, a_n]$ and $w'' = wt_i = [a''_1, a''_2, \cdots, a''_n]$ in $\widetilde{C}_n$ with $i \in [0, n]$. For any $j \in [n]$*

(1) *In the case $i = 0$,* $$a''_j = \begin{cases} a_j + 2, & \text{if } \langle a_j \rangle = 2n, \\ a_j - 2, & \text{if } \langle a_j \rangle = 1, \\ a_j, & \text{otherwise}. \end{cases}$$

(2) *In the case $i \in [n]$,* $$a''_j = \begin{cases} a_j + 1, & \text{if } \langle a_j \rangle \in \{\overline{i+1}, i\}, \\ a_j - 1, & \text{if } \langle a_j \rangle \in \{i+1, \bar{i}\}, \\ a_j, & \text{otherwise}. \end{cases}$$

For any $i \in [0, 2n]$. Let $\widetilde{D}_R(i)$ be the set of all $w \in \widetilde{A}_{2n}$ with $|\{s_i, s_{i+1}\} \cap \widetilde{\mathcal{R}}(w)| = 1$. For any $w \in \widetilde{D}_R(i)$, we see that there is exactly one element (denote by w^*) of $\{ws_i, ws_{i+1}\}$ in $\widetilde{D}_R(i)$. The transformation from w to w^* in $\widetilde{D}_R(i)$ is called a *right* $\{s_i, s_{i+1}\}$*-star operation* (or a *right star operation* in short) on w. Similarly. We can define a *left star operation* on $w \in \widetilde{D}_R(i)$. In the present paper, when we mention a right star operation (resp., a left star operation) on some $w \in \widetilde{C}_n$, we always mean that w is regarded as an element of $\widetilde{A}_{2n}$. We make such a convention once and forever.

For any $w \in \widetilde{C}_n$. Let $M(w)$ be the set of all $x \in \widetilde{C}_n$ satisfying that there exists a sequence $x_0 = w, x_1, \cdots, x_r = x$ in $\widetilde{C}_n$ with $r \geqslant 0$ satisfying that for any $i \in [r]$, $x_i^{-1} x_{i-1} \in S$ and x_i can be obtained from x_{i-1} by one or two right star operations.

Let $w \in \widetilde{C}_n$. We can define a graph $\mathcal{M}(w)$ by the following requirements:

(i) The vertex set of the graph $\mathcal{M}(w)$ is $M(w)$, each vertex $x \in M(w)$ with label $\mathcal{R}(x)$;

(ii) Let x, y in $M(w)$. We draw a solid edge from x to y if $x^{-1}y \in S$ and x can be obtained from y by one or two right star operations.

A path in $\mathcal{M}(w)$ is, by definition, a sequence $x_0, x_1, \cdots, x_r$ in $M(w)$ with $r > 0$ such that x_{i-1} and x_i are joined by a solid edge for any $i \in [r]$. Let $x, y \in \widetilde{C}_n$. x, y are called have the *same generalized τ-invariants* if for any path $x = x_0, x_1, \cdots, x_r$ in $\mathcal{M}(x)$, there exists a corresponding path $y=y_0,y_1,\cdots,y_r$ in $\mathcal{M}(y)$ with $\mathcal{R}(x_i)=\mathcal{R}(y_i)$ for any $i\in[r]$ and the condition still holds if the roles of x and y are interchanged. The graph $\mathcal{M}(w)$ can be used in proving that two elements of $\widetilde{C}_n$ have the different generalized τ-invariants (see [2]). Let x,y in $\widetilde{C}_n$. We draw a dashed edge from x to y if $M(x)\cap M(y)=\emptyset$ and $x^{-1}y\in S$.

For example. In Fig. 1, the vertex $[5,4,6]$ with label $\{t_2,t_3\}$, the vertex $[6,4,5]$ with label $\{t_1,t_3\}$ and the vertex $[8,4,5]$ with label $\{t_0,t_3\}$. The notion $\boxed{\mathbf{32}}$ denotes the set $\{t_2,t_3\}$ and the notion $\boxed{\mathbf{32}}$—$\boxed{\mathbf{31}}$ denotes two vertices $[5,4,6]$ and $[6,4,5]$ are joined by a solid edge. There is a dashed edge joined the vertices $[6,4,5]$ and $[8,4,5]$ since $[6,4,5]=[8,4,5]t_0$ and $M([8,4,5])\cap M([6,4,5])=\emptyset$. We see that no two elements in Fig. 2 have the same generalized τ-invariants.

Lemma 2.4(see [1, Lemma 16.14]) *Let x,y in $\widetilde{C}_n$.*

(1) *$x\underset{L}{\sim}y$ (resp., $x\underset{R}{\sim}y$) in $\widetilde{C}_n$ if and only if $x\underset{L}{\sim}y$ (resp., $x\underset{R}{\sim}y$) in $\widetilde{A}_{2n}$;*

(2) *$x\underset{L}{\leqslant}y$ (resp., $x\underset{L}{\sim}y$) implies $\mathcal{R}(x)\supseteq\mathcal{R}(y)$ (resp., $\mathcal{R}(x)=\mathcal{R}(y)$);*

(3) *$x\underset{R}{\leqslant}y$ (resp., $x\underset{R}{\sim}y$) implies $\mathcal{L}(x)\supseteq\mathcal{L}(y)$ (resp., $\mathcal{L}(x)=\mathcal{L}(y)$).*

Suppose that E is a non-empty subset of a Coxeter group W. E is called *left-connected* (resp., *right-connected*) if for any x,y in E, there exists a sequence $x=x_0,x_1,\cdots,x_r=y$ in E with $x_{i-1}x_i^{-1}\in S$ (resp., $x_i^{-1}x_{i-1}\in S$) for any $i\in[r]$ and called *two-sided-connected* if for any x,y in E, there exists a sequence $x=x_0,x_1,\cdots,x_r=y$ in E satisfying that either $x_{i-1}x_i^{-1}\in S$ or $x_i^{-1}x_{i-1}\in S$ for any $i\in[r]$. Let F be a non-empty subset of E. We call F a *left-connected component* of E if F is the maximal left-connected subset of E.

Lemma 2.5(see [2, Lemma 2.18]) (1) *Any left-connected (resp., right-connected, two-sided-connected) set of $\psi^{-1}(\lambda)$ is in a left (resp., right, two-sided) cell of $\widetilde{A}_{2n}$;*

(2) *Any left-connected (resp., right-connected, two-sided-connected) set of E_λ is in a left (resp., right, two-sided) cell of $\widetilde{C}_n$;*

(3) *The set E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$.*

Lemma 2.6(see [2, Corollary 2.19]) (1) *Let x,y in $\psi^{-1}(\lambda)$ with $\lambda\in\Lambda_{2n+1}$. If $\widetilde{l}(y)=\widetilde{l}(x)+\widetilde{l}(yx^{-1})$ (resp., $\widetilde{l}(y)=\widetilde{l}(x)+\widetilde{l}(x^{-1}y)$), then x and y are in the same left-connected (resp., right-connected) component of $\psi^{-1}(\lambda)$ and hence $x\underset{L}{\sim}y$ (resp., $x\underset{R}{\sim}y$).*

(2) *Let x,y in E_λ with $\lambda\in\Lambda_{2n+1}$. If $l(y)=l(x)+l(yx^{-1})$ (resp., $l(y)=l(x)+l(x^{-1}y)$), then x and y are in the same left-connected (resp., right-connected) component of E_λ and hence $x\underset{L}{\sim}y$ (resp., $x\underset{R}{\sim}y$).*

3 Partial order $\preceq_w$ on $[2n+1]$ determined by $w\in\widetilde{C}_n$

Two technical tools from Shi in [2, Section 3] will be introduced in this section. One is used in proving the left-connectedness of a left cell of $\widetilde{C}_n$ (see Theorem 3.2). The other is used in checking whether two elements of $\widetilde{C}_n$ are in the same left cell (see Lemma 3.3).

Lemma 3.1(see [7, Lemma 3.2]) *Fix $w\in\widetilde{C}_n$.*

(i) *Let $i\neq j$ in $[2n]$, then $j\prec_w k$ if and only if $\bar{k}\prec_w\bar{j}$;*

Suppose that $j\neq k$ in $[2n]$ are both w-wild heads and $i\in[2n]$ is w-tame head, we have

(ii) *$\bar{j}\prec_w k$ if and only if $j,\bar{k}$ are w-comparable if and only if j,k are w-comparable;*

(iii) i, k *are* w*-uncomparable if and only if* $i \neq k$;

(iv) $j, i, \bar{j}$ *forms a* w*-chain if and only if* i, $\bar{i}$ *are both* w*-comparable with* j;

(v) $j, k, \bar{j}, \bar{k}$ *forms a* w*-chain if and only if* j, k *are* w*-comparable*;

(vi) *If* $(j)w > 2n+1$, *we have* $\bar{j} \prec_w 2n+1 \prec_w j$. *Suppose that* $(j)w < 2n+1$, *then* $j, \bar{j}$ *are both* w*-uncomparable with* $2n+1$;

(vii) $i, \bar{i}$ *are both* w*-uncomparable with* $2n+1$.

Theorem 3.2(see [7, 3.3]) *For any* $k \in \mathbb{Z}$, *let* $t'_k = t_{\langle k\rangle}$ *if* $\langle k\rangle \in [n]$, $t'_k = t_{\langle k\rangle-1}$ *if* $\langle k\rangle \in [n+1, 2n]$ *and* $t'_k = 1$ *if otherwise. Let* $t_{i,j} = t'_{i+j-1} \cdots t'_{i+1} t'_i$ *for any* $i \in \mathbb{Z}$ *and* $j \in \mathbb{N}^*$. *Suppose that* $x \in \widetilde{C}_n$ *and* $i \in \mathbb{Z}$ *satisfy* $(i)x - 2n - 1 > (j)x$ *for any* $j \in [i+1, i+a]$ *with some* $a \in [2n]$. *Let* $x' = t_{i,a}x$, *we have* $l(x') = l(x) - l(t_{i,a})$ *and* $\psi(x) = \psi(x')$. *In addition, suppose that* $(i)x - 2n - 1 > (j)x$ *for any* $j \in [i+1, i+2n]$. *Let* $x'' = t_{i,2n+1}$, *we have* $l(x'') = l(x) - 2n - 1$ *and* $\psi(x) = \psi(x'')$. *Moreover, for any* $m \in \mathbb{Z}$, *we have*

$$(m)x'' = \begin{cases} (m)x - 2n - 1, & \text{if } \langle m\rangle = \langle i\rangle, \\ (m)x + 2n + 1, & \text{if } \langle m\rangle = \langle \bar{i}\rangle, \\ (m)x, & \text{otherwise.} \end{cases}$$

Fix $w \in \widetilde{C}_n$. *Let* $E_1 = \{i_1, i_2, \cdots, i_a\}$ *and* $E_2 = \{j_1, j_2, \cdots, j_b\}$ *with* $a \in \mathbb{N}^*$, $b \in \mathbb{N}$ *and* $a + b = n$ *be two subsets of* $[2n]$ *such that*:

(i) $i_1 < i_2 < \cdots < i_a$ *and* $j_1 < j_2 < \cdots < j_b$;

(ii) *The integers of* $E_1 \cup E_2$ *are pairwise not* $2n$*-dual*;

(iii) $(\bar{i})w < (i)w$ *for any* $i \in E_1 \cup E_2$;

(iv) *There exists some* $l \in \mathbb{N}^*$ *such that* $(i)w - (j)w > l(2n+1)$ *for any* $i \in E_1$ *and* $j \in E_2 \cup \{2n+1\}$.

By repeatedly left multiplying various elements of the form $t_{i,j}$ *on* w, *we can get some* $w' \in \widetilde{C}_n$ *such that there exist some* $1 \leqslant k_1 < k_2 < \cdots < k_b \leqslant 2b$ *satisfying*:

(1) $l(w') = l(w) - l(ww'^{-1})$;

(2) *If* $b > 0$, *then* $[2b] = \{k_1, k_2, \cdots, k_b, 2b+1-k_1, 2b+1-k_2, \cdots, 2b+1-k_b\}$. *The map* $\phi : \{j_1, j_2, \cdots, j_b, \bar{j}_1, \bar{j}_2, \cdots, \bar{j}_b\} \longrightarrow [2b]$ *defined by* $\phi(j_m) = k_m$ *and* $\phi(\bar{j}_m) = 2b + 1 - k_m$ *for any* $m \in [b]$ *is an order-preserving bijection*;

(3) *There exists some* $l' \geqslant l$ *with* $(p)w' = (i_p)w - l'(2n+1)$ *and* $(a + k_q)w' = (j_q)w$ *for any* $p \in [a]$ *and* $q \in [b]$;

(4) $(\bar{i})w' < (i)w'$ *for any* $i \in [a] \cup \{a + k_m | m \in [b]\}$;

(5) $0 < \min\{(i)w' - (a + k_m)w' | i \in [a], m \in [b]\} < 2n + 1$ *if* $b > 0$;

(6) w', w *with* $\psi(w') = \psi(w)$ *are in the same left-connected component of* E_λ.

An r-tuple $(a_1, a_2, \cdots, a_r)$ with $a_1, a_2, \cdots, a_r$ in $\mathbb{N}^*$ is called a *composition* of positive integer n with rank $r \geqslant 1$ if $a_1 + a_2 + \cdots + a_r = n$. Let $\widetilde{\Lambda}_n$ be the set of all compositions of n. A *generalized tabloid* of rank n is an r-tuple $T = (T_1, T_2, \cdots, T_r)$ satisfying that $[n]$ is a disjoint union of its non-empty subsets $T_1, T_2, \cdots, T_r$. Let $\mathcal{C}_{2n+1}$ be the set of all generalized tabloids of rank $2n+1$. Let $T' = (T'_1, T'_2, \cdots, T'_r)$ and $T'' = (T''_1, T''_2, \cdots, T''_t)$ in $\mathcal{C}_{2n+1}$. We write $T' = T''$ if $r = t$ and $T'_i = T''_i$ for any $i \in [r]$. For any $T \in \mathcal{C}_{2n+1}$, let $\xi(T) = (|T_1|, |T_2|, \cdots, |T_r|) \in$

$\widetilde{\Lambda}_{2n+1}$ and $\zeta(T)=(|T_{i_1}|,|T_{i_2}|,\cdots,|T_{i_r}|)\in\Lambda_{2n+1}$, where the integer sequence $i_1,i_2,\cdots,i_r$ is a permutation of $1,2,\cdots,r$ with $|T_{i_1}|\geqslant|T_{i_2}|\geqslant\cdots\geqslant|T_{i_r}|$. We see by [1] that $\xi:\mathcal{C}_{2n+1}\longrightarrow\widetilde{\Lambda}_{2n+1}$ and $\zeta:\widetilde{\Lambda}_{2n+1}\longrightarrow\Lambda_{2n+1}$ are both surjective maps. Let Ω be the set of all $w\in\widetilde{A}_{2n}$ such that there exists a generalized tabloid $T=(T_1,T_2,\cdots,T_r)$ in $\mathcal{C}_{2n+1}$ satisfying that $a\prec_w b$ for any $a\in T_i$ and $b\in T_j$ with $i<j$ in $[r]$ and that any $a\neq b$ in T_i are w-uncomparable with $i\in[r]$. We see that T is entirely determined by the element $w\in\Omega$, then the generalized tabloid T can be denoted by $T(w)$. The map $T:\Omega\longrightarrow\mathcal{C}_{2n+1}$ is surjective by a result in [1].

Lemma 3.3(see [5, Lemma 19.4.6]) *Let w,w' in Ω with $\xi(T(w))=\xi(T(w'))$. Then $w\underset{L}{\sim}w'$ in $\widetilde{A}_{2n}$ if and only if $T(w)=T(w')$.*

4 The main results

We give the description for all the cells of the specific weighted Coxeter group $(\widetilde{C}_3,\widetilde{l}_6)$ in the present section (see Theorem 4.1). Let $\lambda\in\Lambda_7$, denote by $n(\lambda)$ the number of left cells of $\widetilde{C}_3$ in E_λ. Suppose that E_λ is a union of two two-sided cells E_λ^1 and E_λ^2 of $\widetilde{C}_3$. Denote by $n_1(\lambda)$ (resp., $n_2(\lambda)$) the number of left cells of $\widetilde{C}_3$ in E_λ^1 (resp., E_λ^2). The main results are as follows:

Theorem 4.1 *In the weighted Coxeter group $(\widetilde{C}_3,\widetilde{l}_6)$, let $\lambda\in\Lambda_7$.*

(1) *The set E_λ forms a single two-sided cell of $\widetilde{C}_3$ if $\lambda\in\{\mathbf{7},\mathbf{61},\mathbf{52},\mathbf{51^2},\mathbf{43},\mathbf{421},\mathbf{41^3},\mathbf{3^21},\mathbf{32^2},\mathbf{31^4},\mathbf{2^31},\mathbf{2^21^3},\mathbf{21^5},\mathbf{1^7}\}$;*

(2) *The set E_λ is a union of two two-sided cells of $\widetilde{C}_3$ if $\lambda=\mathbf{321^2}$;*

(3) *The set E_λ is infinite if $\lambda\in\{\mathbf{7},\mathbf{61},\mathbf{52},\mathbf{51^2},\mathbf{43},\mathbf{421},\mathbf{41^3},\mathbf{3^21},\mathbf{32^2},\mathbf{31^4}\}$;*

(4) *The set E_λ is finite if $\lambda\in\{\mathbf{321^2},\mathbf{2^31},\mathbf{2^21^3},\mathbf{21^5},\mathbf{1^7}\}$;*

(5) *The numbers $n(\lambda)$ for all $\lambda\in\Lambda_7$ are listed in the following Tab 1, where $n_1(\mathbf{321^2})=\mathbf{4}$ and $n_2(\mathbf{321^2})=\mathbf{1}$;*

(6) *Each left cell of $\widetilde{C}_3$ is left-connected;*

(7) *Each two-sided cell of $\widetilde{C}_3$ is two-sided-connected.*

Tab. 1　The numbers $n(\lambda)$ for all $\lambda\in\Lambda_7$

λ	$\mathbf{7}$	$\mathbf{61}$	$\mathbf{52}$	$\mathbf{51^2}$	$\mathbf{43}$	$\mathbf{421}$	$\mathbf{41^3}$	$\mathbf{3^21}$	$\mathbf{32^2}$	$\mathbf{321^2}$	$\mathbf{31^4}$	$\mathbf{2^31}$	$\mathbf{2^21^3}$	$\mathbf{21^5}$
$n(\lambda)$	48	24	24	24	12	12	12	12	8	5	6	3	3	1

Theorem 4.1 will be proved in section 5 by case-by-case argument. Let $\Delta:=\{\mathbf{421},\mathbf{3^21},\mathbf{321^2}\}$. For any $\lambda\in\Lambda_7$, the set E_λ with $\lambda\notin\Delta$ has been described in [7-11]. We need only to consider the sets $E_{\mathbf{421}}$, $E_{\mathbf{3^21}}$ and $E_{\mathbf{321^2}}$.

Let $\lambda\in\Delta$. We will find a subset F_λ of E_λ such that the set F_λ has a non-empty intersection with each left-connected component of E_λ (by Theorem 3.2 and various left star operations) and that no two elements in F_λ are in the same left cell of $\widetilde{C}_3$ (by Lemma 2.4 and Lemma 3.3). Then by Lemma 2.5 and Lemma 3.3 we see that the set F_λ can be seen as a representative set for the left cells of $\widetilde{C}_3$ in E_λ. Then the number $n(\lambda)$ is just the cardinal of the set F_λ. We usually prove that the set E_λ forms a single two-sided cell of $\widetilde{C}_3$ by proving that the set E_λ is two-sided-connected. Lemma 1.1 will be used in proving that E_λ is a union of two two-sided cells of $\widetilde{C}_3$.

5 The proof of Theorem 4.1

Theorem 4.1 will be proved by case-by-case argument in the following part of this section (see Proposition 5.3, Proposition 5.6 and Proposition 5.10).

Case 1 The set $E_{\mathbf{421}}$

By Lemma 3.1 we see that for any $w \in \widetilde{C}_3$, $w \in E_{\mathbf{421}}$ if and only if w satisfies one of the following conditions (a)-(c):

(a) There exist some pairwise not 6-dual i, j, k in [6] with i, j are both w-tame heads and k is w-wild head, satisfying $\bar{k} \prec_w i \prec_w j \prec_w k$.

(b) There exist some pairwise not 6-dual i, j, k in [6] with i is w-tame head and j, k are both w-wild heads, satisfying $3 < (j)w < 7$ and either $\bar{j} \not\prec_w i \prec_w j \prec_w k$ or $\bar{j} \not\prec_w \bar{i} \prec_w j \prec_w k$.

(c) There exist some pairwise not 6-dual i, j, k in [6] with i, j, k are all w-wild heads, satisfying:

(c1) $j \prec_w k$ but i, j, k is not a w-chain;

(c2) $3 < (i)w < 7$ and $3 < (j)w < 7$;

(c3) Either $j \not\prec_w i$ or $7 \not\prec_w k$.

Let $E^1_{\mathbf{421}}$ (resp., $E^4_{\mathbf{421}}$) be the set of all $w \in E_{\mathbf{421}}$ satisfying the condition (a) (resp., condition (c)). Let $E^2_{\mathbf{421}}$ (resp., $E^3_{\mathbf{421}}$) be the set of all $w \in E_{\mathbf{421}}$ satisfying the condition (b) with $\bar{j} \not\prec_w i \prec_w j$ (resp., $\bar{j} \not\prec_w \bar{i} \prec_w j$). We have $E_{\mathbf{421}} = E^1_{\mathbf{421}} \cup E^2_{\mathbf{421}} \cup E^3_{\mathbf{421}} \cup E^4_{\mathbf{421}}$.

Proposition 5.1 *The set $E_{\mathbf{421}}$ is infinite.*

Proof. The result follows from the fact that $\{w|w = [6 + 7p, 3, 2], p \in \mathbb{N}\} \subset E_{\mathbf{421}}$. □

Lemma 5.2 *There exists a subset $F_{\mathbf{421}}$ of $E_{\mathbf{421}}$ such that each left-connected component of $E_{\mathbf{421}}$ contains some $w \in F_{\mathbf{421}}$.*

Proof. We need only to find a subset $F_{\mathbf{421}}$ of $E_{\mathbf{421}}$ such that for any $w' \in E_{\mathbf{421}}$, there exists some $w \in F_{\mathbf{421}}$ such that w', w are in the same left-connected component of $E_{\mathbf{421}}$.

(i) Let $F'_{\mathbf{421}}$ be the set of all $w \in E_{\mathbf{421}}$ satisfying the following condition:

(a$'$) $3 < (5)w < (3)w < 7$ and $(3)w < (1)w < (3)w + 7$.

We see from Theorem 3.2 that by repeatedly left multiplying various elements with the form $t_{i,j}$ on any $w' \in E^1_{\mathbf{421}} \cup E^2_{\mathbf{421}}$, we can get some $w \in F'_{\mathbf{421}}$ such that w', w are in the same left-connected component of $E_{\mathbf{421}}$. One can check that

$$F'_{\mathbf{421}} = \{[6, 3, 5], [8, 3, 5], [9, 3, 6], [10, 2, 6], [11, 2, 6], [12, 3, 6]\}.$$

(ii) Let $F''_{\mathbf{421}}$ be the set of all $w \in E_{\mathbf{421}}$ satisfying the following condition:

(a$''$) $3 < (2)w < (3)w < 7$ and $(2)w < (1)w < (2)w + 7$.

Similarly. For any $w' \in E^3_{\mathbf{421}} \cup E^4_{\mathbf{421}}$, we can get some $w \in F''_{\mathbf{421}}$ such that w', w are in the same left-connected component of $E_{\mathbf{421}}$. We have

$$F''_{\mathbf{421}} = \{[5, 4, 6], [6, 4, 5], [8, 4, 5], [9, 4, 6], [10, 5, 6], [11, 5, 6]\}.$$

Let $F_{\mathbf{421}} = F'_{\mathbf{421}} \cup F''_{\mathbf{421}}$. The result is proved. □

Proposition 5.3 (1) *The infinite set $E_{\mathbf{421}}$ is two-sided-connected, which forms a single two-sided cell of $\widetilde{C}_3$;*

(2) *The set $E_{\mathbf{421}}$ contains 12 left cells of $\widetilde{C}_3$, each of which is left-connected.*

Proof. Let $x_1 = [9,4,6], x_2 = [8,4,5], x_3 = [6,4,5]$ and $x_4 = [10,2,6]$. We see that $F_{\mathbf{421}} = M(x_1) \cup M(x_2) \cup M(x_3) \cup M(x_4)$ (see Fig. 1). It implies by $x_2 = x_1 t_1 = x_3 t_0$ and $[9,3,6] = x_4 t_2 \in M(x_2)$ that the set $F_{\mathbf{421}}$ is right-connected.

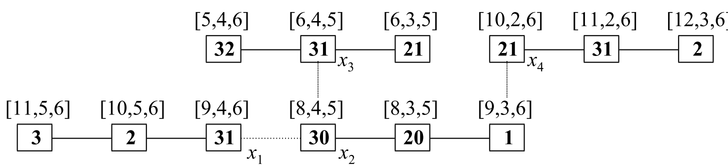

Fig. 1 The right-connectedness of the set $F_{\mathbf{421}}$

We see from Fig. 1 that no two elements in $F_{\mathbf{421}}$ have the same generalized tabloids. The result follows from Lemma 2.5, Lemma 3.3, Proposition 5.1 and Lemma 5.2. □

Case 2 The set $E_{\mathbf{3^21}}$

Let $w \in \widetilde{C}_3$. We see from Lemma 3.1 that $w \in E_{\mathbf{3^21}}$ if and only if w satisfies the following condition (a):

(a) There exist some pairwise not 6-dual i, j, k in $[6]$ with $j < k$ and i is w-tame head, satisfying at least one of the following conditions:

(a1) $\bar{j} \prec_w i \prec_w k$ and $0 < (j)w < (k)w < 7$;

(a2) $\bar{j} \prec_w i \prec_w j$ and $0 < (j)w < 7 < (k)w < (j)w + 7$;

(a3) $\bar{j} \prec_w i \prec_w k$ and $0 < (j)w < 7 < (k)w < (j)w + 7$;

(a4) $7 < (j)w < (k)w < (j)w + 7$.

Let $E^1_{\mathbf{3^21}}$ (resp., $E^2_{\mathbf{3^21}}$, $E^3_{\mathbf{3^21}}$, $E^4_{\mathbf{3^21}}$) be the subset of $E_{\mathbf{3^21}}$, elements of which satisfy condition (a1) (resp., condition (a2), condition (a3), condition (a4)) of condition (a). We have $E_{\mathbf{3^21}} = E^1_{\mathbf{3^21}} \cup E^2_{\mathbf{3^21}} \cup E^3_{\mathbf{3^21}} \cup E^4_{\mathbf{3^21}}$.

Proposition 5.4 *The set $E_{\mathbf{3^21}}$ is infinite.*

Proof. The result follows from the fact that $\{[8+7p, 9+7p, 3] | p \in \mathbb{N}\} \subset E_{\mathbf{3^21}}$. □

Lemma 5.5 *There exists a subset $F_{\mathbf{3^21}}$ of $E_{\mathbf{3^21}}$ such that each left-connected component of $E_{\mathbf{3^21}}$ contains some $w \in F_{\mathbf{3^21}}$.*

Proof. We will find a subset $F_{\mathbf{3^21}}$ of $E_{\mathbf{3^21}}$ satisfying the requirement above.

(i) Let $F^1_{\mathbf{3^21}}$ be the set of all $w \in E^1_{\mathbf{3^21}}$ satisfying the following condition:

(b1) $0 < (5)w < (6)w < (4)w < (2)w < 7$ and $(4)w > 3$.

By applying various left star operations on any $w' \in E^1_{\mathbf{3^21}}$, we can get some $w \in F^1_{\mathbf{3^21}}$ such that w', w are in the same left-connected component of $E_{\mathbf{3^21}}$.

(ii) Let $F^2_{\mathbf{3^21}}$ be the set of all $w \in E^2_{\mathbf{3^21}}$ satisfying the following condition:

(b2) $0 < (6)w < (4)w < (1)w < 7 < (2)w < (1)w + 7$ and $(4)w > 3$.

Similarly, for any $w' \in E^2_{\mathbf{3^21}}$, we can get some element $w \in F^2_{\mathbf{3^21}}$ such that w', w are in the same left-connected component of $E_{\mathbf{3^21}}$.

(iii) Let $F^3_{\mathbf{3^21}}$ be the set of all $w \in E^3_{\mathbf{3^21}}$ satisfying the following condition:

(b3) $0 < (6)w < (4)w < 7 < (2)w < (1)w + 7$ and $(4)w > 3$.

One can check that for any $w' \in E^3_{\mathbf{3^21}}$, there exists some $w \in F^3_{\mathbf{3^21}}$ such that w' and w are in the same left-connected component of $E_{\mathbf{3^21}}$. We have

$$F^3_{\mathbf{3^21}} = \{[3,9,1],[4,9,1],[5,11,1],[5,10,1],[3,8,2],[4,8,2],$$

$$[5,8,3],[6,9,3],[6,10,2],[6,11,2],[6,12,3]\}.$$

(iv) Let $F''^4_{\mathbf{3^21}}$ be the subset of $E^4_{\mathbf{3^21}}$, the element of which satisfies the following condition:

(b$''$4) $3 < (4)w < 7 < (1)w < 14$ and $(1)w < (2)w < (1)w + 7$.

We see from Theorem 3.2 that for any $w' \in E^4_{\mathbf{3^21}}$, there exists some $w \in F''^4_{\mathbf{3^21}}$ such that w, w' are in the same left-connected component of $E_{\mathbf{3^21}}$.

Let $F'^4_{\mathbf{3^21}}$ be the subset of $F''^4_{\mathbf{3^21}}$, elements of which satisfy the following condition:

(b$'$4) $3 < (4)w < 7 < (1)w < (2)w < 14$.

Let $w' \in F''^4_{\mathbf{3^21}}$, we have $(1)w' < (2)w' < (1)w' + 7$ and $7 < (1)w' < 14$. If $(2)w' < 14$, we have $w' \in F'^4_{\mathbf{3^21}}$. Suppose that $(2)w' > 14$, we see that $7 < (2)w' - 7 < (1)w' < 14$. Let $w = t_0t_1t_2t_3t_2w'$, we have $w \in F'^4_{\mathbf{3^21}}$ and $l(w') = l(t_0t_1t_2t_3t_2) + l(w)$. Then w' and w are in the same left-connected component of $E_{\mathbf{3^21}}$ by Lemma 2.6.

Let $F^4_{\mathbf{3^21}}$ be the set of all $w \in F'^4_{\mathbf{3^21}}$ satisfying the following condition:

(b4) $3 < (4)w < 7 < (1)w < (2)w < (3)w + 7$.

Let $w' \in F'^4_{\mathbf{3^21}}$, we have $0 < (2)w' - 7 < 7$ and $0 < (3)w' < 4$. If $(2)w' - 7 < (3)w'$, then $w' \in F^4_{\mathbf{3^21}}$. Suppose that $(2)w' - 7 > (3)w'$. Let $w = t_0t_1t_2t_3t_2w'$, we see that $w \in F^3_{\mathbf{3^21}}$ and w' and w are in the same left-connected component of $E_{\mathbf{3^21}}$. By $0 < (1)w - 7 < (2)w - 7 < (3)w < 4$ we have $F^4_{\mathbf{3^21}} = \{[8,9,3]\}$.

(v) For any $w' \in F^1_{\mathbf{3^21}}$, let $w = t_2t_3t_2t_1t_0w'$, we have $w \in F^3_{\mathbf{3^21}}$. Then w' and w are in the same left-connected component of $E_{\mathbf{3^21}}$ by Lemma 2.6. We see by the conditions (b2) and (b3) that $F^2_{\mathbf{3^21}} \subseteq F^3_{\mathbf{3^21}}$. Let $F_{\mathbf{3^21}} = F^3_{\mathbf{3^21}} \cup \{[8,9,3]\}$, the result is proved. □

Proposition 5.6 (1) *The infinite set $E_{\mathbf{3^21}}$ is two-sided-connected, which forms a single two-sided cell of $\widetilde{C}_3$.*

(2) *The set $E_{\mathbf{3^21}}$ contains 12 left cells of $\widetilde{C}_3$, each of which is left-connected.*

Proof. Let $x_1 = [5,8,3], x_2 = [4,8,2], x_3 = [6,10,2]$ and $x_4 = [4,9,1]$. We have $F^3_{\mathbf{3^21}} = M(x_1) \cup M(x_2) \cup M(x_3) \cup M(x_4)$ (see Fig. 2). We see from $x_2 = x_1t_2 = x_4t_1$, $[5,10,1] = x_3t_1 \in M(x_4)$ and $[6,9,3] = [8,9,3]t_0 \in M(x_1)$ that the set $F_{\mathbf{3^21}}$ is right-connected.

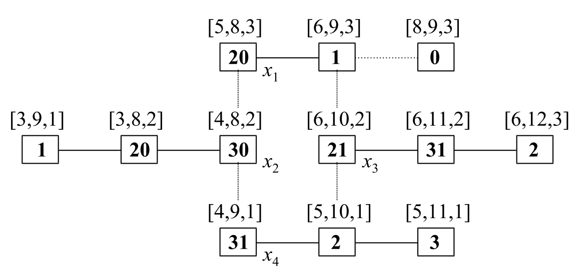

Fig. 2 The right-connectedness of the set $F_{\mathbf{3^21}}$

By Fig. 2 we see that no two elements in $F_{\mathbf{3^21}}$ have the same generalized tabloids. The result follows from Lemma 2.5, Lemma 3.3, Proposition 5.4 and Lemma 5.5. $\square$

Case 3 The set $E_{\mathbf{321^2}}$

Let $w \in \widetilde{C}_3$. We see that $w \in E_{\mathbf{321^2}}$ if and only if w satisfies the following condition (a):

(a) There exist some pairwise not 6-dual i, j, k in [6] with i is w-tame head and j and k are both w-wild heads, satisfying:

(a1) $(j)w > 7$ and j, k are w-uncomparable;

(a2) There exactly one of the following conditions holds:

(a21) The integers of $\{i, \bar{i}, k, 7\}$ are pairwise w-uncomparable;

(a22) $(k)w > 7$ and the integers of $\{i, \bar{i}, j, k\}$ are pairwise w-uncomparable.

Let $E^1_{\mathbf{321^2}}$ (resp., $E^2_{\mathbf{321^2}}$) be the set of all $w \in E_{\mathbf{321^2}}$ satisfying the condition (a21) (resp., condition (a22)) of the condition (a). Then we have $E_{\mathbf{321^2}} = E^1_{\mathbf{321^2}} \cup E^2_{\mathbf{321^2}}$.

Proposition 5.7 *The set $E_{\mathbf{321^2}}$ is finite.*

Proof. For any $w \in E_{\mathbf{321^2}}$ and $t \in [6]$, we always have $-4 < (t)w < 21$. $\square$

Lemma 5.8 $(E^1_{\mathbf{321^2}})^{-1} = E^1_{\mathbf{321^2}}$ *and* $(E^2_{\mathbf{321^2}})^{-1} = E^2_{\mathbf{321^2}}$.

Proof. By closely observing the matrix forms of the elements in $E_{\mathbf{321^2}}$, we see that if $w \in E_{\mathbf{321^2}}$ satisfying the condition (a21) (resp., the condition (a22)) of the condition (a), so does w^{-1}. $\square$

Lemma 5.9 *There exists a subset $F^1_{\mathbf{321^2}}$ (resp.,$F^2_{\mathbf{321^2}}$) of $E^1_{\mathbf{321^2}}$ (resp.,$E^2_{\mathbf{321^2}}$) such that each left-connected component of $E^1_{\mathbf{321^2}}$ (resp.,$E^2_{\mathbf{321^2}}$) contains some w in $F^1_{\mathbf{321^2}}$ (resp.,$F^2_{\mathbf{321^2}}$).*

Proof. (i) Let $F^1_{\mathbf{321^2}}$ be the set of all $w \in E^1_{\mathbf{321^2}}$ satisfying the following condition:

(b1) $3 < (3)w < (5)w < 7 < (6)w < (3)w + 7$.

By applying various left star operations on any $w' \in E^1_{\mathbf{321^2}}$, we can get some $w \in F^1_{\mathbf{321^2}}$ such that w' and w are in the same left-connected component of $E_{\mathbf{321^2}}$. Then we have

$F^1_{\mathbf{321^2}} = \{[-1, 2, 4], [-2, 1, 4], [-3, 1, 5], [-4, 1, 5]\}$.

(ii) Let $F^2_{\mathbf{321^2}}$ be the set of all $w \in E^2_{\mathbf{321^2}}$ satisfying the following condition:

(b2) $3 < (4)w < 7 < (5)w < (6)w < (3)w + 7$.

For any $w' \in E^2_{\mathbf{321^2}}$, we see that there exists some $w \in F^2_{\mathbf{321^2}}$ in a left-connected component of $E_{\mathbf{321^2}}$ containing w'. We see from $0 < (5)w - 7 < (6)w - 7 < (3)w < 4$ that $F^2_{\mathbf{321^2}} = \{[-2, -1, 3]\}$. $\square$

Proposition 5.10 (1) *The finite set $E_{\mathbf{321^2}}$ is a union of two two-sided cells $E^1_{\mathbf{321^2}}$ and $E^2_{\mathbf{321^2}}$ of $\widetilde{C}_3$, each of which is two-sided-connected*;

(2) *The set $E^1_{\mathbf{321^2}}$ contains 4 left cells of $\widetilde{C}_3$, each of which is left-connected*;

(3) *The set $E^2_{\mathbf{321^2}}$ is left-connected, which forms a single left cell of $\widetilde{C}_3$.*

Proof. It implies by $F^1_{\mathbf{321^2}} = M([-2, 1, 4]) \cup \{[-1, 2, 4]\}$ and $[-2, 1, 4] = [-1, 2, 4]t_1$ that the set $F^1_{\mathbf{321^2}}$ is right-connected.

By $\mathcal{R}([-2, -1, 3]) = \{t_0\}$ and Fig. 3 we see that no two elements in $F_{\mathbf{321^2}}$ have the same generalized tabloids. The result follows from Lemma 1.1, Lemma 2.5, Lemma 3.3, Proposition 5.7 and Lemmas 5.8—5.9. $\square$

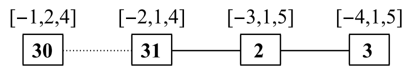

Fig. 3 The right-connectedness of the set $F^{1}_{\mathbf{321^2}}$

So far we have proved all the assertions in Theorem 4.1.

Acknowledgment The author thanks Professor J. Y. Shi of the Department of Mathematics in East China Normal University for his hospitality and crucial guidance.

[References]

[1] LUSZTIG G. Hecke algebra with unequal parameters [M]. Providence: American Mathmatical Society, 2003.

[2] SHI J Y. The cells of the affine Weyl group $\widetilde{C}_n$ in a certain quasi-split case [J]. J Algebra, 2015(422): 697-729.

[3] HUANG Q, SHI J Y. Some cells in the weighted Coxeter group $(\widetilde{C}_n, \widetilde{l}_{2n+1})$ [J]. J Algebra, 2013(395): 63-81.

[4] 岳明仕. 拟分裂情形下仿射 Weyl 群 $\widetilde{C}_4$ 的胞腔 [J]. 华东师范大学学报: 自然科学版, 2013(1): 61-75.

[5] SHI J Y. The Kazhdan-Lusztig cells in certain affine Weyl groups [M]. Berlin: Springer-Verlag, 1986.

[6] GREENE C. Some partitions associated with a partially ordered set [J]. J Comb Theory (A), 1976(20): 69-79.

[7] HUANG Q. Left cells in the weighted Coxeter group $\widetilde{C}_n$ [J]. J East China Norm Univ, 2013(1): 91-103.

[8] 黄谦. 某些加权的 Coxeter 群的左胞腔 [D]. 上海: 华东师范大学数学系, 2013.

[9] YUE M S. Cells of the affine Weyl group $\widetilde{C}_n$ in quasi-split case [J]. J East China Norm Univ, 2014(3): 77-92.

[10] YUE M S. Cells of the weighted Coxeter group $\widetilde{C}_n$ [J]. J East China Norm Univ, 2015(3): 38-46.

[11] YUE M S. Left cells of weighted Coxeter group $(\widetilde{C}_n, \widetilde{l}_{2n})$ [J]. Advances in Mathematics (China), 2015, 44(4): 505-518.

(责任编辑 林 磊)

(上接第 26 页)

[6] BOBODZHANOV A A, SAFONOV V F. An internal transition layer in a linear optimal control problem [J]. Differential Equations, 2001, 37(3): 332-345.

[7] VASIL'EVA A B, BUTUZOV V F. The Asymptotic Method of Singularly Perturbed Theory [M]. Moscow: Nauka, 1990.

[8] VASIL'EVA A B, BUTUZOV V F. Asymptotic Expansions of Singularly Perturbed Differential Equations [M]. Moscow: Nauka, 1973.

[9] VASIL'EVA A B. Contrast structures of step-like type for a second-order singularly perturbed quasilinear differential equation [J]. Zh Vychisl Mat Mat Fiz, 1995, 35: 520-531.

[10] VASIL'EVA A B. Contrast structure in the three systems of singularly perturbed [J]. Comput Math Math Phys, 1999, 39: 2007-2018.

[11] 倪明康, 林武忠. 奇异摄动问题中的渐近理论 [M]. 北京: 高等教育出版社, 2009.

[12] LIN X B. Construction and asymptotic stability of structurally stable internal layer solutions [J], Trans Amer Math Soc, 2001, 353: 2983-3043.

[13] TIN S K, KOPELL N, JONES C K R T. Invariant manifolds and singularly perturbed boundary value problems [J]. SIAM J Numer Anal, 1994, 31: 1558-1576.

[14] NI M K, WANG Z M. On higher-dimensional contrast structure of singularly perturbed Dirichlet problem [J]. Science China Mathematics, 2012, 55(3): 495-507.

[15] FENICHEL N. Geometric singular perturbation theory for ordinary differential equations [J]. Journal of Differential Equations, 1979, 31(1): 53-98.

(责任编辑 林 磊)