Article ID: 1000-5641(2010)06-0186-13

Hopf bifurcation analysis of a predator-prey system with delay and harvesting

RAO Feng[1], WANG Wei-ming[2], LI Zhi-bin[1]

(1. *Department of Computer Science and Technology, East China Normal University, Shanghai* 200241, *China*;
2. *College of Mathematics and Information Science, Wenzhou University, Wenzhou Zhejiang* 325035, *China*)

Abstract: In this paper, by using the analysis of qualitative method and bifurcation theory, we investigated the dynamics of the Monod-Haldane type predator-prey system with constant prey harvesting and a single time delay. It is shown that the Hopf bifurcation can occur as the delay τ crosses some critical values. Furthermore, the direction and stability of the Hopf bifurcation are determined by deriving normal form theory and center manifold theorem. Finally, numerical simulations are performed to illustrate the analytical results.

Key words: predator-prey system; time delay; Hopf bifurcation; stability
CLC number: O175.1 **Document code:** A

一类含时滞与收获的捕食系统的 Hopf 分支分析

饶 凤[1], 王玮明[2], 李志斌[1]

(1. 华东师范大学 计算机科学技术系, 上海 200241;
2. 温州大学 数学与信息科学学院, 浙江 温州 325035)

摘要: 运用定性分析和分支理论, 研究了一类含时滞与收获的Monod-Haldane型捕食系统的动力学行为, 确定了Hopf分支发生时的时滞τ的临界条件, 并通过规范型理论和中心流行定理, 研究了Hopf分支的方向与稳定性等. 最后, 利用数值模拟验证了研究结果.

关键词: 捕食系统; 时滞; Hopf分支; 稳定性

0 Introduction

In almost every situation, time-delays occur so often that to ignore them is to ignore reality. For example, animals must take time to digest their food before further activities and responses

收稿日期: 2010-01
基金项目: 国家自然科学基金(10771072); 浙江省科学基金(Y7080041)
第一作者: 饶风, 女, 博士生, 研究方向为随机斑图动力学的符号计算问题.
E-mail: raofeng2002@163.com.
通讯作者: 李志斌, 男, 教授, 研究方向为非线性系统及计算机符号计算. E-mail: lizb@cs.ecnu.edu.cn.

take place, and hence any model of species dynamics without delays is an approximation at best. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate[1−15].

We are to study the stability and bifurcation of the following delayed Monod-Haldane predator-prey system with harvesting of prey at a constant rate

$$\begin{aligned}\dot{x}(t) &= x(t)(1-x(t-\tau)) - \frac{mx(t)y(t)}{1+ax(t)+bx^2(t)} - h,\\ \dot{y}(t) &= y(t)\Big(-d + \frac{nx(t)}{1+ax(t)+bx^2(t)}\Big),\end{aligned} \tag{0.1}$$

where $x(t)$ and $y(t)$ denote the density of the prey and the predator populations at time t, respectively; a, b, d, m, n are positive constants by biological terms; m the maximal predator per capita consumption rate, n the conversion rate, d the predator death rate. The parameter $\tau \geqslant 0$ is a time delay due to regulation of the prey; $h > 0$ represents the rate of harvesting.

The organization of this paper is as follows. In Section 1, we would consider the existence of Hopf bifurcation of the delayed system (0.1). In Section 2, we investigate the direction and stability of the Hopf bifurcation. Finally, we give some discussions.

1 Existence of Hopf bifurcation

By analyzing, it is easy to know that system (0.1) has one positive equilibrium,

$$\begin{aligned}(x^*, y^*) =&\Big(\frac{n-ad-\sqrt{(ad-n)^2-4bd^2}}{2bd}\\ &\cdot\frac{n((bd(n-d(bh+a-1))-(ad-n)^2)x^*+d(n-d(a+b)))}{b^2d^3mx^*}\Big).\end{aligned}$$

Set

$$\begin{aligned}f^{(1)} &= x(t)(1-x(t-\tau)) - \frac{mx(t)y(t)}{1+ax(t)+bx^2(t)} - h,\\ f^{(2)} &= y(t)\Big(-d+\frac{nx(t)}{1+ax(t)+bx^2(t)}\Big),\\ p_1 &= 1-x^* - \Big(\frac{\sqrt{(ad-n)^2-4bd^2}}{n}\Big)\\ &\quad\cdot\Big(1-\frac{n-ad-\sqrt{(ad-n)^2-4bd^2}}{2bd} - \frac{2bdh}{n-ad-\sqrt{(ad-n)^2-4bd^2}}\Big),\\ p_2 &= -\frac{md}{n},\\ p_3 &= \Big(1-\frac{n-ad-\sqrt{(ad-n)^2-4bd^2}}{2bd} - \frac{2bdh}{n-ad-\sqrt{(ad-n)^2-4bd^2}}\Big)\\ &\quad\cdot\Big(\frac{n}{m}-d-\frac{d\sqrt{(ad-n)^2-4bd^2}}{n}\Big),\\ p_4 &= 0, \quad p_5 = -x^*.\end{aligned}$$

And linearizing system (0.1) at the positive equilibrium (x^*, y^*), we derive that

$$\dot{x}(t) = p_1 x(t) + p_2 y(t) + p_5 x(t-\tau), \quad \dot{y}(t) = p_3 x(t) + p_4 y(t). \tag{1.1}$$

Hence, the characteristic equation of system (1.2) at the positive equilibrium (x^*, y^*) takes the form

$$\lambda^2 - p_1\lambda - p_5\lambda \mathrm{e}^{-\lambda\tau} = 0. \tag{1.2}$$

We now investigate the distribution of roots of Eq. (1.3).

If $\mathrm{i}\omega(\omega > 0)$ is a root of Eq. (1.3), then

$$-\omega^2 - \mathrm{i}\omega(p_1 + p_5\mathrm{e}^{-\mathrm{i}\omega\tau}) = 0. \tag{1.3}$$

separating the real and imaginary parts, we obtain

$$-\omega^2 = p_5\omega\sin\omega\tau, \quad -p_1\omega = p_5\omega\cos\omega\tau. \tag{1.4}$$

It follows from Eq. (1.5) that

$$\omega^4 + (p_1^2 - p_5^2)\omega^2 = 0. \tag{1.5}$$

Clearly, Eq. (1.6) has only one positive root ω_0 defined by

$$\omega_0 = \sqrt{p_5^2 - p_1^2}, \tag{1.6}$$

and

$$\tau_j = \frac{1}{\omega_0}\arccos\Big(\frac{-p_1}{p_5}\Big) + \frac{2\pi j}{\omega_0}, \quad j = 0, 1, 2, \cdots. \tag{1.7}$$

Then (τ_j, ω_0) is the solution of Eq. (1.5). This means that when $\tau = \tau_j$, Eq. (1.3) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_0$.

Now, let's consider the behavior of roots of Eq. (1.3) near τ_j. Denote

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau). \tag{1.8}$$

The roots of Eq.(1.3) satisfy

$$\alpha(\tau_j) = 0, \quad \omega(\tau_j) = \omega_0. \tag{1.9}$$

Substituting $\lambda(\tau)$ into Eq. (1.3) and differentiating both sides of it with respect to τ, we have

$$\Big(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\Big)^{-1} = -\frac{(2\lambda - p_1)\mathrm{e}^{\lambda\tau}}{p_5\lambda^2} + \frac{1 - \lambda\tau}{\lambda^2}. \tag{1.10}$$

Then from Eq. (1.5) and (1.7), we derive that

$$\mathrm{Re}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^{-1}_{\tau=\tau_j} = \frac{1}{p_5^2}. \tag{1.11}$$

Therefore, we can obtain

$$\operatorname{sign}\Big\{\operatorname{Re}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)_{\tau=\tau_j}\Big\}=\operatorname{sign}\Big\{\operatorname{Re}\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^{-1}_{\tau=\tau_j}\Big\}=\operatorname{sign}\Big\{\frac{1}{p_5^2}\Big\}>0. \tag{1.12}$$

When the delay τ gradually near τ_j, the root of Eq. (1.3) crosses the imaginary axis from left to right. In addition, note that when $\tau=0$, Eq. (1.3) has roots with negative real parts only if $p_1+p_5<0$.

From the above discussion, we can obtain the following theorem.

Theorem 1.1 If $0<4bd^2\leqslant(ad-n)^2$, $0<h<\min\{h_0,\frac{1}{4}\}$, and $p_1+p_5<0$, let ω_0 and τ_j $(j=0,1,2,\cdots)$ be defined as in Eq. (1.7) and (1.8), respectively.

(i) The positive equilibrium (x^*,y^*) of system (0.1) is asymptotically stable for all $\tau\in[0,\tau_0)$ and unstable for $\tau>\tau_0$;

(ii) system (0.1) undergoes a Hopf bifurcation at the positive equilibrium (x^*,y^*) when $\tau=\tau_j$ $(j=0,1,2,\cdots)$.

2 The direction and stability of Hopf bifurcation

In this section, we will study the direction and stability of the Hopf bifurcation by applying the normal form theory[16]. By translation $u_1(t)=x(t)-x^*, u_2(t)=y(t)-y^*$ and rescaling the time by $t\to t/\tau$, system (0.1) can be transformed into

$$\begin{aligned}\dot{u}_1(t)&=p_1u_1(t)+p_2u_2(t)+p_5u_2(t-\tau)+\sum_{i+j+l\geqslant2}\frac{1}{i!j!l!}f_{ijl}^{(1)}u_1^i(t)u_2^j(t)u_1^l(t-\tau),\\ \dot{u}_2(t)&=p_3u_1(t)+\sum_{i+j\geqslant2}\frac{1}{i!j!}f_{ij}^{(2)}u_1^i(t)u_2^j(t).\end{aligned} \tag{2.1}$$

Then we rewrite (2.1) as

$$\dot{z}(t)=L(\tau)z_t+F(z_t,\tau). \tag{2.2}$$

in the phase space $\mathbf{C}=\mathbf{C}([-1,0];\mathbb{R}^2)$, where, for $\varphi=(\varphi_1,\varphi_2)^{\mathrm{T}}\in\mathbf{C}$,

$$L(\tau)(\varphi)=\begin{pmatrix}p_1\varphi_1(0)+p_2\varphi_2(0)+p_5\varphi_1(-1)\\ p_3\varphi_1(0)\end{pmatrix}, \tag{2.3}$$

$$F(\varphi,\tau)=\tau\begin{pmatrix}\displaystyle\sum_{i+j+l\geqslant2}\frac{1}{i!j!l!}f_{ijl}^{(1)}\varphi_1^i(0)\varphi_2^j(0)\varphi_1^l(-1)\\ \displaystyle\sum_{i+j\geqslant2}\frac{1}{i!j!}f_{ij}^{(2)}\varphi_1^i(0)\varphi_2^j(0)\end{pmatrix}, \tag{2.4}$$

and we expand L and F as Taylor expansions

$$\begin{aligned}L(\tau)&=L_0+\tau L_1+\frac{\tau^2}{2!}L_2+O(\tau^3),\\ F(\varphi,\tau)&=\frac{1}{2}F_2(\varphi,\tau)+\frac{1}{3!}F_3(\varphi,\tau)+O(|(\varphi,\tau)|^4),\end{aligned} \tag{2.5}$$

where

$$\frac{1}{p!}F_p(\varphi,\tau)=\tau\begin{pmatrix}\displaystyle\sum_{i+j+l\geqslant 2}\frac{1}{i!j!l!}f_{ijl}^{(1)}\varphi_1^i(0)\varphi_2^j(0)\varphi_1^l(-1)\\ \displaystyle\sum_{i+j\geqslant 2}\frac{1}{i!j!}f_{ij}^{(2)}\varphi_1^i(0)\varphi_2^j(0)\end{pmatrix}. \tag{2.6}$$

Introducing new parameter $\tau=\tau_k+\beta$, then $\beta=0$ is a Hopf bifurcation value of (2.2), and we rewrite (2.2) as

$$\dot{z}(t)=L(\tau_k)z_t+F_0(z_t,\beta), \tag{2.7}$$

where $F_0(z_t,\beta)=L(\beta)z_t+F(z_t,\tau_k+\beta)$.

From the discussions in the previous section, we assume that A_0 is the infinitesimal generator of the characteristic equation of $\dot{z}(t)=L(\tau_k)z_t$. Then A_0 has a pair of simple imaginary roots $\pm\mathrm{i}\omega_k$, $\omega_k=\tau_k\omega_0$. Consider $\Lambda=\{-\mathrm{i}\omega_k,\mathrm{i}\omega_k\}$ and denote by P the invariant space of A_0 associated with Λ, where $\dim P=2$. We can decompose $\mathbf{C}=P\oplus Q$ by applying the formal adjoint theory for functional differential equation[17], and consider the complex coordinates $\mathbf{C}([-1,0];\mathbb{C}^2)$. Let $\Phi=(\phi_1,\phi_2)$ be a matrix whose columns form a basis of the center space P of $\dot{z}(t)=L(\tau_k)z_t$ with $\phi_1(\theta)=\mathrm{e}^{\mathrm{i}\omega_k\theta}v,\phi_2(\theta)=\mathrm{e}^{-\mathrm{i}\omega_k\theta}\bar{v}$, $-1\leqslant\theta\leqslant 0$, where the bar means complex conjugation, and $v=\mathrm{col}(v_1,v_2)$ is a vector in $\mathbb{C}^2$ that satisfies

$$L(\tau_k)\phi_1=\mathrm{i}\omega_k v. \tag{2.8}$$

Choose $\Psi=\mathrm{col}(\psi_1,\psi_2)$ as a matrix whose rows form a basis for the adjoin space P^*, with $\psi_1(s)=\mathrm{e}^{-\mathrm{i}\omega_k s}u^{\mathrm{T}}$, $\psi_2(s)=\mathrm{e}^{\mathrm{i}\omega_k s}\bar{u}^{\mathrm{T}}$, $0\leqslant s\leqslant 1$, $u=\mathrm{col}(u_1,u_2)\in\mathbb{C}^2$. Let $(\Psi,\Phi)=I$, and $(\psi_1,\phi_1)=1,(\psi_1,\phi_2)=0$, where $(.,.)$ is a adjoint bilinear form

$$(\psi,\phi)=\psi(0)\phi(0)-\int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,\quad\forall\phi\in P,\psi\in P^*.$$

Following Ref. [17] and by directly computing, we can obtain

$$v=\begin{pmatrix}1\\ \dfrac{p_3}{\mathrm{i}\omega_0}\end{pmatrix},\quad u=\frac{1}{1-\dfrac{p_2p_3}{\omega_0^2}+p_5\mathrm{e}^{\mathrm{i}\omega_0}}\begin{pmatrix}1\\ -\dfrac{p_2}{\mathrm{i}\omega_0}\end{pmatrix}. \tag{2.9}$$

From Ref. [16], we know that (2.2) can be written as an infinite dimensional ordinary differential equation on the Banach space BC of functional from $[-1,0]$ into $\mathbf{R}$ which are uniformly continuous on $[-1,0)$ and with a jump discontinuity at 0. Elements of BC are of the form $\phi+X_0\beta$ and $X_0(\theta)$ is defined by

$$X_0(\theta)=\begin{cases}I, & \theta=0,\\ 0, & -1\leqslant\theta<0.\end{cases}$$

Let $\pi:BC\to P$ denote the projection $\pi(\phi+X_0\beta)=\Phi[(\Psi,\phi)+\Psi(0)\beta]$, where $\phi\in\mathbf{C},\beta\in\mathbb{R}^2$. We can decomposition $BC=P\oplus\mathrm{Ker}\pi$ with the property that $Q\subset\mathrm{Ker}\pi$, where

Q is an infinite dimensional complementary subspace of P in $\mathbf{C}$. Let A be the infinitesimal generator for the flow of the linear system $\dot{z}(t) = L(\tau_k)z_t$. Decompose z_t in (2.2) according to the decomposition of BC, in the form $z_t = \Phi x(t) + y_t$, with $x \in \mathbb{R}^2$ and $y_t \in \mathrm{Ker}\pi \cap D(A) = Q \cap \mathbf{C^1} = Q^1$, where $D(A)$ is the domain of A. Define a diagonal matrix

$$\boldsymbol{B} = \begin{pmatrix} \mathrm{i}\omega_k & 0 \\ 0 & -\mathrm{i}\omega_k \end{pmatrix}, \tag{2.10}$$

then Eq.(2.2) is equivalent to the following system

$$\begin{aligned} \dot{x} &= \boldsymbol{B}x + \Psi(0)F_0(\Phi x + y, \beta) = \boldsymbol{B}x + \sum_{j\geqslant 2} \frac{1}{j!} f_j^1(x, y, \beta), \\ \dot{y} &= A_{Q^1} y + (I - \pi)X_0 F_0(\Phi x + y, \beta) = A_{Q^1} y + \sum_{j\geqslant 2} \frac{1}{j!} f_j^2(x, y, \beta), \end{aligned} \tag{2.11}$$

where $f_j^1(x, y, \beta)$ and $f_j^2(x, y, \beta)$ are homogeneous polynomials in (x, y, β) of degree j, with coefficients in $\mathbb{C}^2$ and Ker π, respectively. The normal form method implies a normal form on the center manifold when $y = 0$ of the origin for (2.7) as

$$\dot{x} = \boldsymbol{B}x + \frac{1}{2} g_2^1(x, 0, \beta) + \frac{1}{3!} g_3^1(x, 0, \beta) + \text{h.o.t.} \tag{2.12}$$

where g_2^1 and g_3^1 are the second and third order terms in $(x, 0, \beta)$, respectively. h.o.t. stands for higher order terms. In the following, $V_j^{m+p}(X)$ denotes the linear space of homogeneous polynomials of degree j with $m + p$ real variables, $x = (x_1, \cdots, x_m)$, $\beta = (\beta_1, \cdots, \beta_p)$ with coefficients in X, and $(M_j^1 p)(x, \beta) = [B, p(., \beta)](x) = \mathrm{D}_x p(x, \beta)Bx - Bp(x, \beta)$. In this case, the operator M_j^1 acts in $V_j^3(\mathbb{C}^2) = \mathrm{Im}(M_j^1) \oplus \mathrm{Ker}(M_j^1)$[16], and

$$\mathrm{Ker}(M_j^1) = \mathrm{span}\Big\{ x^q \beta^l e_k : (q, \lambda) = \lambda_k, \quad k = 1, 2, \quad q \in \mathbf{N_0^2}, \quad l \in \mathbf{N_0}, \quad |(q, l)| = j \Big\},$$

where $\lambda = (\lambda_1, \lambda_2) = (\mathrm{i}\omega_k, -\mathrm{i}\omega_k)$ and $\{e_1, e_2\}$ is the canonical basis for $\mathbb{C}^2$. Hence, we have

$$\mathrm{Ker}(M_2^1) = \mathrm{span}\left\{ \begin{pmatrix} x_1\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\beta \end{pmatrix} \right\}, \tag{2.13}$$

$$\mathrm{Ker}(M_3^1) = \mathrm{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\beta^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\beta^2 \end{pmatrix} \right\}. \tag{2.14}$$

Considering (2.4), (2.7) and (2.11), we have

$$f_2^1(x, y, \beta) = \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} [2\beta L_1(\Phi x + y) + F_2(\Phi x + y, \beta)], \tag{2.15}$$

and can obtain

$$f_2^1(x, 0, \beta) = \begin{pmatrix} 2A_1\beta x_1 + 2A_2\beta x_2 + B_{200}x_1^2 + 2B_{110}x_1x_2 + B_{020}x_2^2 \\ 2\bar{A}_1\beta x_1 + 2\bar{A}_2\beta x_2 + \bar{B}_{200}x_1^2 + 2\bar{B}_{110}x_1x_2 + \bar{B}_{020}x_2^2 \end{pmatrix}, \tag{2.16}$$

where

$$
\begin{aligned}
A_1 &= \frac{\mathrm{i}\omega_k}{\tau_k}(u_1v_1+u_2v_2), \quad A_2 = -\frac{\mathrm{i}\omega_k}{\tau_k}(u_1\bar{v}_1+u_2\bar{v}_2),\\
\bar{A}_1 &= \frac{\mathrm{i}\omega_k}{\tau_k}(\bar{u}_1v_1+\bar{u}_2v_2), \quad \bar{A}_2 = -\frac{\mathrm{i}\omega_k}{\tau_k}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2),\\
B_{200} &= u_1\tau_k(f^{(1)}_{200}v_1^2+f^{(1)}_{020}v_2^2+f^{(1)}_{002}\mathrm{e}^{-2\mathrm{i}\omega_k}v_1^2+2f^{(1)}_{110}v_1v_2+2f^{(1)}_{101}\mathrm{e}^{-\mathrm{i}\omega_k}v_1^2+\\
&\quad +2f^{(1)}_{011}\mathrm{e}^{\mathrm{i}\omega_k}v_1v_2)+u_2\tau_k(f^{(2)}_{20}v_1^2+f^{(2)}_{02}v_2^2+2f^{(2)}_{11}v_1v_2),\\
B_{110} &= u_1\tau_k(f^{(1)}_{200}v_1\bar{v}_1+f^{(1)}_{020}v_2\bar{v}_2+f^{(1)}_{002}v_1\bar{v}_1+f^{(1)}_{110}(v_1\bar{v}_2+\bar{v}_1v_2)\\
&\quad +f^{(1)}_{101}(\mathrm{e}^{\mathrm{i}\omega_k}v_1\bar{v}_1+\mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_1)+f^{(1)}_{011}(\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1v_2+\mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_2))\\
&\quad +u_2\tau_k(f^{(2)}_{20}v_1\bar{v}_1+f^{(2)}_{02}v_2\bar{v}_2+f^{(1)}_{11}(v_1\bar{v}_2+\bar{v}_1v_2)),\\
B_{020} &= u_1\tau_k(f^{(1)}_{200}\bar{v}_1^2+f^{(1)}_{020}\bar{v}_2^2+f^{(1)}_{002}\mathrm{e}^{2\mathrm{i}\omega_k}\bar{v}_1^2+2f^{(1)}_{110}\bar{v}_1\bar{v}_2+2f^{(1)}_{101}\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1^2\\
&\quad +2f^{(1)}_{011}\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1\bar{v}_2)+u_2\tau_k(f^{(2)}_{20}\bar{v}_1^2+f^{(2)}_{02}\bar{v}_2^2+2f^{(1)}_{11}\bar{v}_1\bar{v}_2). \qquad (2.17)
\end{aligned}
$$

Thus, the second order term as described in (2.12) of the normal form on the center manifold is given by

$$
\frac{1}{2}g_2^1(x,0,\beta)=\frac{1}{2}\mathrm{Proj}_{\mathrm{Ker}(M_2^1)}f_2^1(x,0,\beta)=\begin{pmatrix} A_1\beta x_1\\ \bar{A}_1\beta x_2\end{pmatrix}. \qquad (2.18)
$$

Next, we compute the cubic term $g_3^1(x,0,\beta)$ as described in (2.12). Note that

$$
g_3^1(x,0,\beta)\in\mathrm{Ker}(M_3^1)=\mathrm{span}\left\{\begin{pmatrix} x_1^2x_2\\ 0\end{pmatrix},\begin{pmatrix} x_1\beta^2\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_1x_2^2\end{pmatrix},\begin{pmatrix} 0\\ x_2\beta^2\end{pmatrix}\right\}.
$$

However, the terms $O(|x|\beta^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, it is only needed to compute the coefficients of

$$
\begin{pmatrix} x_1^2x_2\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0\\ x_1x_2^2\end{pmatrix}.
$$

It follows that

$$
\frac{1}{3!}g_3^1(x,0,\beta)=\frac{1}{3!}\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\bar{f}_3^1(x,0,\beta)=\frac{1}{3!}\mathrm{Proj}_S\bar{f}_3^1(x,0,0)+O(|x|\beta^2),
$$

where

$$
S:=\mathrm{span}\left\{\begin{pmatrix} x_1^2x_2\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_1x_2^2\end{pmatrix}\right\},
$$

and

$$
\bar{f}_3^1(x,0,0)=f_3^1(x,0,0)+\frac{3}{2}[(\mathrm{D}_xf_2^1)U_2^1-(\mathrm{D}_xU_2^1)g_2^1](x,0,0)+\frac{3}{2}[(\mathrm{D}_yf_2^1)h](x,0,0),
$$

here $\bar{f}_3^1(x,0,0)$ is the third order term of the equation which is obtained after computing the second order term of the normal form.

Now, we compute $\frac{1}{3!}g_3^1(x,0,\beta)$ step by step.

(i) We compute $[(\mathrm{D}_xU_2^1)g_2^1](x,0,0)$. From (2.18), we get $g_2^1(x,0,0)=0$, then

$$[(\mathrm{D}_xU_2^1)g_2^1](x,0,0)=0.$$

(ii) We compute $\mathrm{Proj}_S[(\mathrm{D}_xf_2^1)U_2^1](x,0,0)$. In view of the definition of M_2^1, we get

$$\begin{aligned}U_2^1(x,0)=(M_2^1)^{-1}f_2^1(x,0,0)&=\begin{pmatrix}\dfrac{\psi_1(0)}{\mathrm{i}\omega_k}\Big(a_{200}x_1^2-a_{110}x_1x_2-\dfrac{1}{3}a_{020}x_2^2\Big)\\ \overline{\dfrac{\psi_1(0)}{\mathrm{i}\omega_k}\Big(a_{200}x_1^2-a_{110}x_1x_2-\dfrac{1}{3}a_{020}x_2^2\Big)}\end{pmatrix}\\&=\begin{pmatrix}\dfrac{\psi_1(0)}{\mathrm{i}\omega_k}\Big(a_{200}x_1^2-a_{110}x_1x_2-\dfrac{1}{3}a_{020}x_2^2\Big)\\ -\dfrac{\mathrm{i}\psi_2(0)}{\omega_k}\Big(\dfrac{1}{3}a_{020}x_2^2-a_{110}x_1x_2-a_{200}x_1^2\Big)\end{pmatrix}.\end{aligned}\tag{2.19}$$

thus, we obtain

$$\mathrm{Proj}_S[(\mathrm{D}_xf_2^1)U_2^1](x,0,0)=\begin{pmatrix}\dfrac{2\psi_1(0)}{\mathrm{i}\omega_k}\Big(a_{200}a_{110}-2|a_{110}|^2-\dfrac{1}{3}|a_{020}|^2\Big)x_1^2x_2\\ \dfrac{-2\mathrm{i}\psi_2(0)}{\omega_k}\Big(\bar{a}_{200}\bar{a}_{110}-2|a_{110}|^2-\dfrac{1}{3}|a_{020}|^2\Big)x_1x_2^2\end{pmatrix}.$$

(iii) We compute $\mathrm{Proj}_S[(\mathrm{D}_yf_2^1)h](x,0,0)$, where h is a second order homogeneous polynomial in (x_1,x_2,β) with coefficients in Q^1 as

$$h=h(x_1,x_2,\beta)=\sum_{|(q,l)|=2}h_{(q,l)}x^q\beta^l=h_{110}x_1x_2+h_{101}x_1\beta+h_{011}x_2\beta+h_{200}x_1^2+h_{020}x_2^2+h_{002}\beta^2,$$

here $h=\mathrm{col}(h^1,h^2)$. Following Ref. [16], since

$$\begin{aligned}(M_2^2h)(x,\beta)&=\mathrm{D}_xh(x,\beta)Bx-A_{Q^1}(h(x,\beta))\\&=\mathrm{D}_xh(x,\beta)Bx-\dot{h}(x,\beta)-X_0[L_0(h(x,\beta))-\dot{h}(x,\beta)(0)],\end{aligned}\tag{2.20}$$

and $h=h(x,\beta)(\theta)$ can be evaluated by the following system

$$\dot{h}(x,\beta)-\mathrm{D}_xh(x,\beta)Bx=\Phi\Psi(0)[2\beta L_1(\Phi x)+F_2(\Phi x,\beta)],\tag{2.21}$$

$$\dot{h}(x,\beta)(0)-L_0(h(x,\beta))=2\beta L_1(\Phi x)+F_2(\Phi x,\beta),\tag{2.22}$$

where $\dot{h}$ denotes the derivative of $h(x,\beta)(\theta)$ with respect to θ. Then we know $h=h(x_1,x_2,\beta)$ is the unique solution in $V_2^3(Q^1)$ of

$$(M_2^2h)(x,\beta)=(I-\pi)X_0[2\beta L_1(\Phi x)+F_2(\Phi x,\beta)].\tag{2.23}$$

From (2.15), we know

$$\begin{aligned}f_2^1(x,y,0)=\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}[F_2(\phi_1x_1+\phi_2x_2+y,\tau_k)]=\begin{pmatrix}u_1&u_2\\ \bar{u}_1&\bar{u}_2\end{pmatrix}\\\cdot\left[\tau_k\begin{pmatrix}f_{200}^{(1)}P_1^2+f_{020}^{(1)}P_2^2+f_{002}^{(1)}L_1^2+2f_{110}^{(1)}P_1P_2+2f_{101}^{(1)}P_1L_1+2f_{011}^{(1)}P_2L_1\\ f_{20}^{(2)}P_1^2+f_{02}^{(2)}P_2^2+2f_{11}^{(2)}P_1P_2\end{pmatrix}\right],\end{aligned}\tag{2.24}$$

where

$$P_1 = v_1x_1 + \bar{v}_1x_2 + y_1(0), \quad P_2 = v_2x_1 + \bar{v}_2x_2 + y_2(0),$$
$$L_1 = \mathrm{e}^{-\mathrm{i}\omega_k}v_1x_1 + \mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1x_2 + y_1(-1). \tag{2.25}$$

Then, we can obtain

$$\mathrm{Proj}_S[(\mathrm{D}_y f_2^1)h](x,0,0) = \begin{pmatrix} 2b_3x_1^2x_2 \\ 2\bar{b}_3x_1x_2^2 \end{pmatrix},$$

where

$$b_3 = u^{\mathrm{T}}\tau_k \begin{pmatrix} Mh_{100}^1(0) + \bar{M}h_{200}^1(0) + f_{110}^{(1)}h_{110}^2(0)v_1 + f_{110}^{(1)}h_{200}^2(0)\bar{v}_1 + f_{101}^{(1)}h_{110}^1(-1)v_1 + f_{101}^{(1)}h_{200}^1(-1)\bar{v}_1 \\ Nh_{110}^1(0) + \bar{N}h_{200}^1(0) + f_{011}^{(2)}(h_{110}^2(0)v_1 + h_{200}^2(0)\bar{v}_1) \end{pmatrix}, \tag{2.26}$$

here

$$\begin{aligned} M &= f_{200}^{(1)}v_1 + f_{020}^{(1)}v_2 + f_{002}^{(1)}\mathrm{e}^{-\mathrm{i}\omega_k}v_1 + f_{110}^{(1)}(v_1+v_2) + f_{101}^{(1)}(v_1 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1) \\ &\quad + f_{011}^{(1)}(v_2 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1), \\ N &= f_{20}^{(2)}v_1 + f_{02}^{(2)}v_2 + f_{11}^{(2)}(v_1+v_2). \end{aligned} \tag{2.27}$$

Since $h_{110}(\theta)$ and $h_{200}(\theta)$ for $\theta \in [-1,0)$ appear in b_3, we still need to compute them. We know that $h_{110} = \mathrm{col}(h_{110}^1, h_{110}^2)$ is the solution of

$$\dot{h}_{110}(\theta) = (\psi_1, \psi_2)\begin{pmatrix} a_{110} \\ \bar{a}_{110} \end{pmatrix},$$

with the boundary condition

$$\dot{h}_{110}(0) - L_0h_{110} = \begin{pmatrix} a_{110} \\ \bar{a}_{110} \end{pmatrix},$$

and $h_{200} = \mathrm{col}(h_{200}^1, h_{200}^2)$ is the solution of

$$\dot{h}_{200}(\theta) - 2\mathrm{i}\omega_k h_{200}(\theta) = (\psi_1, \psi_2)\begin{pmatrix} a_{200} \\ \bar{a}_{200} \end{pmatrix},$$

with the boundary condition

$$\dot{h}_{200}(0) - L_0h_{200} = \begin{pmatrix} a_{200} \\ \bar{a}_{200} \end{pmatrix},$$

where

$$\begin{aligned}
a_{110} &= \tau_k(2f_{200}^{(1)}v_1\bar{v}_1 + 2f_{020}^{(1)}v_2\bar{v}_2 + 2f_{002}^{(1)}v_1\bar{v}_1 + 2f_{110}^{(1)}(v_1\bar{v}_2 + \bar{v}_1v_2) \\
&\quad + 2f_{101}^{(1)}(\mathrm{e}^{\mathrm{i}\omega_k}v_1\bar{v}_1 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_1) + 2f_{011}^{(1)}(\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1v_2 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_2)), \\
\bar{a}_{110} &= \tau_k(2f_{20}^{(2)}v_1\bar{v}_1 + 2f_{02}^{(2)}v_2\bar{v}_2 + 2f_{11}^{(2)}(v_1\bar{v}_2 + \bar{v}_1v_2)), \\
a_{200} &= \tau_k(f_{200}^{(1)}v_1^2 + f_{020}^{(1)}v_2^2 + f_{002}^{(1)}\mathrm{e}^{-2\mathrm{i}\omega_k}v_1^2 + 2f_{110}^{(1)}v_1v_2 + 2f_{101}^{(1)}\mathrm{e}^{-\mathrm{i}\omega_k}v_1^2 + 2f_{011}^{(1)}\mathrm{e}^{\mathrm{i}\omega_k}v_1v_2), \\
\bar{a}_{200} &= \tau_k(f_{20}^{(2)}v_1^2 + f_{02}^{(2)}v_2^2 + 2f_{11}^{(2)}v_1v_2).
\end{aligned} \tag{2.28}$$

Solving for $h_{110}(\theta)$ and $h_{200}(\theta)$, we have

$$h_{110}(\theta) = \frac{1}{\mathrm{i}\omega_k}(a_{110}\mathrm{e}^{\mathrm{i}\omega_k\theta}v - \bar{a}_{110}\mathrm{e}^{-\mathrm{i}\omega_k\theta}\bar{v}) + D_1, \tag{2.29}$$

$$h_{200}(\theta) = \frac{1}{-\mathrm{i}\omega_k}(a_{200}\mathrm{e}^{\mathrm{i}\omega_k\theta}v + \frac{1}{3}\bar{a}_{200}\mathrm{e}^{-\mathrm{i}\omega_k\theta}\bar{v}) + \mathrm{e}^{2\mathrm{i}\omega_k\theta}D_2, \tag{2.30}$$

where $D_1 = \mathrm{col}(D_1^{(1)}, D_1^{(2)}), D_2 = \mathrm{col}(D_2^{(1)}, D_2^{(2)})$ are respectively the solutions of the following linear algebra equations

$$\begin{pmatrix} -p_1 - p_5 & -p_2 \\ -p_3 & 0 \end{pmatrix} D_1 = \begin{pmatrix} a_{110} \\ \bar{a}_{110} \end{pmatrix},$$

$$\begin{pmatrix} 2\mathrm{i}\omega_k - p_1 - p_5\mathrm{e}^{-2\mathrm{i}\omega_k} & -p_2 \\ -p_3 & 2\mathrm{i}\omega_k \end{pmatrix} D_2 = \begin{pmatrix} a_{200} \\ \bar{a}_{200} \end{pmatrix}.$$

Hence, we know that

$$\begin{aligned}
D_1^{(1)} &= -\frac{\tau_k(2f_{20}^{(2)}v_1\bar{v}_1 + 2f_{02}^{(2)}v_2\bar{v}_2 + 2f_{11}^{(2)}(v_1\bar{v}_2 + \bar{v}_1v_2))}{p_3}, \\
D_1^{(2)} &= -\frac{a_{110} + (p_1 + p_5)D_1^{(1)}}{p_2}, \\
D_2^{(1)} &= \frac{2\mathrm{i}\omega_k a_{200} + p_2\bar{a}_{200}}{-4{\omega_k}^2 - 2\mathrm{i}\omega_k(p_1 + p_5\mathrm{e}^{-2\mathrm{i}\omega_k}) - p_2p_3}, \\
D_2^{(2)} &= \frac{2\mathrm{i}\omega_k\bar{a}_{200} - (p_1 + p_5\mathrm{e}^{-2\mathrm{i}\omega_k})\bar{a}_{200} + p_3a_{200}}{-4{\omega_k}^2 - 2\mathrm{i}\omega_k(p_1 + p_5\mathrm{e}^{-2\mathrm{i}\omega_k}) - p_2p_3}.
\end{aligned} \tag{2.31}$$

(iv) We compute $\mathrm{Proj}_S f_3^1(x,0,0)$. Because $f_3^1(x,0,0)$ is given by

$$f_3^1(x,0,0) = \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}[F_3(\phi_1x_1 + \phi_2x_2, \tau_k)].$$

By computing, F_3 can be represented in

$$\begin{aligned}
&F_3(\phi_1x_1 + \phi_2x_2, \tau_k) \\
&\quad = \tau_k\begin{pmatrix} f_{300}^{(1)}{Q_1}^3 + f_{030}^{(1)}{Q_2}^3 + 3f_{210}^{(1)}{Q_1}^2Q_2 + 3f_{120}^{(1)}Q_1{Q_2}^2 + 3f_{021}^{(1)}{Q_2}^2T_1 + 3f_{201}^{(1)}{Q_1}^2T_1 \\ f_{30}^{(2)}{Q_1}^3 + f_{03}^{(2)}{Q_2}^3 + 3f_{21}^{(2)}{Q_1}^2Q_2 + 3f_{12}^{(2)}Q_1{Q_2}^2 \end{pmatrix},
\end{aligned} \tag{2.32}$$

where $Q_1 = v_1x_1 + \bar{v}_1x_2$, $Q_2 = v_2x_1 + \bar{v}_2x_2, T_1 = \mathrm{e}^{-\mathrm{i}\omega_k}v_1x_1 + \mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1x_2$. Thus, we obtain

$$\mathrm{Proj}_S f_3^1(x,0,0) = \begin{pmatrix} 3c_3x_1^2x_2 \\ 3\bar{c}_3x_1x_2^2 \end{pmatrix},$$

where

$$\begin{aligned} c_3 = {} & \tau_k u_1(f_{300}^{(1)}v_1^2\bar{v}_1 + f_{030}^{(1)}v_2^2\bar{v}_2 + f_{210}^{(1)}(v_1^2\bar{v}_2 + 2v_1\bar{v}_1v_2) + f_{120}^{(1)}(2v_1v_2\bar{v}_2 + v_2^2\bar{v}_1) \\ & + f_{021}^{(1)}(\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1v_2^2 + 2\mathrm{e}^{-\mathrm{i}\omega_k}v_1v_2\bar{v}_2) + f_{201}^{(1)}(\mathrm{e}^{\mathrm{i}\omega_k}v_1^2\bar{v}_1 + 2\mathrm{e}^{-\mathrm{i}\omega_k}v_1^2\bar{v}_1)) + \tau_k u_2(f_{30}^{(2)}v_1^2\bar{v}_1 \\ & + f_{03}^{(2)}v_2^2\bar{v}_2 + f_{21}^{(2)}(v_1^2\bar{v}_1 + 2v_1v_2\bar{v}_1) + f_{12}^{(2)}(2v_1v_2\bar{v}_2 + v_2^2\bar{v}_1)), \qquad (2.33) \\ \bar{c}_3 = {} & \tau_k \bar{u}_1(f_{300}^{(2)}v_1\bar{v}_1^2 + f_{030}^{(2)}v_2\bar{v}_2^2 + f_{210}^{(2)}(2v_1\bar{v}_1\bar{v}_2 + \bar{v}_1^2v_2) + f_{120}^{(2)}(v_1\bar{v}_2^2 + 2\bar{v}_1v_2\bar{v}_2) \\ & + f_{021}^{(2)}(2\mathrm{e}^{\mathrm{i}\omega_k}\bar{v}_1\bar{v}_2 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_2) + f_{201}^{(2)}(2\mathrm{e}^{\mathrm{i}\omega_k}v_1\bar{v}_1^2 + \mathrm{e}^{-\mathrm{i}\omega_k}v_1\bar{v}_1^2)) + \tau_k\bar{u}_2(f_{30}^{(2)}v_1\bar{v}_1^2 \\ & + f_{03}^{(2)}v_2\bar{v}_2^2 + f_{21}^{(2)}(2v_1\bar{v}_1\bar{v}_2 + \bar{v}_1^2v_2) + f_{12}^{(2)}(v_1\bar{v}_2^2 + 2\bar{v}_1v_2\bar{v}_2)). \qquad (2.34) \end{aligned}$$

Summarizing (i)-(iv), we obtain

$$\frac{1}{3!}g_3^1(x,0,0) = \begin{pmatrix} A_2x_1^2x_2 \\ \bar{A}_2x_1x_2^2 \end{pmatrix},$$

where $A_2 = \frac{1}{2\mathrm{i}\omega_k}(a_{200}a_{110} - 2|a_{110}|^2 - \frac{1}{3}|a_{020}|^2) + \frac{1}{2}(b_3 + c_3)$. Thus, the normal form (2.12) has the form

$$\begin{aligned} \dot{x} &= \boldsymbol{B}x + \frac{1}{2}g_2^1(x,0,\beta) + \frac{1}{3!}g_3^1(x,0,\beta) + \text{h.o.t.} \\ &= \boldsymbol{B}x + \begin{pmatrix} A_1\beta x_1 \\ \bar{A}_1\beta x_2 \end{pmatrix} + \begin{pmatrix} A_2x_1^2x_2 \\ \bar{A}_2x_1x_2^2 \end{pmatrix} + O(|x|\beta^2 + |x|^4), \qquad (2.35) \end{aligned}$$

which can be written in real coordinates (w_1, w_2) through the change of variables $x_1 = w_1 - \mathrm{i}w_2$, $x_2 = w_1 + \mathrm{i}w_2$. Setting $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this normal form becomes

$$\dot{\rho} = l_1\beta\rho + l_2\rho^3 + O(\beta^2\rho + |(\rho,\beta)|^4), \quad \dot{\xi} = -\omega_k + O(|(\rho,\beta)|), \qquad (2.36)$$

where $l_1 = \mathrm{Re}A_1, l_2 = \mathrm{Re}A_2$. Following Ref. [18], we know that the sign of l_1l_2 determines the direction of the bifurcation and that the sign of l_2 determines the stability of the nontrivial periodic orbits.

Summarizing the above discussions, we can obtain the following theorem.

Theorem 2.1 The flow of (2.7) on the center manifold of the origin at $\beta = 0$ is given by (2.36). Hopf bifurcation is supercritical if $l_1l_2 < 0$ and subcritical if $l_1l_2 > 0$. Moreover, the nontrivial periodic solution is stable if $l_2 < 0$ and unstable if $l_2 > 0$.

3 Discussion

We consider the system (0.1) with $a = 2.2, b = 0.63$, $d = 0.45$, $n = 2.1$, $m = 1$, $h = 0.1$, that is,

$$\begin{aligned} \dot{x}(t) &= x(t)(1 - x(t-\tau)) - \frac{x(t)y(t)}{1 + 2.2x(t) + 0.63x(t)^2} - 0.1, \\ \dot{y}(t) &= y(t)\Big(-0.45 + \frac{2.1x(t)}{1 + 2.2x(t) + 0.63x(t)^2}\Big). \qquad (3.1) \end{aligned}$$

By computing, we get the equilibrium $(x^*, y^*) = (0.459\,280\,184, 0.692\,262\,152\,6)$. In the following, we only consider the direction and stability of Hopf bifurcation at $\tau_0 = 2.258\,612\,153$, we have $\omega_0 = 0.206\,875\,025\,4$.

From Theorem 1.1, we obtain that the characteristic equation of (3.1) at the equilibrium (x^*, y^*) has two simple roots $\pm i\omega_0$ for $\tau = \tau_0$ and the other roots $\lambda(\neq \pm i\omega_0)$ have non-zero real parts. Furthermore, if $\tau \in [0, \tau_0)$, all roots of the characteristic equation of (3.1) at the equilibrium (x^*, y^*) have negative real parts and the equilibrium (x^*, y^*) is locally asymptotically stable. At $\tau = \tau_0$, Hopf bifurcation occurs for system (3.1). In the following, we omit the complicated expressions and directly obtain the numerical results by Maple.

$$v = \begin{pmatrix} 1 \\ -i1.326\,436\,531 \end{pmatrix}, \quad u = \begin{pmatrix} 0.518\,379\,287\,9 + i0.025\,410\,967\,11 \\ 0.026\,321\,240\,20 - i0.536\,948\,699\,7 \end{pmatrix},$$

$$a_{200} = -3.223\,583\,807 + i2.030\,757\,661, \quad a_{110} = -1.813\,669\,110 - i0.370\,799\,194\,4,$$
$$a_{020} = -0.569\,561\,250\,9 + i0.046\,288\,286\,66, \quad b_3 = -10.312\,056\,17 + i1.984\,773\,990,$$
$$c_3 = 0.210\,578\,807\,5 + i5.871\,645\,710.$$

Then, we have $A_1 = 0.001\,965\,827\,328 - i0.040\,102\,533\,91$, $A_2 = -2.388\,547\,367 + i3.539\,654\,600$. Thus, $l_1 = 0.001\,965\,827\,328$, $l_2 = -2.388\,547\,367$. From Theorem 2.1, we know that there is a supercritical Hopf bifurcation at $\tau = \tau_0$ and the nontrivial periodic solution associated with Hopf bifurcation at $\tau = \tau_0$ is stable in the center manifold. The computer simulation is depicted in the Fig. 1.

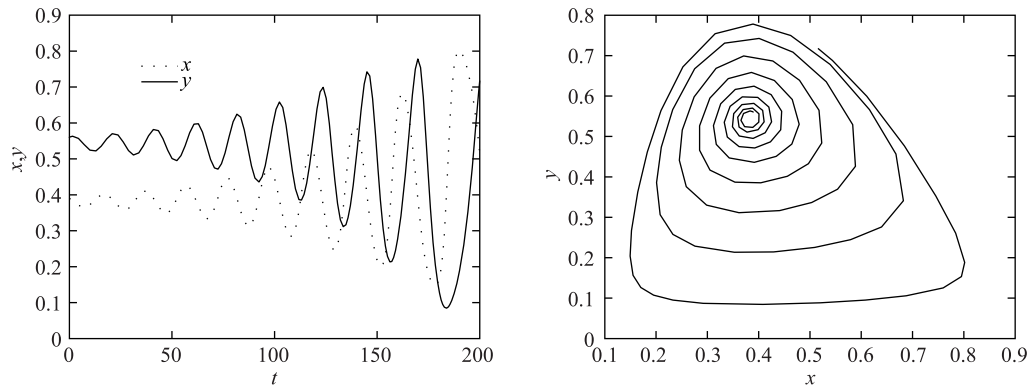

Fig. 1　Dynamic behavior of system (3.1) with initial values $x_0 = 0.39$, $y_0 = 0.56$, (Left) Time series of the prey population x and the predator population y; (Right) Phase portraits.

[References]

[1] CHEN Y Y, YU J, SUN C J. Stability and Hopf bifurcation analysis in a three-level food chain system with delay [J]. Chaos, Solitons and Fractals, 2007, 31: 683-694.

[2] CHEN Y Y, SONG C M. Stability and Hopf bifurcation analysis in a prey-predator system with stage-structure for prey and time delay [J]. Chaos, Solitons and Fractals, 2008, 38: 1104-1114.

[3] LIN Y P, LI J B, ZHAO X H. Local stability and bifurcation in a three-unit delayed neural network [J]. Journal of Systems Science and Complexity, 2003, 16: 46-52.

[4] DING X Q. Periodicity in a delayed simi-ratio-dependent predator-prey system [J]. Applied Mathematics, 2005, 20: 151-158.
[5] FAN M, WANG K. Periodicity in a delayed ratio-dependent predator-prey system [J]. Journal of Mathematical Analysis and Applications, 2001, 262: 179-190.
[6] GAN Q T, XU R, YANG P H. Bifurcation and chaos in a ratio-dependent predator-prey system with time delay [J]. Chaos, Solitons and Fractals, 2009(9): 1883-1895.
[7] LING L, WANG W M. Dynamics of a Ivlev-type predator-prey system with constant rate harvesting [J]. Chaos, Solitons and Fractals, 2009, 41: 2139-2153.
[8] LIN Y P, LI J B, CAO J D. The Hopf bifurcation direction of a four dimensional electronic neural network system [J]. Journal of Systems Science and Complexity, 1997(10): 337-343.
[9] MAITI A, PAL A K, SAMANTA G P. Effect of time-delay on a food chain model [J]. Applied Mathematics and Computation, 2008, 200: 189-203.
[10] MARTIN A, RUAN S G. Predator-prey models with delay and prey harvesting [J]. Journal of Mathematical Biology, 2001, 43: 247-267.
[11] PECELLI G. Prey-predator systems with delay: Hopf bifurcation and stable oscillations [J]. Mathematical and Computer Modelling, 1997, 25: 77-98.
[12] SUN C J, HAN M A, LIN Y P, et al. Global qualitative analysis for a predator-prey system with delay [J]. Chaos, Solitons and Fractals, 2007, 32: 1582-1596.
[13] SONG Y L, WEI J J. Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system [J]. Journal of Mathematical Analysis and Applications, 2005, 301: 1-21.
[14] YAN X P, LI W T. Hopf bifurcation and global periodic solutions in a delayed predator-prey system [J]. Applied Mathematics and Computation, 2006, 177: 427-445.
[15] YAN X P, ZHANG C H. Hopf bifurcation in a delayed Lokta-Volterra predator-prey system[J]. Nonlinear Analysis: Real World Applications, 2008(9): 114-127.
[16] FARIA T, MAGALHAES L T. Normal form for retarded functional differential equations with parameters and applications to Hopf bifurcation [J]. Journal of Differential Equations, 1995, 122: 181-200.
[17] HALE J K, LUNEL S M. Introduction to Functional Differentail Equation [M]. New York: Springer-Verlag, 1993.
[18] CHOW S, HALE J. Methods of Bifurcation Theory [M]. New York: Springer-Verlag, 1982.

(上接第 185 页)

[References]

[6] WANG L L, LI W T. Periodic solutions and permanence for a delayed nonautonomous ratio-dependent predator–prey model with Holling type functional response [J]. J Comput Appl Math, 2004, 162: 341-357.
[7] ZHANG Z Q, HOU Z T, WANG L. Multiplicity of positive periodc solutions to a generalized delayed predator-prey system with stocking [J]. Nonlinear analysis, 2008, 68: 2608-2622.
[8] MA Z E. Mathematical Modeling and Studying on Species Ecology [M]. Hefei: Education Press, 1996 (in chinese).
[9] GAINESE R E, MAWHIN J L. Coincidence Degree and Nonliear Differential Equations [M]. Berin: Springer, 1977.