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A note on Kronecker's double sum and non-holomorphic Eisenstein series

SHEN Li-chien

(Department of Mathematics, University of Florida, Gainesville FL 32611-8105, USA)

Abstract: A family of non-holomorphic Eisenstein series of weight k and level N is generated from twisting of the Kronecker double series by Dirichlet characters and from which we will derive its representation in terms of the Whittaker function and the functional equation for the Eisenstein series.

Keywords: Dirichlet character; conductor; Gaussian sum; twisted Poisson summation formula; Whittaker function

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关于克罗内克二重和与非全纯艾森斯坦级数的一个注记

沈力健

(佛罗里达大学 数学系, 佛罗里达 盖恩斯维尔 32611-8105, 美国)

摘要: 由狄利克雷特征对克罗内克二重级数的扭曲生成了一族权为 k 级别为 N 的非全纯艾森斯坦级数. 由此, 我们推导出了它的惠特克函数表示以及艾森斯坦级数的泛函方程.

关键词: 狄利克雷特征; 导子; 高斯和; 扭曲的泊松求和公式; 惠特克函数

According to Andre Weil (cf. [5, p. 69]), the double series with which Kronecker was dealing in his later years were all of the form

$$\sum_{w \in W} \Phi(w)(\bar{x} + \bar{w})^a |x + w|^{-2s},$$

where W is a lattice in the complex plane and u, v are the generators of W , $w = mu + nv$, a an integer, and Φ is a character of the additive group W :

$$\Phi(mu + nv) = e^{2i\pi(m\mu + n\nu)},$$

where $0 \leq \mu, \nu < 1$.

In this note, we shall study the twisting of the above series by a pair of primitive Dirichlet characters.

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作者简介: 沈力健, 男, 教授, 研究方向为函数论. E-mail: shen@ufl.edu.

The paper is organized as follows.

In Section 0, the relevant properties of the Dirichlet characters will be given. In Section 1, we introduce a family of non-holomorphic Eisenstein series. In Sections 2, 3 and 4, we deal with the Whittaker functions and an integral representation of them. Fourier transform technique is employed in Section 5 to express the non-holomorphic Eisenstein series in terms of the Whittaker functions. The last three sections deal with lifting the Eisenstein series to the group $GL^+(2, \mathbb{R})$ and derivation of a functional equation for Eisenstein series, and we end the paper with a brief account of the connection between the Kronecker's double sum and the imaginary quadratic fields.

Notation. The symbols \mathbb{Z} , \mathbb{R} , \mathbb{C} denote respectively, the set of integers, real numbers and complex numbers. For a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $\operatorname{Re} z = x$, the real part of z .

0 Basic properties of characters modulo N

Let N be a positive integer, and χ a Dirichlet character modulo N . Extend χ to the set integers \mathbb{Z} so that, for all integers m and n ,

- (1) $\chi(1) = 1$;
- (2) $\chi(n + N) = \chi(n)$;
- (3) $\chi(mn) = \chi(m)\chi(n)$;
- (4) $\chi(n) = 0$ if (n, N) , the gcd of n and N , is > 1 .

Let N' be a positive integer which is divisible by N . For any character χ modulo N , we can form a character χ' modulo N' as follows:

$$\chi'(k) = \begin{cases} \chi(k), & \text{if } (k, N') = 1, \\ 0, & \text{if } (k, N') > 1. \end{cases}$$

We say that χ' is induced by the character χ . Let χ be a character modulo N . If there is a proper divisor d of N and a character modulo d which induces χ , then the character χ is said to be non-primitive, otherwise it is called primitive, and we say N is the conductor of χ if it is primitive modulo N . We note that if N is the conductor of a character, then either $N = 1$ or $N \geq 3$.

The Gaussian sum $g_k(\chi)$ associated with the character χ is defined as

$$g_k(\chi) = \sum_{n=1}^{N-1} \chi(n) e^{2ikn\pi/N}.$$

We denote $g(\chi) = g_1(\chi)$. We need a lemma.

Lemma 0.1^{[1]5} Let χ be a primitive character modulo N . Then for any integer k

$$g_k(\chi) = \overline{\chi(k)} g(\chi),$$

or equivalently,

$$\chi(m) = \frac{1}{g(\overline{\chi})} \sum_{k=1}^{N-1} \overline{\chi(k)} e^{2i\pi km/N}.$$

1 A non-holomorphic Eisenstein series

Let $\mathfrak{h} = \{\tau = x + iy : y > 0\}$ be the upper half-plane.

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, N \mid c \right\}.$$

We shall consider the following series:

$$\sum_{m,n=-\infty}^{\infty} \frac{\varphi(m)\overline{\psi(n)}}{(m\tau + n)^k} \cdot \frac{y^s}{|m\tau + n|^{2s}}, \quad (1.1)$$

where we assume that one of the conductors of φ , ψ is ≥ 3 . This means $\varphi(0)\psi(0) = 0$, thus, the term in the summands corresponding to $(m, n) = (0, 0)$ is not present.

Using Lemma 0.1, we rewrite the sum (1.1) in terms of the Kronecker double sums for $x = 0$:

$$\begin{aligned} & \sum_{m,n=-\infty}^{\infty} \frac{\varphi(m)\overline{\psi(n)}}{(m\tau + n)^k} \cdot \frac{y^s}{|m\tau + n|^{2s}} \\ &= \frac{1}{g(\overline{\varphi})g(\psi)} \sum_{k=1}^{M-1} \sum_{l=1}^{N-1} \overline{\varphi(k)}\psi(l) \sum_{m,n=-\infty}^{\infty} \frac{e^{2i\pi(mk/M+nl/N)}}{(m\tau + n)^k} \cdot \frac{y^s}{|m\tau + n|^{2s}} \\ &= \frac{y^s}{g(\overline{\varphi})g(\psi)} \sum_{k=1}^{M-1} \sum_{l=1}^{N-1} \overline{\varphi(k)}\psi(l) \left(\sum_{m,n=-\infty}^{\infty} e^{2i\pi(mk/M+nl/N)} (m\overline{\tau} + n)^k |m\tau + n|^{-2(s+k)} \right). \end{aligned}$$

We construct a family of non-holomorphic Eisenstein series.

Theorem 1.1 Suppose φ and ψ are primitive characters of modulo M and N , respectively, $M \geq 3$ or $N \geq 3$, and $\varphi\psi(-1) = (-1)^k$. Then

$$E_{\varphi,\psi}(\tau; k, s) := \sum_{m,n=-\infty}^{\infty} \frac{\varphi(m)\overline{\psi(n)}y^s}{(mN\tau + n)^k |mN\tau + n|^{2s}}$$

converges absolutely on $\{s : \operatorname{Re} s > 1 - \frac{k}{2}\}$ and satisfies

$$E_{\varphi,\psi}\left(\frac{a\tau + b}{c\tau + d}; k, s\right) = \varphi(d)\psi(d)(c\tau + d)^k E_{\varphi,\psi}(\tau; k, s)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(MN)$, where $\tau = x + iy$, $y > 0$ and $\varphi\psi(-1) = (-1)^k$.

Proof We note that since $\operatorname{Re} s > 1 - \frac{k}{2}$, the double sum converges absolutely.

We observe that since ψ has conductor N ,

$$\overline{\psi(mNb + nd)}\psi(d) = \overline{\psi(n)}.$$

Since $MN \mid c$ and φ has conductor M ,

$$\overline{\varphi(a)}\varphi\left(ma + n\frac{c}{N}\right) = \varphi(m),$$

$$\frac{\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)^s}{\left|mN\left(\frac{a\tau+b}{c\tau+d}+n\right)\right|^{2s}} = \frac{y^s}{\left|(ma+n\frac{c}{N})N\tau+(mNb+nd)\right|^{2s}}.$$

From which we derive

$$\begin{aligned} E_{\varphi,\psi}\left(\frac{a\tau+b}{c\tau+d}; k, s\right) &= (c\tau+d)^k \sum_{m,n=-\infty}^{\infty} \frac{\varphi(m)\overline{\psi(n)}}{\left((ma+n\frac{c}{N})N\tau+(mNb+nd)\right)^k} \cdot \frac{y^s}{\left|(ma+n\frac{c}{N})N\tau+(mNb+nd)\right|^{2s}} \\ &= (c\tau+d)^k \sum_{m,n=-\infty}^{\infty} \frac{\overline{\varphi(a)}\varphi(ma+n\frac{c}{N})\overline{\psi(mNb+nd)}\psi(d)}{\left((ma+n\frac{c}{N})N\tau+(mNb+nd)\right)^k} \cdot \frac{y^s}{\left|(ma+n\frac{c}{N})N\tau+(mNb+nd)\right|^{2s}}. \end{aligned}$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(MN)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $\begin{pmatrix} a & bN \\ c/N & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and from this fact and $\overline{\varphi(a)} = \varphi(d)$, we deduce

$$E_{\varphi,\psi}\left(\frac{a\tau+b}{c\tau+d}; k, s\right) = \varphi(d)\psi(d)(c\tau+d)^k E_{\varphi,\psi}(\tau; k, s).$$

We remark that if $\varphi\psi(-1) \neq (-1)^k$, then $E_{\varphi,\psi}(\tau; k, s)$ is identically zero.

Corollary 1.2 Define

$$A_{\varphi,\psi}(\tau; k, s) := \sum_{m,n=-\infty}^{\infty} \frac{\varphi(m)\overline{\psi(n)}y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}}.$$

Then

$$A_{\varphi,\psi}\left(\frac{a\tau+b}{c\tau+d}; k, s\right) = \varphi(d)\psi(d)A_{\varphi,\psi}(\tau; k, s) \left(\frac{c\tau+d}{|c\tau+d|}\right)^k$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(MN)$.

2 Eigenfunctions of Δ , Δ_μ and Whittaker functions

Let

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial \theta \partial x}$$

and

$$\Delta_\mu = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\mu y \frac{\partial}{\partial x}.$$

Lemma 2.1 Suppose $f(x, y, \theta) = e^{2\pi i \lambda x} Y(y) e^{i\mu \theta}$, $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, satisfies the equation

$$\Delta f = s(1-s)f$$

and $|Y(y)| = O(y^N)$ for some $N > 0$ as $y \rightarrow \infty$. Then

$$Y(y) = cW_{\frac{\operatorname{sgn}(\lambda)\mu}{2}, s-\frac{1}{2}}(4|\lambda|\pi y)$$

for some constant c , and

$$u := e^{2\pi i \lambda x} W_{\frac{\operatorname{sgn}(\lambda)\mu}{2}, s-\frac{1}{2}}(4|\lambda|\pi y)$$

satisfies

$$\Delta_\mu u = s(1-s)u,$$

where $\operatorname{sgn}(\lambda) = \frac{\lambda}{|\lambda|}$.

The function $W_{\alpha, \nu}(y)$ is called the Whittaker function which is a solution of the differential equation

$$u'' + \left(-\frac{1}{4} + \frac{\alpha}{z} + \frac{\frac{1}{4} - \nu^2}{z^2}\right)u = 0,$$

where $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{C}$.

Since

$$W_{\alpha, \nu}(y) = y^\alpha e^{-y}(1 + O(1/y)),$$

$Y(y)$ is rapidly decreasing as $y \rightarrow \infty$.

We also mention the fact:

$$W_{\alpha, \nu} = W_{\alpha, -\nu}. \quad (2.1)$$

An exposition on Whittaker functions can be found in [6, Chapter XVI].

3 An integral representation of Whittaker functions

For $\operatorname{Re} s > \frac{1}{2}$, k an integer and $\lambda \in \mathbb{R}$, define

$$w(\lambda; k, s) := \int_{-\infty}^{\infty} \left(\frac{|t+i|}{t+i} \right)^k \frac{e^{-i\lambda t}}{(1+t^2)^s} dt.$$

Then we have the following results.

Lemma 3.1 (1) $\overline{w(\lambda; k, s)} = (-1)^k w(\lambda; k, \bar{s})$;

(2) $w(-\lambda; k, s) = (-1)^k w(\lambda; -k, s)$;

(3) $w(0; k, s) = \frac{2}{i^k} \int_0^{\pi/2} \cos(k\theta) \cos^{2s-2} \theta d\theta = \pi(-i)^k 4^{1-s} \frac{\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)}$;

(4) $\int_{-\infty}^{\infty} \left(\frac{|t+\tau|}{t+\tau} \right)^k \left(\frac{y}{|t+\tau|^2} \right)^s e^{-2i\pi\lambda t} dt = y^{1-s} e^{2i\pi\lambda x} w(2\pi\lambda y; k, s)$, where $\tau = x + iy \in \mathfrak{h}$;

(5) $w(2\pi\lambda y; k, s) = \frac{(-i)^k \pi^s}{\Gamma(s+\operatorname{sgn}(\lambda)k/2)} |\lambda|^{s-1} y^{s-1} W_{\frac{\operatorname{sgn}(\lambda)k}{2}, s-\frac{1}{2}}(4\pi|\lambda|y)$.

Proof It is easy to verify (1)–(4) and see [3, p. 12 (30)] for evaluation of the integral in (3).

We sketch the proof of (5).

Recall the following fact (cf. [2, p. 86]): Suppose $c, d \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$. Then

$$\Delta_k \left(\frac{|cz + d|^k}{(cz + d)^k} \cdot \frac{y^s}{|cz + d|^{2s}} \right) = s(1-s) \frac{|cz + d|^k}{(cz + d)^k} \cdot \frac{y^s}{|cz + d|^{2s}}. \quad (3.1)$$

Choose $z = \tau \in \mathfrak{h}$, $c = 1$, $d = t$ in (3.1). Leaving the justification to readers, we have

$$\begin{aligned} \Delta_k \int_{-\infty}^{\infty} \left(\frac{|t + \tau|}{t + \tau} \right)^k \left(\frac{y}{|t + \tau|^2} \right)^s e^{-2i\pi\lambda t} dt &= \int_{-\infty}^{\infty} \Delta_k \left(\frac{|t + \tau|}{t + \tau} \right)^k \left(\frac{y}{|t + \tau|^2} \right)^s e^{-2i\pi\lambda t} dt \\ &= s(1-s) \int_{-\infty}^{\infty} \left(\frac{|t + \tau|}{t + \tau} \right)^k \left(\frac{y}{|t + \tau|^2} \right)^s e^{-2i\pi\lambda t} dt. \end{aligned}$$

From (4), we see that $e^{2i\pi\lambda x}(y^{1-s}w(2\pi\lambda y; k, s))$ is an eigenfunction of

$$\Delta_k u = s(1-s)u$$

and since

$$|y^{1-s}w(2\pi\lambda y; k, s)| = O(1)$$

as $y \rightarrow \infty$, we conclude from Lemma 2.1,

$$y^{1-s}w(2\pi\lambda y; k, s) = c(k, s, \lambda) W_{\frac{\operatorname{sgn}(\lambda)k}{2}, s-\frac{1}{2}}(4\pi|\lambda|y) \quad (3.2)$$

for some constant $c(k, s, \lambda)$.

We now show that $c(k, s, \lambda) = (\operatorname{sgn}(\lambda))^k |\lambda|^{s-1} c(\operatorname{sgn}(\lambda)k, s, 1)$.

Assume $\lambda > 0$. Let $y = 1$ in (3.2):

$$w(2\pi\lambda; k, s) = c(k, s, \lambda) W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\lambda).$$

In (3.2), set $\lambda = 1$ and replace y by λ :

$$w(2\pi\lambda; k, s) = c(k, s, 1) \lambda^{s-1} W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\lambda). \quad (3.3)$$

Hence,

$$c(k, s, \lambda) = \lambda^{s-1} c(k, s, 1)$$

if $\lambda > 0$.

Using the fact^{[3]262} that

$$x^{\mu-1/2} W_{\kappa, \mu}(x) \rightarrow \frac{\Gamma(2\mu)}{\Gamma(\mu - \kappa + 1/2)}$$

as $x \rightarrow 0^+$, we deduce from (3.3),

$$w(0; k, s) = c(k, s, 1) (4\pi)^{1-s} \frac{\Gamma(2s-1)}{\Gamma(s-k/2)}.$$

Hence, from (3),

$$c(k, s, 1) = \frac{(-i)^k \pi^s}{\Gamma(s + k/2)}.$$

Assume $\lambda < 0$. From (2) and (3.3),

$$w(2\pi\lambda; k, s) = (-1)^k w(2\pi|\lambda|; -k, s) = (-1)^k c(-k, s, 1) |\lambda|^{s-1} W_{\frac{-k}{2}, s-\frac{1}{2}}(4\pi|\lambda|).$$

Thus

$$w(2\pi\lambda; k, s) = \frac{(-i)^k \pi^s}{\Gamma(s + \operatorname{sgn}(\lambda)k/2)} |\lambda|^{s-1} W_{\frac{\operatorname{sgn}(\lambda)k}{2}, s-\frac{1}{2}}(4\pi|\lambda|).$$

Let

$$C(k, s, \lambda) = \frac{(-i)^k \pi^s}{\Gamma(s + \operatorname{sgn}(\lambda)k/2)}.$$

Thus, from (4) and (5), we have an integral representation of Whittaker functions. For $\operatorname{Re} s > 1/2$,

$$\int_{-\infty}^{\infty} \left(\frac{|t + \tau|}{t + \tau} \right)^k \left(\frac{y}{|t + \tau|^2} \right)^s e^{-2i\pi\lambda t} dt = C(k, s, \lambda) |\lambda|^{s-1} e^{2i\pi\lambda x} W_{\frac{\operatorname{sgn}(\lambda)k}{2}, s-\frac{1}{2}}(4\pi|\lambda|y).$$

4 Fourier transform of $(t + i)^{-k}(1 + t^2)^{-s}$

Define, for an integer k , $\operatorname{Re} s > \frac{1}{2} - \frac{k}{2}$ and $\lambda \in \mathbb{R}$,

$$W(\lambda; k, s) = \int_{-\infty}^{\infty} \frac{e^{-i\lambda t}}{(t + i)^k (1 + t^2)^s} dt.$$

We list well-known special cases:

$$W(\lambda; k, 0) = \begin{cases} \frac{2\pi(-i)^k}{(k-1)!} \lambda^{k-1} e^{-\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases} \quad (4.1)$$

where k is an integer ≥ 2 ;

$$W(2\pi\lambda; 0, s) = \begin{cases} \frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}, & \text{if } \lambda = 0, \\ \frac{2\pi^s}{\Gamma(s)} |\lambda|^{s-1/2} K_{s-1/2}(2\pi\lambda), & \text{if } \lambda \neq 0, \end{cases}$$

where $\operatorname{Re} s > 1/2$ and $K_\nu(z)$ is the K -Bessel function defined by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-\frac{z}{2}(t+1/t)} t^\nu \frac{dt}{t}$$

for $\operatorname{Re} z > 0$. It is an entire function with respect to the variable ν and

$$W_{0,\nu}(y) = \left(\frac{y}{\pi} \right)^{1/2} K_\nu(y/2).$$

See [1, p. 12 and p. 67] and [4, p. 270] for the derivations of the above identities. Notably, for an integer $n \geq 0$,

$$W(\lambda; 2n+1, -n) = \begin{cases} -2\pi L_n(2\lambda)e^{-\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

where $L_n(x)$, $n = 0, 1, 2, 3, \dots$, are the Laguerre polynomials.

It is easy to verify the following properties of $W(\lambda; k, s)$.

- Lemma 4.1** (1) $\overline{W(\lambda; k, s)} = (-1)^k W(\lambda; k, \bar{s})$;
 (2) $W(-\lambda; k, s) = (-1)^k W(\lambda; -k, s+k)$;
 (3) $W(0; k, s) = 2(-i)^k \int_0^{\pi/2} \cos(k\theta) \cos^{2s+k-2} \theta d\theta$;
 (4) $\int_{-\infty}^{\infty} \frac{1}{(t+\tau)^k} \left(\frac{y}{|t+\tau|^2} \right)^s e^{-2i\pi\lambda t} dt = y^{-s-k+1} e^{2i\pi\lambda x} W(2\pi\lambda y; k, s)$, where $\tau = x + iy \in \mathfrak{h}$;
 (5) $W(2\pi\lambda y; k, s) = C(k, s+k/2, \lambda) |\lambda|^{s-1+k/2} y^{s-1+k/2} W_{\frac{\text{sgn}(\lambda)k}{2}, s-\frac{1}{2}+\frac{k}{2}}(4\pi|\lambda|y)$;
 (6) $\Delta_k(y^{1-s-k/2} e^{2i\pi\lambda x} W(2\pi\lambda y; k, s)) = (s+k/2)(1-s-k/2) y^{1-s-k/2} e^{2i\pi\lambda x} W(2\pi\lambda y; k, s)$.

Proof To prove (5), we write

$$\frac{1}{(t+i)^k (1+t^2)^s} = \left(\frac{|t+i|}{t+i} \right)^k \frac{1}{(1+t^2)^{s+k/2}},$$

from which we derive

$$W(\lambda; k, s) = w \left(\lambda; k, s + \frac{k}{2} \right)$$

and (5) follows from Lemma 3.1(5).

For (6), we observe that

$$\frac{y^{k/2}}{(cz+d)^k} \cdot \frac{y^s}{|cz+d|^{2s}} = \frac{|cz+d|^k}{(cz+d)^k} \cdot \frac{y^{s+\frac{k}{2}}}{|cz+d|^{2s+k}}.$$

Then, from (2.1),

$$\Delta_k \left(\frac{y^{k/2}}{(cz+d)^k} \cdot \frac{y^s}{|cz+d|^{2s}} \right) = \left(s + \frac{k}{2} \right) \left(1 - s - \frac{k}{2} \right) \frac{y^{k/2}}{(cz+d)^k} \cdot \frac{y^s}{|cz+d|^{2s}}.$$

From (4), we derive that

$$\begin{aligned} \Delta_k(y^{1-s-k/2} e^{2i\pi\lambda x} W(2\pi\lambda y; k, s)) \\ = (s+k/2)(1-s-k/2) y^{1-s-k/2} e^{2i\pi\lambda x} W(2\pi\lambda y; k, s). \end{aligned} \quad (4.2)$$

5 Twisted Poisson summation formula and a representation of $A_{\varphi, \psi}(\tau; k, s)$ in terms of Whittaker functions

Recall

$$A_{\varphi, \psi}(\tau; k, s) := \sum_{m, n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}}.$$

Theorem 5.1 Suppose φ, ψ are primitive modulo $M \geq 3$ and $N \geq 3$, $\varphi\psi(-1) = (-1)^k$ and $\operatorname{Re} s > -\frac{k}{2} + 1$. Then

$$A_{\varphi, \psi}(\tau; k, s) = \frac{2g(\bar{\psi})}{N^{2s+k}} \sum_{l=-\infty}^{\infty} C(k, s+k/2, l) |l|^{s-1/2+k/2} \sigma_{1-k-2s}(l; \varphi, \psi) \\ \times \frac{1}{\sqrt{|l|}} e^{2i\pi lx} W_{\frac{\operatorname{sgn}(l)k}{2}, s-\frac{1}{2}+\frac{k}{2}}(4\pi|l|y), \quad (5.1)$$

where

$$\sigma_{\beta}(l; \varphi, \psi) = \sum_{mn=l, m \geq 1} m^{\beta} \varphi(m) \psi(n).$$

Proof We observe first that, since $\varphi\psi(-1) = (-1)^k$,

$$\sum_{m=-\infty}^{-1} \sum_{n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}} = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}}.$$

Hence,

$$\sum_{m, n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}} = 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}}. \quad (5.2)$$

Recall the χ -twisted Poisson summation formula^{[1]8}:

Let χ be a primitive character modulo N . Then

$$\sum_{n=-\infty}^{\infty} \chi(n) f(n) = \frac{g(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \hat{f}(n/N).$$

From which, with $f(t) = \frac{1}{(t+\tau)^k |t+\tau|^{2s}}$, we derive from Lemma 4.1(4),

$$\sum_{n=-\infty}^{\infty} \frac{\overline{\psi(n)}}{(\tau+n)^k |\tau+n|^{2s}} = \frac{g(\bar{\psi})}{N} y^{1-k-2s} \sum_{n=-\infty}^{\infty} \psi(n) e^{2i\pi nx/N} W(2\pi ny/N; k, s)$$

and from (5.2),

$$\sum_{m, n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}} \\ = \frac{g(\bar{\psi})}{N^{2s+k}} y^{1-s-k/2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} m^{1-k-2s} \varphi(m) \psi(n) e^{2i\pi mnx} W(2\pi mny; k, s) \\ = \frac{g(\bar{\psi})}{N^{2s+k}} y^{1-s-k/2} \sum_{l=-\infty}^{\infty} \left(\sum_{mn=l, m \geq 1} m^{1-k-2s} \varphi(m) \psi(n) \right) e^{2i\pi lx} W(2\pi ly; k, s). \quad (5.3)$$

Hence from Lemma 4.1(5),

$$\sum_{m, n=-\infty}^{\infty} \frac{\varphi(m) \overline{\psi(n)} y^{s+k/2}}{(mN\tau+n)^k |mN\tau+n|^{2s}} = \frac{2g(\bar{\psi})}{N^{2s+k}} \sum_{l=-\infty}^{\infty} C(k, s+k/2, l) |l|^{s-1/2+k/2} \sigma_{1-k-2s}(l; \varphi, \psi) \\ \times \frac{1}{\sqrt{|l|}} e^{2i\pi lx} W_{\frac{\operatorname{sgn}(l)k}{2}, s-\frac{1}{2}+\frac{k}{2}}(4\pi|l|y).$$

For the special case $s = 0$, we derive from (4.1) and (5.3) the following result.

Corollary 5.2 Suppose k is an integer > 2 , $M \geq 3$, $N \geq 3$ and $\varphi\psi(-1) = (-1)^k$. Then

$$E_{\varphi,\psi}(\tau; k, 0) = \frac{2g(\overline{\psi})}{N^k} \cdot \frac{(-2i\pi)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{1-k}(n; \varphi, \psi) n^{k-1} q^n,$$

where $q = e^{2i\pi\tau}$.

From (4.2), we have

Corollary 5.3

$$\Delta_k A_{\varphi,\psi}(\tau; k, s) = (s + k/2)(1 - s - k/2) A_{\varphi,\psi}(\tau; k, s).$$

6 Lifting of $E_{\varphi,\psi}(\tau; k, s)$ from \mathfrak{h} to $GL^+(2, \mathbb{R})$

Let $GL^+(2, \mathbb{R})$ denote the set which consists of 2×2 real matrices with positive determinant.

Let

$$f : \mathfrak{h} \rightarrow \mathbb{C}.$$

For a fixed $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$, we define the slash operator $|_k$ by

$$(f|_k \gamma)(\tau) = \frac{|\gamma|^{k/2}}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

where $|\gamma|$ denotes the determinant of γ . Define the lift \tilde{f} of f to $GL^+(2, \mathbb{R})$ as

$$\tilde{f}(\gamma) := (f|_k \gamma)(i).$$

We shall lift $E_{\varphi,\psi}(\tau; k, s)$ from $\tau \in \mathfrak{h}$ to $GL^+(2, \mathbb{R})$. First we need a lemma.

Lemma 6.1^{[2]105} (Iwasawa decomposition) Every $\gamma \in GL^+(2, \mathbb{R})$ has a unique factorization

$$\gamma = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

where $r > 0$, $x \in \mathbb{R}$, $y > 0$ and $0 \leq \theta < 2\pi$.

Let $\tilde{E}_{\varphi,\psi}(\gamma; k, s)$ denote the lift of $E_{\varphi,\psi}(\tau; k, s)$ and $r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$.

Theorem 6.2 Suppose

$$\gamma = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in GL^+(2, \mathbb{R})$$

and $\tau = x + iy$. Then

$$(1) \tilde{E}_{\varphi,\psi}(\gamma; k, s) = A_{\varphi,\psi}(\tau; k, s) e^{ik\theta};$$

$$(2) \tilde{E}_{\varphi,\psi}(\rho\gamma; k, s) = \varphi(d)\psi(d)\tilde{E}_{\varphi,\psi}(\gamma; k, s) \text{ for } \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(MN);$$

$$(3) \tilde{E}_{\varphi,\psi}(\gamma r_\phi; k, s) = e^{ik\phi} \tilde{E}_{\varphi,\psi}(\gamma; k, s);$$

$$(4) \Delta \tilde{E}_{\varphi,\psi}(\gamma; k, s) = (s + k/2)(1 - s - k/2) \tilde{E}_{\varphi,\psi}(\gamma; k, s).$$

Proof The proof of property (1) is based on the crucial fact of the slash operator^{[2]84}

$$f|_k \gamma_1 \gamma_2 = f|_k \gamma_1 |_k \gamma_2.$$

Thus

$$\begin{aligned} \tilde{E}_{\varphi,\psi}(\gamma; k, s) &= (E_{\varphi,\psi}(\cdot; k, s)|_k \gamma)(i) \\ &= \left(E_{\varphi,\psi}(\cdot; k, s)|_k \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \Big|_k r_\theta \right) (i) \\ &= e^{ik\theta} \left(E_{\varphi,\psi}(\cdot; k, s)|_k \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) (i) \\ &= e^{ik\theta} y^{k/2} E_{\varphi,\psi}(x + iy; k, s) \\ &= e^{ik\theta} A_{\varphi,\psi}(\tau; k, s). \end{aligned}$$

From Theorem 1.1, we have

$$(E_{\varphi,\psi}(\cdot; k, s)|_k \rho)(\tau) = \varphi(d)\psi(d)E_{\varphi,\psi}(\tau; k, s).$$

Thus

$$\begin{aligned} \tilde{E}_{\varphi,\psi}(\rho\gamma; k, s) &= (E_{\varphi,\psi}(\cdot; k, s)|_k \rho|_k \gamma)(i) \\ &= \varphi(d)\psi(d)(E_{\varphi,\psi}(\cdot; k, s)|_k \gamma)(i) \\ &= \varphi(d)\psi(d)\tilde{E}_{\varphi,\psi}(\gamma; k, s). \end{aligned}$$

The property (3) is obvious and (4) follows from Lemma 2.1 and Corollary 5.3.

7 An alternative form of Eisenstein series

We now give a slightly different formulation of the Eisenstein series which yields more elegant statements and keener to the language of automorphic representations.

In (5.1), let $\nu = s + \frac{k}{2}$ and define

$$\mathfrak{E}_{\varphi,\psi}(\tau; k, \nu) := \sum_{m,n=-\infty}^{\infty} \varphi(m) \overline{\psi(n)} \left(\frac{|mN\tau + n|}{mN\tau + n} \right)^k \frac{y^\nu}{|mN\tau + n|^{2\nu}}$$

and

$$\tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu) = e^{ik\theta} \mathfrak{E}_{\varphi,\psi}(\tau; k, \nu).$$

Then

$$\tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu) = \tilde{E}_{\varphi,\psi}(\gamma; k, \nu - k/2),$$

and from Theorems 5.1 and 6.2, we have

Corollary 7.1 Suppose φ, ψ are primitive modulo $M \geq 3$ and $N \geq 3$, $\varphi\psi(-1) = (-1)^k$ and $\operatorname{Re} \nu > 1$. Then

$$\begin{aligned} & \mathfrak{E}_{\varphi,\psi}(\tau; k, \nu) \\ &= \frac{2g(\overline{\psi})}{N^{2\nu}} \sum_{l=-\infty}^{\infty} C(k, \nu, l) |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \varphi, \psi) \frac{1}{\sqrt{|l|}} e^{2i\pi lx} W_{\frac{\operatorname{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y), \end{aligned} \quad (7.1)$$

$$\tilde{\mathfrak{E}}_{\varphi,\psi}(\rho\gamma; k, \nu) = \varphi(d)\psi(d)\tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu)$$

$$\text{for } \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(MN),$$

$$\tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma r_\phi; k, \nu) = e^{ik\phi} \tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu),$$

and

$$\Delta \tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu) = \nu(1-\nu) \tilde{\mathfrak{E}}_{\varphi,\psi}(\gamma; k, \nu).$$

To bring out the connection of $\mathfrak{E}_{\varphi,\psi}(\tau; k, \nu)$ between different weights k , we introduce Maass raising and lowering operators:

$$R_k = (\tau - \overline{\tau}) \frac{\partial}{\partial \tau} + \frac{k}{2}$$

and

$$L_k = -(\tau - \overline{\tau}) \frac{\partial}{\partial \overline{\tau}} - \frac{k}{2}.$$

We add that

$$\Delta_k = -L_{k+2}R_k + \frac{k}{2} \left(1 + \frac{k}{2}\right) = -R_{k-2}L_k + \frac{k}{2} \left(1 - \frac{k}{2}\right).$$

It is straightforward to verify that

$$R_k \left(\frac{|t+\tau|}{t+\tau} \right)^k \left(\frac{y}{|t+\tau|^2} \right)^s = (s+k/2) \left(\frac{|t+\tau|}{t+\tau} \right)^{k+2} \left(\frac{y}{|t+\tau|^2} \right)^s$$

and

$$L_k \left(\frac{|t+\tau|}{t+\tau} \right)^k \left(\frac{y}{|t+\tau|^2} \right)^s = (s-k/2) \left(\frac{|t+\tau|}{t+\tau} \right)^{k-2} \left(\frac{y}{|t+\tau|^2} \right)^s.$$

Applying R_k and L_k to $\mathfrak{E}_{\varphi,\psi}(\tau; k, \nu)$, we obtain

$$R_k \mathfrak{E}_{\varphi,\psi}(\tau; k, \nu) = (\nu + k/2) \mathfrak{E}_{\varphi,\psi}(\tau; k+2, \nu)$$

and

$$L_k \mathfrak{E}_{\varphi, \psi}(\tau; k, \nu) = (\nu - k/2) \mathfrak{E}_{\varphi, \psi}(\tau; k - 2, \nu).$$

The justification of differentiation term by term follows easily from the assumption that the series converges absolutely when $\operatorname{Re} \nu > 1$ for any $k \in \mathbb{Z}$.

To remove the dependency of k of the Maass operators, we let

$$R = e^{2i\theta} \left((\tau - \bar{\tau}) \frac{\partial}{\partial \tau} + \frac{1}{2i} \cdot \frac{\partial}{\partial \theta} \right)$$

and

$$L = e^{-2i\theta} \left(-(\tau - \bar{\tau}) \frac{\partial}{\partial \bar{\tau}} - \frac{1}{2i} \cdot \frac{\partial}{\partial \theta} \right).$$

We have

$$R \tilde{\mathfrak{E}}_{\varphi, \psi}(\tau; k, \nu) = (\nu + k/2) \tilde{\mathfrak{E}}_{\varphi, \psi}(\tau; k + 2, \nu)$$

and

$$L \tilde{\mathfrak{E}}_{\varphi, \psi}(\tau; k, \nu) = (\nu - k/2) \tilde{\mathfrak{E}}_{\varphi, \psi}(\tau; k - 2, \nu).$$

Hence, it suffices to study $\mathfrak{E}_{\varphi, \psi}(\tau; 0, \nu)$ and $\mathfrak{E}_{\varphi, \psi}(\tau; 1, \nu)$, since all other ones can be generated from these two using the Maass operators.

8 Functional equation for $\mathfrak{E}_{\varphi, \psi}(\tau; k, \nu)$

We remark that since Whittaker function $W_{\alpha, z}(y)$ decreases exponentially as $y \rightarrow \infty$ for any $z \in \mathbb{C}$, as a function of ν , we can extend $\mathfrak{E}_{\varphi, \psi}(\tau; k, \nu)$ analytically to the entire complex plane via its series expansion (7.1), obtained in the previous section, in terms of the Whittaker function, and we will establish its functional equation using the property:

$$W_{\alpha, z}(y) = W_{\alpha, -z}(y).$$

We begin with

Lemma 8.1 Suppose $\varphi\psi(-1) = (-1)^k$. Then

$$|l|^r \sigma_{-2r}(l; \varphi, \psi) = \begin{cases} |l|^{-r} \sigma_{2r}(l; \psi, \varphi), & \text{if } l > 0, \\ (-1)^k |l|^{-r} \sigma_{2r}(l; \psi, \varphi), & \text{if } l < 0. \end{cases}$$

Replacing ν with $1 - \nu$ in (7.1) and recalling

$$C(k, \nu, l) = \frac{(-i)^k \pi^\nu}{\Gamma(\nu + \operatorname{sgn}(l)k/2)},$$

$$\Gamma(1 - z) \Gamma(z) = \frac{\pi}{\sin \pi z},$$

and

$$(-1)^k \sin(\pi(\nu + k/2)) = \sin(\pi(\nu - k/2)),$$

we derive from the lemma above,

$$\begin{aligned} & \left(\frac{2g(\bar{\psi})\pi^{1-\nu}}{i^k N^{2(1-\nu)}} \right)^{-1} \mathfrak{E}_{\varphi, \psi}(\tau; k, 1-\nu) \\ &= \frac{1}{\Gamma(1-\nu+k/2)} \sum_{l=1}^{\infty} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \\ & \quad + \frac{(-1)^k}{\Gamma(1-\nu-k/2)} \sum_{l=-\infty}^{-1} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \\ &= \frac{\sin(\pi(\nu-k/2))}{\pi} \Gamma(\nu-k/2) \sum_{l=1}^{\infty} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \\ & \quad + \frac{(-1)^k \sin(\pi(\nu+k/2))}{\pi} \Gamma(\nu+k/2) \sum_{l=-\infty}^{-1} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \\ &= \frac{\sin(\pi(\nu-k/2))}{\pi} \Gamma(\nu+k/2) \Gamma(\nu-k/2) \\ & \quad \times \left(\frac{1}{\Gamma(\nu+k/2)} \sum_{l=1}^{\infty} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \right) \\ & \quad + \frac{\sin(\pi(\nu-k/2))}{\pi} \Gamma(\nu+k/2) \Gamma(\nu-k/2) \\ & \quad \times \left(\frac{1}{\Gamma(\nu-k/2)} \sum_{l=-\infty}^{-1} |l|^{\nu-1/2} \sigma_{1-2\nu}(l; \psi, \varphi) \frac{1}{\sqrt{|l|}} e^{2i\pi l x} W_{\frac{\text{sgn}(l)k}{2}, \nu-\frac{1}{2}}(4\pi|l|y) \right) \\ &= \frac{\sin(\pi(\nu-k/2))}{\pi} \Gamma(\nu+k/2) \Gamma(\nu-k/2) \left(\frac{2\varphi(-1)g(\bar{\varphi})\pi^\nu}{i^k M^{2\nu}} \right)^{-1} \mathfrak{E}_{\psi, \varphi}(\tau; k, \nu). \end{aligned}$$

Hence

$$\begin{aligned} & \left(\frac{2g(\bar{\psi})\pi^{1-\nu}}{i^k N^{2(1-\nu)}} \right)^{-1} \mathfrak{E}_{\varphi, \psi}(\tau; k, 1-\nu) \\ &= \frac{\sin(\pi(\nu-k/2))}{\pi} \Gamma(\nu+k/2) \Gamma(\nu-k/2) \left(\frac{2\varphi(-1)g(\bar{\varphi})\pi^\nu}{i^k M^{2\nu}} \right)^{-1} \mathfrak{E}_{\psi, \varphi}(\tau; k, \nu) \\ &= \frac{\Gamma(\nu+k/2)}{\Gamma(1-\nu+k/2)} \left(\frac{2g(\bar{\varphi})\pi^\nu}{i^k M^{2\nu}} \right)^{-1} \mathfrak{E}_{\psi, \varphi}(\tau; k, \nu). \end{aligned} \tag{8.1}$$

If χ is a primitive character modulo N , we define $\mathfrak{f}(\chi) = N$. It is well-known^{[1]81}:

$$g(\chi)g(\bar{\chi}) = \chi(-1)\mathfrak{f}(\chi).$$

Let

$$\mathfrak{L}_{\varphi, \psi}(\tau; k, \nu) = \frac{g(\psi)\mathfrak{f}(\psi)^{2\nu-1}}{\pi^\nu} \Gamma(\nu+k/2) \mathfrak{E}_{\varphi, \psi}(\tau; k, \nu).$$

From (8.1), we derive the following functional equation for $\mathfrak{L}_{\varphi,\psi}(\tau; k, \nu)$.

Theorem 8.2 Suppose φ and ψ are primitive characters modulo, respectively, $M \geq 3$ and $N \geq 3$, $\varphi\psi(-1) = (-1)^k$. Then

$$\mathfrak{L}_{\varphi,\psi}(\tau; k, 1 - \nu) = (-1)^k \mathfrak{L}_{\psi,\varphi}(\tau; k, \nu).$$

9 Remarks

The non-holomorphic Eisenstein series for the case $M = N = 1$ and $k = 0$:

$$E(\tau, s) = \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}}$$

has been study extensively. Interested readers can consult [1-2], [4] and [5] for additional properties.

In this paper, we have been investigating the cases $M, N \geq 3$, readers should have no difficulty in completing the remaining cases $M = 1, N \geq 3$ and $M \geq 3, N = 1$ based on the methodology developed in the paper.

To end the paper, it seems appropriate to give a brief account of the connection between of the Kronecker's sum and the imaginary quadratic fields.

Let \mathbb{F} be an imaginary quadratic field of discriminant $-d$, and $\mathbb{O}_{\mathbb{F}}$ the ring of integers in \mathbb{F} .

Recall that the zeta function associated with the field \mathbb{F} is defined as:

$$\zeta_{\mathbb{F}}(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s},$$

where the sum is over all non-zero ideals in $\mathbb{O}_{\mathbb{F}}$.

Define the partial zeta function associated with the ideal class \mathfrak{A} :

$$\zeta(s, \mathfrak{A}) = \sum_{\mathfrak{a} \in \mathfrak{A}} \frac{1}{(N\mathfrak{a})^s}.$$

Then

$$\zeta_{\mathbb{F}}(s) = \sum_{\mathfrak{A}} \zeta(s, \mathfrak{A}).$$

It is well-known that

$$\zeta_{\mathbb{F}}(s) = \zeta(s) L(s, \chi_{-d}),$$

where

$$L(s, \chi_{-d}) = \sum_{n=1}^{\infty} \frac{\chi_{-d}(n)}{n^s}$$

and $\chi_{-d}(n) = \left(\frac{-d}{n}\right)$ is the Kronecker symbol. Thus

$$\zeta_{\mathbb{F}}(s) = \sum_{n=1}^{\infty} \left(\sum_{k|n} \chi_{-d}(k) \right) \frac{1}{n^s}.$$

Let $\mathfrak{b} \in \mathfrak{A}^{-1}$, the inverse of the ideal class \mathfrak{A} . With no loss of generality, we assume $\mathfrak{b} = [\tau, 1]$. Then

$$\zeta(s, \mathfrak{A}) = \frac{1}{w} \left(\frac{2}{\sqrt{d}} \right)^s \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}},$$

where w is the number of roots of unity in $\mathbb{O}_{\mathbb{F}}$. See [4, p. 280] for the details and derivation of the above identity.

It is well-known^{[1]68} that

$$\begin{aligned} \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}} &= 2\zeta(2s)y^s + 2\sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1)y^{1-s} \\ &\quad + \frac{4\pi^s \sqrt{y}}{\Gamma(s)} \sum_{n \neq 0} \sigma_{1-2s}(|n|) |n|^{s-1/2} K_{s-1/2}(2\pi|n|y) e^{2i\pi nx}, \end{aligned}$$

where $\sigma_c(n) = \sum_{k|n} k^c$.

For $-d = -3, -4, -7, -11, -19, -43, -67, -163$, the class number of the fields is one, we have

$$\zeta_{\mathbb{F}}(s) = \frac{1}{w} \left(\frac{2}{\sqrt{d}} \right)^s \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m\tau + n|^{2s}},$$

where $\mathfrak{b} = [\tau, 1]$ is chosen as $\mathbb{O}_{\mathbb{F}}$.

We have the following representations of the zeta functions of the fields in terms of the infinite sums of the K -Bessel functions:

For $d = -4$, $w = 4$ and $\tau = i$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k|n} \chi_{-4}(k) \right) \frac{1}{n^s} &= \frac{1}{4} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^s} \\ &= \frac{1}{2} \zeta(2s) + \frac{1}{2} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) \\ &\quad + \frac{\pi^s}{\Gamma(s)} \sum_{n \neq 0} \sigma_{1-2s}(|n|) |n|^{s-1/2} K_{s-1/2}(\pi|n|). \end{aligned}$$

For $d = -3$, $w = 6$ and $\tau = \frac{1+i\sqrt{3}}{2}$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k|n} \chi_{-d}(k) \right) \frac{1}{n^s} &= \frac{1}{6} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + mn + n^2)^s} \\ &= \frac{1}{3} \zeta(2s) + \frac{1}{3} 2^{2s} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) 3^{1/2-s} \\ &\quad + \frac{1}{3} \cdot \frac{2^{1/2+s} 3^{1/4-s/2} \pi^s}{\Gamma(s)} \sum_{n \neq 0} (-1)^n \sigma_{1-2s}(|n|) |n|^{s-1/2} K_{s-1/2}(\pi |n| \sqrt{3}). \end{aligned}$$

For $d = 7, 11, 19, 43, 67, 163$, we have $w = 2$ and $\tau = \frac{1+i\sqrt{d}}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k|n} \chi_{-d}(k) \right) \frac{1}{n^s} &= \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + mn + \frac{1+d}{4} n^2)^s} \\ &= \zeta(2s) + 2^{2s} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) d^{1/2-s} \\ &\quad + \frac{2^{1/2+s} d^{1/4-s/2} \pi^s}{\Gamma(s)} \sum_{n \neq 0} (-1)^n \sigma_{1-2s}(|n|) |n|^{s-1/2} K_{s-1/2}(\pi |n| \sqrt{d}). \end{aligned}$$

We remark that, since

$$K_{\nu}(y) \sim \sqrt{\frac{\pi}{2y}} e^{-y}$$

as $y \rightarrow \infty$, the above series converges rapidly.

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