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Note on representations of Cartan type Lie algebras over a finite field

YAO Yu-feng

(*Department of Mathematics, Shanghai Maritime University, Shanghai* 201306, *China*)

Abstract: A sufficient and necessary condition was obtained for an irreducible generalized χ-reduced module of a Cartan type Lie algebra over $k=\bar{\mathbb{F}}_q$ being split over $\mathbb{F}_q$, where the height of the character χ is no more than 0. For the Witt algebra, the corresponding result for general χ was given.

Key words: Cartan type Lie algebra; generalized restricted Lie algebra; modular representation; Frobenius morphism; $\mathbb{F}_q$-form

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有限域上 Cartan 型李代数表示的一点注记

姚裕丰

(上海海事大学 数学系, 上海 201306)

摘要: 得到了当特征函数 χ 的高度小于或等于 0 时, Cartan 型李代数在代数封闭域 $k=\bar{\mathbb{F}}_q$上的不可约广义χ-约化表示分裂的一个充分必要条件. 在Witt代数情形, 对于一般的特征函数χ, 得到了相应的结论.

关键词: Cartan型李代数; 广义限制李代数; 模表示; Frobenius 同态; $\mathbb{F}_q$-型

0 Introduction

Recall that all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero arise from simple algebraic groups. While in the case of prime characteristic, there are finite-dimensional simple Lie algebras of so-called Cartan type besides classical Lie algebras arising from simple algebraic groups. These simple Lie algebras of Cartan type fall into four classes: types W, S, H and K (cf. [1]). One can also define the Cartan type Lie algebras over a finite field. More works on representations of Cartan type Lie algebras are over algebraically closed fields (cf. [2-23]). It seems to be necessary to consider representations of Cartan type Lie algebras over a finite field. Jie Du and Bin Shu[24] developed an approach to

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作者简介: 姚裕丰, 男, 博士, 讲师, 研究方向为李代数及表示理论. E-mail: yfyao@shmtu.edu.cn.

investigate representations of a finite Lie algebra L^F over a finite field $\mathbb{F}_q$ through representations of a Lie algebra L with a Frobenius morphism F over the algebraic closure $k = \bar{\mathbb{F}}_q$. Initiated by their work, we give a sufficient and necessary condition for irreducible generalized χ-reduced modules of Cartan type Lie algebras over $k = \bar{\mathbb{F}}_q$ being split over $\mathbb{F}_q$, where the height of the character χ is no more than 0. For the Witt algebra, we can give some further results for more general χ.

Throughout this paper, we define q as a given power of a prime $p > 2$, $\mathbb{F}_q$ the finite field of q elements and k the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. All the vector spaces and modules are of finite dimensional.

1 Preliminaries

1.1 Lie algebras of Cartan type

Fix a positive integer m and an m-tuple $\mathbf{n} = (n_1, \cdots, n_m)$ of positive integers. Denote by $A(m;\mathbf{n})$ the index set $\{\alpha = (\alpha_1, \cdots, \alpha_m) \mid 0 \leqslant \alpha_i \leqslant p^{n_i-1}, i = 1, 2, \cdots, m\}$. We have the divided power algebra $\mathfrak{A}(m;\mathbf{n})$ with a basis $\{x^\alpha \mid \alpha \in A(m;\mathbf{n})\}$.

Let $D_i\,(1 \leqslant i \leqslant m)$ be the linear partial derivation of $\mathfrak{A}(m;\mathbf{n})$ with respect to the i-th invariant x_i such that $D_i(x^\alpha) = x^{\alpha-\varepsilon_i}$, $\forall\, \alpha \in A(m;\mathbf{n})$. In the following, we will recall the four classes of Cartan type Lie algebras, drawing most of the notations and results from [25].

(1) The generalized Jacobson-Witt algebra $W(m;\mathbf{n})$ is by definition a collection of all special derivations of the divided power algebra $\mathfrak{A}(m;\mathbf{n})$. By [25, Proposition 2.2, Chapter 4], $W(m;\mathbf{n}) = k\text{-span}\{x^\alpha D_i \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i \leqslant m\}$. In the following, the standard basis of $W(m;\mathbf{n})$ is always referred to $\{x^\alpha D_i \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i \leqslant m\}$ denoted by $\{E_i^W \mid i = 1, 2, \cdots, t_W\}$ such that $E_i^W = D_i$ for $1 \leqslant i \leqslant m$, where $t_W = \dim W(m;\mathbf{n}) = mp^{\sum n_i}$.

It is worth mentioning that $W(m;\mathbf{n})_0 = F\text{-span}\{x^\alpha D_j \mid |\alpha| \geqslant 1, j = 1, 2, \cdots, m\}$ admits a structure of a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, and the zero-graded component $W(m;\mathbf{n})_{[0]} \cong \mathfrak{gl}(m)$. Moreover, $\mathfrak{h}^W := k$-span$\{h_i^W := x_i D_i \mid i = 1, 2, \cdots m\}$ is a canonical torus of $W(m;\mathbf{n})_{[0]}$.

(2) Here in this case we assume $m \geqslant 3$. Define the divergence map div from the generalized Jacobson-Witt algebra $W(m;\mathbf{n})$ to the divided power algebra $\mathfrak{A}(m;\mathbf{n})$ as follows

$$\begin{aligned} \text{div}: \qquad & W(m;\mathbf{n}) \longrightarrow \mathfrak{A}(m;\mathbf{n}), \\ & \sum_{i=1}^m f_i D_i \longmapsto \sum_{i=1}^m D_i(f_i). \end{aligned}$$

Set $\widetilde{S(m;\mathbf{n})} = \{D \in W(m;\mathbf{n}) \mid \text{div } D = 0\}$. Then by definition, the derived algebra of $\widetilde{S(m;\mathbf{n})}$ is called the special algebra $S(m;\mathbf{n})$. By [25, Proposition 3.3, Chapter 4], $S(m;\mathbf{n}) = k$-span$\{D_{ij}(x^\alpha) \mid \alpha \in A(m;\mathbf{n}), 1 \leqslant i < j \leqslant m\}$, where D_{ij} is a linear map from $\mathfrak{A}(m;\mathbf{n})$ to $W(m;\mathbf{n})$ defined as follows

$$\begin{aligned} D_{ij}: \qquad & \mathfrak{A}(m;\mathbf{n}) \longrightarrow W(m;\mathbf{n}) \\ & x^\alpha \longmapsto D_{ij}(x^\alpha) = x^{\alpha-\varepsilon_j} D_i - x^{\alpha-\varepsilon_i} D_j. \end{aligned}$$

A standard basis of $S(m;\mathbf{n})$ is taken from $\{D_{ij}(x^\alpha)\mid\alpha\in A(m;\mathbf{n}),1\leqslant i<j\leqslant m\}$ denoted by $\{E_i^S\mid i=1,2,\cdots,t_S\}$ such that $E_i^S=D_i$ for $1\leqslant i\leqslant m$, where $t_S=\dim S(m;\mathbf{n})=(m-1)(p^{\sum n_i}-1)$. It is obvious that $S(m;\mathbf{n})$ is a graded subalgebra of $W(m;\mathbf{n})$.

It is worth mentioning that $S(m;\mathbf{n})_0=F\text{-span}\{D_{ij}(x^\alpha)\mid|\alpha|\geqslant 2,1\leqslant i<j\leqslant m\}$ admits a structure of a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, and the zero-graded component $S(m;\mathbf{n})_{[0]}\cong\mathfrak{sl}(m)$. Moreover, $\mathfrak{h}^S:=k\text{-span}\{h_i^S:=x_iD_i-x_{i+1}D_{i+1}\mid i=1,2,\cdots m-1\}$ is a canonical torus of $S(m;\mathbf{n})_{[0]}$.

(3) Here in this case we assume $m=2r$ is even. Define the Hamiltonian operator D_H from $\mathfrak{A}(2r;\mathbf{n})$ to $W(2r;\mathbf{n})$ as follows

$$\begin{aligned}D_H:\quad &\mathfrak{A}(2r;\mathbf{n})\longrightarrow W(2r;\mathbf{n}),\\ &f\longmapsto D_H(f)=\sum_{i=1}^{2r}\sigma(i)D_i(f)D_{i'},\end{aligned}$$

where $\sigma(i):=\begin{cases}1, & \text{if}\quad 1\leqslant i\leqslant r,\\ -1, & \text{if}\quad r+1\leqslant i\leqslant 2r,\end{cases}$ and $i':=\begin{cases}i+r, & \text{if}\quad 1\leqslant i\leqslant r,\\ i-r, & \text{if}\quad r+1\leqslant i\leqslant 2r.\end{cases}$

Then by definition $H(2r;\mathbf{n})=k\text{-span}\{D_H(x^\alpha)\mid 0\prec\alpha\prec\tau\}$ is the Hamiltonian algebra. The standard basis is always referred to $\{D_H(x^\alpha)\mid\alpha\in A(2r;\mathbf{n})\}$ denoted by $\{E_i^H\mid i=1,2,\cdots,t_H\}$ such that $E_i^H=D_i$ for $1\leqslant i\leqslant 2r$, where $t_H=\dim H(m;\mathbf{n})=p^{\sum n_i}-2$. It is obvious that $H(2r;\mathbf{n})$ is a graded subalgebra of $W(2r;\mathbf{n})$.

It's specially worth mentioning that $H(2r;\mathbf{n})_0=k\text{-span}\{D_H(x^\alpha)\mid|\alpha|\geqslant 2\}$ admits a structure of a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, and the zero-graded component $H(2r;\mathbf{n})_{[0]}\cong\mathfrak{sp}(2r)$. Moreover, $\mathfrak{h}^H:=k\text{-span}\{h_i^H:=x_iD_i-x_{i+r}D_{i+r}\mid i=1,2,\cdots r\}$ is a canonical torus of $H(2r;\mathbf{n})_{[0]}$.

(4) Here in this case we assume that $m=2r+1$ is odd. Define the contact operator D_K from $\mathfrak{A}(2r+1;\mathbf{n})$ to $W(2r+1;\mathbf{n})$ as follows

$$\begin{aligned}D_K:\quad &\mathfrak{A}(2r+1;\mathbf{n})\longrightarrow W(2r+1;\mathbf{n}),\\ &f\longmapsto D_K(f)=\sum_{i=1}^{2r+1}f_iD_i.\end{aligned}$$

where

$$f_j=x_jD_{2r+1}(f)+\sigma(j')D_{j'}(f),\ j\leqslant 2r,$$

$$f_{2r+1}=2f-\sum_{i=1}^{2r}\sigma(j)x_jf_{j'}.$$

Set $\widetilde{K(2r+1;\mathbf{n})}=k\text{-span}\{D_K(x^\alpha)\mid\alpha\in A(m;\mathbf{n})\}$. The contact algebra $K(2r+1;\mathbf{n})$ is the derived algebra of $\widetilde{K(2r+1;\mathbf{n})}$. The standard basis is always referred to $\{D_K(x^\alpha)\mid 0\preceq\alpha\preceq\tau\}$ when $2r+4\not\equiv 0\,(\mathrm{mod}\,p)$ or $\{D_K(x^\alpha)\mid 0\preceq\alpha\prec\tau\}$ when $2r+4\equiv 0\,(\mathrm{mod}\,p)$ denoted by $\{E_i^K\mid i=1,2,\cdots,t_K\}$ such that $E_i^K=D_K(x_i)$ for $1\leqslant i\leqslant 2r$, $E_{2r+1}^K=D_K(1)$, and

$$t_K=\dim K(m;\mathbf{n})=\begin{cases}p^{\sum n_i}, & \text{if}\quad 2r+4\not\equiv 0\,(\mathrm{mod}\,p),\\ p^{\sum n_i}-1, & \text{if}\quad 2r+4\equiv 0\,(\mathrm{mod}\,p).\end{cases}$$

It is obvious that $K(2r+1;\mathbf{n})$ is a subalgebra of $W(2r+1;\mathbf{n})$, but it is not a graded subalgebra. One can, however, define a new gradation on $K(2r+1;\mathbf{n})$ which doesn't inherit from the gradation of $W(2r+1;\mathbf{n})$. For that, define $||\alpha|| = \sum\limits_{i=1}^{2r}\alpha_i + \alpha_{2r+1} - 2$ for $\alpha \in A(2r+1;\mathbf{n})$ and $K(2r+1;\mathbf{n})_{[i]} = k\text{-span}\{D_K(x^\alpha) \mid ||\alpha|| = i\}$. Then $K(2r+1;\mathbf{n}) = \bigoplus\limits_{i\geqslant -2} K(2r+1;\mathbf{n})_{[i]}$ is a gradation of $K(2r+1;\mathbf{n})$. Associated with this gradation, one can obtain the corresponding filtration.

Notice that $K(2r+1;\mathbf{n})_0 = F\text{-span}\{D_K(x^\alpha) \mid ||\alpha|| \geqslant 0\}$ admits a structure of a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, and the zero-graded component $K(2r+1;\mathbf{n})_{[0]} \cong \mathfrak{sp}(2r)\bigoplus kI$. Moreover, $\mathfrak{h}^K := k\text{-span}\{h_i^K := x_iD_i - x_{i+r}D_{i+r}, h_{r+1}^K = D_K(1) \mid i = 1,2,\cdots r\}$ is a canonical torus of $K(2r+1;\mathbf{n})_{[0]}$.

Remark 1.1 The structure constants of the four classes of Cartan type Lie algebras with respect to the standard basis chosen above are integers. This property is important to our discussion below.

1.2 Generalized restricted Lie algebras and their generalized reduced representations

As is well known that not all of the graded Lie algebras of Cartan type are restricted Lie algebras, but there are generalized restricted Lie algebras in the following sense, among which restricted Lie algebras are.

Definition 1.1[9] A generalized restricted Lie algebra L is a Lie algebra associated with an ordered basis $E = (e_i)_{i\in I}$ and a so-called generalized restricted map $\varphi_{\mathbf{s}} : E \to L$ sending $e_i \longmapsto e_i^{\varphi_{\mathbf{s}}}$ with $\mathbf{s} = (s_i)_{i\in I}, s_i \in \mathbb{Z}_+$ such that $\operatorname{ad} e_i^{\varphi_{\mathbf{s}}} = (\operatorname{ad} e_i)^{p^{s_i}}$ for all $i \in I$.

Let us demonstrate how the four graded Cartan type Lie algebras $X(m;\mathbf{n})$, $X \in \{W,S,H,K\}$, are endowed with a generalized restricted structure.

Example 1.1 In $L = X(m;\mathbf{n}), X \in \{W,S,H,K\}$, there is a standard basis $\{e_i := E_i^X \mid i = 1,2,\cdots t_X\}$ of L (see §1.1). Then, associated with this basis and $\mathbf{s} := (n_1,n_2,\cdots,n_m,1,1,\cdots,1)$, L is a generalized restricted Lie algebra with a generalized restricted mapping $\varphi_{\mathbf{s}}$ such that $e_i^{\varphi_{\mathbf{s}}} = 0$ for $i = 1,\cdots,m$ and $e_j^{\varphi_{\mathbf{s}}} = e_j^{[p]}$ for $j > m$, because L_0 is a restricted Lie algebra with $[p]$-mapping defined just as the p-th power as usual derivations, as well as $\operatorname{ad}(e_i)^{p^{n_i}} = 0$ for $i = 1,\cdots,m$.

We have the following basic fact by Schur lemma.

Lemma 1.1 Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$. Let $(L,\varphi_{\mathbf{s}})$ be a generalized restricted Lie algebra over $\mathbb{F}$ associated with a basis $E = (e_i)_{i\in I}$ and $\varphi_{\mathbf{s}}, \mathbf{s} = (s_i)_{i\in I}$. If (V,ρ) is an irreducible representation of L, then there exists a unique $\chi \in L^*$ such that

$$\rho(e_i)^{p^{s_i}} - \rho(e_i^{\varphi_{\mathbf{s}}}) = \chi(e_i)^{p^{s_i}}\,\mathbf{i}d_V,\ \ \forall\, x \in L. \tag{1.1}$$

Here the function χ is also called a (generalized) character of V. A representation (module) of L satisfying (1.1) is called a generalized χ-reduced representation (module), all of which constitute a full subcategory of the Lie algebra representation category.

Let's continue to recall some facts. Assume L is a generalized restricted Lie algebra. For

$\chi \in L^*$, we define $U_{p^{\mathbf{s}}}(L,\chi) := U(L)/\langle e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}}\rangle$ where $\langle e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}}\rangle$ means the ideal in $U(L)$ generated by those central elements $e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}}$ for all $e_i \in E$. Call $U_{p^{\mathbf{s}}}(L,\chi)$ the generalized χ-reduced enveloping algebra of L. A generalized χ-reduced module category of L coincides with the unitary $U_{p^{\mathbf{s}}}(L,\chi)$-module category. Especially, in the case when $\chi = 0$, we have the generalized restricted enveloping algebra $U_{p^{\mathbf{s}}}(L) := U_{p^{\mathbf{s}}}(L,0)$ (cf. [9, 10]).

Remark 1.2 (1) We know that a restricted Lie algebra $(\mathfrak{g},[p])$ can be a generalized restricted Lie algebra associated with an arbitrary given basis E and $\mathbf{s} = \mathbf{1}$. Furthermore, it's easily seen that in this sense, a generalized χ-reduced module category and a generalized χ-reduced enveloping algebra coincide with the ones arising from a restricted Lie algebra.

(2) The invariance of filtration for $L = X(m;\mathbf{n})$ under $\mathrm{Aut}(L)$, $X \in \{W,S,H,K\}$, enables us to define the height of a nonzero $\chi \in L^*$ via $\mathrm{ht}(\chi) := \max\{i \mid \chi(L_{i-1}) \neq 0\}$, and $\mathrm{ht}(0) := -1-\delta_{XK}$. Then the height function on L^* is invariant under the action of $\mathrm{Aut}(L)$ defined by $\sigma\cdot\chi = \chi\circ\sigma^{-1}$ for $\sigma \in \mathrm{Aut}(L)$ and $\chi \in L^*$.

1.3 Frobenius morphism and module Frobenius morphism

Let V be a k-vector space. A map $f : V \longrightarrow V$ is called a $(q-)$semilinear map if $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(av_1) = a^q f(v_1)$, $\forall\, v_1, v_2 \in V$ and $a \in k$. A $(q-)$semilinear map which is invertible is called a $(q-)$semilinear isomorphism. If F is a $(q-)$semilinear isomorphism and in addition, for any $v \in V$, there always exists $n \in \mathbb{N}$ such that $F^n(v) = v$, then F is called a $(q-)$Frobenius map on V.

A Frobenius map F on V gives rise to an $\mathbb{F}_q$-subspace V_0 of V, where V_0 is the fixed points of F on V, i.e. $V_0 = V^F := \{v \in V \mid F(v) = v\}$. Moreover V_0 is an $\mathbb{F}_q$-structure of V, i.e. $V_0 \bigotimes_{\mathbb{F}_q} k \cong V$.

Remark 1.3 In our assumption that V is of finite dimensional, any $(q-)$semilinear isomorphism is a Frobenius map (see [26, 2.2]).

Let L be a Lie algebra over k. A Frobenius map F on L is called a Frobenius morphism if F keeps the Lie bracket structure of L, i.e. $F[x,y] = [F(x),F(y)]$, $\forall\, x,y \in L$. Assume M is an L-module with a Frobenius map F_M, if additionally $F_M(xm) = F(x)F_M(m)$, $\forall\, x \in L, m \in M$, then F_M is called a module Frobenius morphism, and M is called an F-stable L-module with module Frobenius morphism F_M.

Remark 1.4 (1) Let $L = X(m;\mathbf{n}), X \in \{W,S,H,K\}$, there is a standard basis $\{E_i^X \mid i = 1,2,\cdots t_X\}$ for L (see §1.1). We can define a Frobenius morphism $F_0 : L \longrightarrow L$, $\sum_{i=1}^{t_X} a_i E_i^X \longmapsto \sum_{i=1}^{t_X} a_i^q E_i^X$. Then one can easily check that $L^{F_0} = \mathbb{F}_q\text{-span}\{E_i^X \mid i = 1,2,\cdots t_X\}$ denoted by $L_{\mathbb{F}_q}$, and $L_{\mathbb{F}_q}$ is the corresponding Cartan type Lie algebra over a finite field $\mathbb{F}_q$. The Frobenius morphism F_0 is called the standard Frobenius morphism on L.

(2) Let $L = X(m;\mathbf{n}), X \in \{W,S,H,K\}$ and let F be an arbitrary Frobenius morphism on L, then one can easily see that FF_0^{-1} is an automorphism of L. So there exists $\sigma \in \mathrm{Aut}(L)$ such that $F = \sigma F_0$. Then $\{\sigma F_0 \mid \sigma \in \mathrm{Aut}(L)\}$ is the set of all Frobenius morphisms on L.

(3) Any Frobenius morphism F on a Lie algebra L can be extended to be defined on the universal enveloping algebra $U(L)$ of L, because F keeps the Lie bracket structure of L.

Denote by L-**mod** finite dimensional L-module category, L^F-**mod** finite dimensional L^F-module category. Denote by L-$\mathbf{mod}^F$ the category with objects consisting of finite dimensional F-stable modules (V, f), where V is an F-stable L-module and f is a module Frobenius morphism, and the set of morphisms between two objects (V, f) and (W, g) is $\mathrm{Hom}((V, f), (W, g)) = \{\varphi \in \mathrm{Hom}_L(V, W) \mid \varphi f = g\varphi\}$.

The following result is important to the discussion below, which asserts that L^F-**mod** can be imbedded into a subcategory of L-**mod**.

Theorem 1.1[24] The category L-$\mathbf{mod}^F$ is equivalent to the category L^F-**mod**. In particular, there is a one-to-one correspondence between isoclasses of simple L^F-modules and that of simple F-stable L-modules.

1.4 Frobenius twist of L-modules

For each k-space V, let $V^{(1)}$ be a new k-vector space obtained from V by a twist of scalar multiplication, i.e. $V^{(1)} = V$ as an abelian group together with a new scalar multiplication: $a \centerdot v = \sqrt[q]{a}v$ for any $a \in k$, $v \in V$. More precisely, if V has a basis $\{v_i \mid i = 1, 2, \cdots t\}$, then $V^{(1)}$ as a k-vector space has a basis $\{v_i^{(1)} \mid i = 1, 2, \cdots, t\}$ such that $(u+v)^{(1)} = u^{(1)} + v^{(1)}$ and $(av)^{(1)} = a^q v^{(1)}, \forall\, u, v \in V, a \in k$. So the canonical map $\tau_V : V \longrightarrow V^{(1)}$ sending any $v \in V$ to $v^{(1)} \in V^{(1)}$ is a q-semilinear isomorphism. A k-linear map $\varphi : V \longrightarrow W$ naturally induces a k-linear map $\varphi^{(1)} : V^{(1)} \longrightarrow W^{(1)}$ such that $\varphi^{(1)}(v^{(1)}) = (\varphi(v))^{(1)}, \forall\, v \in V$.

Definition 1.2 Let L be a Lie algebra over k with a Frobenius morphism F and Let V be an L-module defined by the Lie algebra homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$. This gives rise to a Lie algebra homomorphism $\rho^{(1)} : L^{(1)} \longrightarrow \mathfrak{gl}(V)^{(1)} \cong \mathfrak{gl}(V^{(1)})$. Thus the composition $\rho^{[1]}$ of the following maps

$$L \xrightarrow{F^{-1}} L \xrightarrow{\tau_L} L^{(1)} \xrightarrow{\rho^{(1)}} \mathfrak{gl}(V^{(1)})$$

defines an L-module structure on $V^{(1)}$ with the following new action

$$x \cdot (v^{(1)}) = (F^{-1}(x)v)^{(1)},\ \forall\, x \in L, v \in V.$$

We denote this module by $V^{[1]}$ and call it the Frobenius twist of V.

The following lemma will be used in the next two sections.

Lemma 1.2 Let $L = X(m; \mathbf{n}), X \in \{W, S, H, K\}$ be a graded Lie algebra of Cartan type with the standard Frobenius morphism F_0 and let V be a (generalized) χ-reduced L-module with a Frobenius map F_V, then

(1) V is F_0-stable if and only if $V \cong V^{[1]}$.

(2) $V^{[1]}$ is a (generalized) $\chi^{[1]}$-reduced L-module, where $\chi^{[1]} \in L^*$ is the twist of χ such that $\chi^{[1]}(x) = (\chi(F^{-1}(x)))^q, \forall\, x \in L$.

By Lemma 1.2, we have the following direct consequence.

Corollary 1.1 Let $L = X(m; \mathbf{n}), X \in \{W, S, H, K\}$ be a graded Lie algebra of Cartan type with the standard Frobenius morphism F_0 and let V be a (generalized) χ-reduced L-module. If V is F_0-stable, then $\chi = \chi^{[1]}$, where $\chi^{[1]}$ is defined as in Lemma 1.2.

2 $\mathbb{F}_q$-forms of Cartan type Lie algebras

In this section, we always assume $L = X(m;\mathbf{n}), X \in \{W, S, H, K\}$. Let k' be a subfield of k. A Lie k'-subalgebra $\widetilde{L}$ of L is called a k'-form of L if the natural homomorphism from $\widetilde{L} \bigotimes_{k'} k$ to L is an isomorphism of Lie algebras. From Remark 1.4, we can see that $L_{\mathbb{F}_q}$ is an $\mathbb{F}_q$-form of L. Moreover, any Frobenius morphism F on L determines an $\mathbb{F}_q$-form L^F of L. On the other hand, given an $\mathbb{F}_q$-form $\widetilde{L}$, there exists a unique Frobenius morphism F' such that $\widetilde{L} = L^{F'}$. Two Frobenius morphisms F_1, F_2 are called equivalent denoted as $F_1 \sim F_2$, if $L^{F_1} \cong L^{F_2}$ as Lie $\mathbb{F}_q$-algebras. So, to determine $\mathbb{F}_q$-forms of L is equivalent to determine the equivalent classes of Frobenius morphisms on L.

The following proposition gives a sufficient condition for two Frobenius morphisms on L being equivalent.

Proposition 2.1　Let $F_1 = \sigma_1 F_0$, $F_2 = \sigma_2 F_0$ be two Frobenius morphisms on L with $\sigma_1, \sigma_2 \in \mathrm{Aut}(L)$. If there exists $\sigma \in \mathrm{Aut}(L)$ keeping $L_{\mathbb{F}_q}$ stable such that $\sigma_1 = \sigma^{-1}\sigma_2\sigma$, then $F_1 \sim F_2$.

Proof　Since σ keeps $L_{\mathbb{F}_q}$ stable, then $\sigma \in \mathrm{Aut}(L_{\mathbb{F}_q})$. Therefore $\sigma F_0 \sigma^{-1} = F_0$ by a direct computation. Hence

$$F_1 = \sigma_1 F_0 = \sigma^{-1}\sigma_2\sigma F_0 \sigma^{-1}\sigma = \sigma^{-1}\sigma_2 F_0 \sigma = \sigma^{-1} F_2 \sigma.$$

Therefore, by [24, Lemma 2.6], $F_1 \sim F_2$. □

Remark 2.1　The condition given in Proposition 2.1 is not a necessary condition for two Frobenius morphisms being equivalent. If it was a necessary condition, take any $\theta \in \mathrm{Aut}(L)$, then $\theta^{-1}F_0\theta \sim F_0$ by [24, Lemma 2.6]. Set $\vartheta = \theta^{-1}F_0\theta F_0^{-1} \in \mathrm{Aut}(L)$, then $\theta^{-1}F_0\theta = \vartheta F_0$, i.e. $\vartheta F_0 \sim F_0$. So $\vartheta = \mathbf{i}d$ by the assumption above. We then obtain that $\theta F_0 = F_0 \theta$ which does not hold for all $\theta \in \mathrm{Aut}(L)$.

We can give the following necessary condition for two Frobenius morphisms being equivalent. Its proof is straightforward.

Proposition 2.2　Let $F_1 = \sigma_1 F_0$, $F_2 = \sigma_2 F_0$ be two Frobenius morphisms on L with $\sigma_1, \sigma_2 \in \mathrm{Aut}(L)$. If $F_1 \sim F_2$, then there exists some $i \geqslant 0$ such that $\sigma_1, \sigma_2 \in G_i$, where $G_i = \{\sigma \in \mathrm{Aut}(L) \mid \sigma(L_j) \subseteq (L_{[j]} + L_{j+i}) \setminus (L_{[j]} + L_{j+i+1}), \forall\, j\}$.

3 Representations of Cartan type Lie algebras over a finite field

In this section, we always assume $L = X(m;\mathbf{n})$, $X \in \{W, S, H, K\}$. We will adopt the theory developed in [24] to study irreducible representations of L over a finite field $\mathbb{F}_q$. For that we first recall some known results about irreducible representations of L over the algebraic closure $k = \bar{\mathbb{F}}_q$ of $\mathbb{F}_q$ (see [2-23]).

Theorem 3.1　Let $L = X(m;\mathbf{n})$, $X \in \{W, S, H, K\}$. Every irreducible L-module M over k corresponds to a unique character $\chi \in L^*$. Assume $\mathrm{ht}(\chi) \leqslant 0$, then

(1) Non-exceptional case. i.e. there doesn't exist an irreducible L_0-submodule of M with an exceptional weight ω_i as its "highest" weight.

In this case, $M \cong U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_p(L_0)} L_0(\lambda)$ for some $\lambda \in (\mathfrak{h}^X)^*$ such that $\lambda(h_i^X) \in \mathbb{F}_p$ for all $i = 1, 2, \cdots, \dim \mathfrak{h}^X$. Here $L_0(\lambda)$ is the irreducible restricted $L_{[0]}$-module with "highest" weight λ which can be viewed as an irreducible L_0-module with trivial action by L_1.

(2) Exceptional case. i.e. there exists an irreducible L_0-submodule of M with an exceptional weight ω_i as its "highest" weight.

In this case, $M \cong \big(U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_p(L_0)} L_0(\omega_i)\big)/\widetilde{M}$, where $L_0(\omega_i)$ is the irreducible restricted $L_{[0]}$-module with the exceptional "highest" weight ω_i which can be viewed as an irreducible L_0-module with trivial action by L_1. And $\widetilde{M}$ is the unique maximal submodule of $U_{p^{\mathbf{s}}}(L,\chi)\bigotimes_{U_p(L_0)} L_0(\omega_i)$.

Remark 3.1 The exceptional weights "ω_i" appearing above for the four classes of Cartan type Lie algebras are actually the fundamental weights for the corresponding classical Lie algebras $\mathfrak{gl}(m)$, $\mathfrak{sl}(m)$, $\mathfrak{sp}(2r)$, $\mathfrak{sp}(2r)\bigoplus KI$. Readers are referred to [5,6,14−16].

Before presenting the main result of this section, we need the following definition.

Definition 3.1 Assume $\mathfrak{g}$ is a Lie algebra over k with a Frobenius morphism F and M is a $\mathfrak{g}$-module. If there exists a $\mathfrak{g}^F$-submodule M_1 of M such that $M_1 \bigotimes_{\mathbb{F}_q} k \cong M$ as $\mathfrak{g}$-modules, then M is called to be split over $\mathbb{F}_q$.

Remark 3.2 If a $\mathfrak{g}$-module M is split over $\mathbb{F}_q$ in the sense of Definition 3.1, then it is obvious that there exists a module Frobenius morphism F_M on M fixing M_1. So M is F-stable. On the other hand, if M is F-stable with module Frobenius morphism F_M, take $M_1 = M^{F_M}$, then M_1 is a $\mathfrak{g}^F$-submodule of M and $M_1 \bigotimes_{\mathbb{F}_q} k \cong M$, so M is split over $\mathbb{F}_q$.

By Theorem 1.1, Corollary 1.1 and Theorem 3.1, we have the following result describing the connection of irreducible representations of L over a finite field $\mathbb{F}_q$ and its algebraic closure $k = \overline{\mathbb{F}}_q$.

Theorem 3.2 Let $L = X(m;\mathbf{n})$, $X \in \{W,S,H,K\}$, be a graded Lie algebra of Cartan type over k with standard Frobenius morphism F_0 defined as in Remark 1.4. Let $\chi \in L^*$ with $\mathrm{ht}(\chi) \leqslant 0$. Then an irreducible L-module M with character χ is split over $\mathbb{F}_q$ if and only if $\chi(E_i^X) \in \mathbb{F}_q$ for all $i = 1, 2, \cdots, t_X = \dim L$.

Proof (1) "$\Rightarrow$": If M is split over $\mathbb{F}_q$, by Remark 3.2, M is F_0-stable. So $\chi = \chi^{[1]}$ by Corollary 1.1. Therefore $\chi(E_i^X) \in \mathbb{F}_q$, $i = 1, 2, \cdots, t_X = \dim L$.

(2) "$\Leftarrow$": Assume $\chi(E_i^X) \in \mathbb{F}_q, i = 1, 2, \cdots, t_X = \dim L$, then $F_0 J_\chi = J_\chi$, where $J_\chi = \langle (E_i^X)^{p^{s_i}} - (E_i^X)^{\varphi_{\mathbf{s}}} - \chi(E_i^X)^{p^{s_i}} \mid i = 1, 2, \cdots, t_X\rangle$. So F_0 can be extended to be defined on the generalized χ-reduced enveloping algebra $U_{p^{\mathbf{s}}}(L,\chi)$. Assume $\lambda \in (\mathfrak{h}^X)^*$ is a restricted weight which is not exceptional, $L_0(\lambda)$ is the irreducible $L_{[0]}$-module with "highest" weight λ. Fix a "highest" weight vector v_λ, then $L_0(\lambda) = U_p(\mathbf{n}^-)v_\lambda = k\text{-span}\{\prod\limits_{i=1}^{s} N_i^{a_i} v_\lambda \mid a_i < p, i = 1, 2, \cdots, s = (\dim L_{[0]} - \dim \mathfrak{h}^X)/2\}$, where the N_i's are standard negative root vectors of the

classical Lie algebra $\mathfrak{gl}(m), \mathfrak{sl}(m)$ or $\mathfrak{sp}(2r)$. Define

$$F_M: \qquad M = U_{p^{\mathbf{s}}}(L,\chi)\bigotimes\nolimits_{U_p(L_0)} L_0(\lambda) \longrightarrow M = U_{p^{\mathbf{s}}}(L,\chi)\bigotimes\nolimits_{U_p(L_0)} L_0(\lambda)$$
$$\sum a_{\mathbf{k},\mathbf{l}}(E_1^X)^{k_1}\cdots(E_m^X)^{k_m}\otimes N_1^{l_1}\cdots N_s^{l_s}v_\lambda \longmapsto \sum a_{\mathbf{k},\mathbf{l}}^q(E_1^X)^{k_1}\cdots(E_m^X)^{k_m}\otimes N_1^{l_1}\cdots N_s^{l_s}v_\lambda.$$

Note that the structure constants with respect to the standard basis of L are integers (see Remark 1.1), $h_i^X\cdot v_\lambda=\lambda(h_i^X)v_\lambda$, $\lambda(h_i^X)\in\mathbb{F}_p$ and $\chi(E_i^X)\in\mathbb{F}_q$. These facts assure that F_M defined above is a module Frobenius morphism. Then M is split over $\mathbb{F}_q$ by Remark 3.2. The argument for the exceptional case is similar by the precise construction of exceptional irreducible modules in [5,7,21−23].

By Theorem 3.2 and Theorem 1.1, we immediately obtain the following:

Corollary 3.1 Let $L=X(m;\mathbf{n})$, $X\in\{W,S,H,K\}$, be a graded Lie algebra of Cartan type over $k=\bar{F}_q$ with standard Frobenius morphism F_0 defined as in Remark 1.4. Let $L_{\mathbb{F}_q}=L^{F_0}=\mathbb{F}_q\text{-span}\{E_i^X\mid i=1,2,\cdots,t_X\}$ be the corresponding Cartan type Lie algebra over a finite field $\mathbb{F}_q$. Let $\chi\in(L_{\mathbb{F}_q})^*$ with $\mathrm{ht}(\chi)\leqslant 0$. Then every irreducible $L_{\mathbb{F}_q}$-module M with character χ is as follows.

(1) Non-exceptional case. i.e. there doesn't exist an irreducible $(L_{\mathbb{F}_q})_0$-submodule of M with an exceptional weight ω_i as its "highest" weight.

In this case, $M\cong U_{p^{\mathbf{s}}}(L_{\mathbb{F}_q},\chi)\bigotimes_{U_p((L_{\mathbb{F}_q})_0)}(L_{\mathbb{F}_q})_0(\lambda)$ for some $\lambda\in(\mathfrak{h}^X)^*$ with $\lambda(h_i^X)\in\mathbb{F}_p$ for all $i=1,2,\cdots,\dim\mathfrak{h}^X$. Here $(L_{\mathbb{F}_q})_0(\lambda)$ is the irreducible restricted $(L_{\mathbb{F}_q})_{[0]}$-module with "highest" weight λ which can be viewed as an irreducible $(L_{\mathbb{F}_q})_0$-module with trivial action by $(L_{\mathbb{F}_q})_1$.

(2) Exceptional case. i.e. there exists an irreducible $(L_{\mathbb{F}_q})_0$-submodule of M with an exceptional weight ω_i as its "highest" weight.

In this case, $M\cong\Big(U_{p^{\mathbf{s}}}(L_{\mathbb{F}_q},\chi)\bigotimes_{U_p((L_{\mathbb{F}_q})_0)}(L_{\mathbb{F}_q})_0(\omega_i)\Big)/M'$, where $(L_{\mathbb{F}_q})_0(\omega_i)$ is the irreducible restricted $(L_{\mathbb{F}_q})_{[0]}$-module with the exceptional "highest" weight ω_i which can be viewed as an irreducible $(L_{\mathbb{F}_q})_0$-module with trivial action by $(L_{\mathbb{F}_q})_1$. And M' is the unique maximal submodule of $U_{p^{\mathbf{s}}}(L_{\mathbb{F}_q},\chi)\bigotimes_{U_p((L_{\mathbb{F}_q})_0)}(L_{\mathbb{F}_q})_0(\omega_i)$.

Remark 3.3 When $X\in\{W,S,H\}$, $\mathrm{ht}(\chi)=-1$, i.e. $\chi=0$. It is obvious that $\chi\in(L_{\mathbb{F}_q})^*$. In this case, the results in the Corollary 3.1 were obtained in [5] by Shen using the techniques of "mixed product" (cf. [5, Remark 4.1]).

4 Further discussion on representations of the Witt algebra over a finite field

In this section, we assume that $L=W(1,\mathbf{1})$ is the Witt algebra. Then L has a standard basis $\{e_i\mid i=-1,0,\cdots,p-2\}$ with the Lie bracket satisfying

$$[e_i,e_j]=\begin{cases}(j-i)e_{i+j}, & \text{if } -1\leqslant i+j\leqslant p-2,\\ 0, & \text{otherwise.}\end{cases}$$

Let $F_0, L_{\mathbb{F}_q}$ be as before, i.e. $F_0\big(\sum\limits_{i=-1}^{p-2} a_i e_i\big) = a_i^q e_i$, and $L_{\mathbb{F}_q} = \mathbb{F}_q\text{-span}\{e_i \mid i = -1, 0, \cdots, p-2\}$. We will study irreducible $L_{\mathbb{F}_q}$-modules over a finite field $\mathbb{F}_q$. For that, we first recall the well-known facts about irreducible representations of L over an algebraically closed field $k = \bar{\mathbb{F}}_q$.

Theorem 4.1[2,27] Every irreducible L-module M corresponds to a unique character $\chi \in L^*$. Assume $\mathrm{ht}(\chi) = r < p-1$. Then

Case I: $r = -1$. In this case, $M \cong M(\lambda) := U(L,\chi) \bigotimes_{U_p(L_0)} L_0(\lambda)$ for some $\lambda \in \{2, 3, \cdots, p-1\}$, or $M \cong M(\mu) := \big(U(L,\chi) \bigotimes_{U_p(L_0)} L_0(\mu)\big)/\widetilde{M}$ for some $\mu \in \{0, 1\}$. Here $L_0(\lambda)$ for $\lambda \in \mathbb{F}_p$ represents a one-dimensional restricted $\mathfrak{h} = ke_0$-module with the action of e_0 as multiplication by the scalar λ, which can be viewed as an L_0-module with trivial action by L_1. And $\widetilde{M}$ is the unique maximal submodule of $U(L,\chi) \bigotimes_{U_p(L_0)} L_0(\mu)$.

Case II: $0 \leqslant r < p-1$. In this case, $M \cong U(L,\chi) \bigotimes_{U(L_s,\chi)} k_\chi$. Here $s = [\frac{r}{2}]$ and k_χ is a one-dimensional $U(L_s,\chi)$-module with a basis v_χ such that $xv_\chi = \chi(x)v_\chi$, $\forall\, x \in L_s \cap L_1$.

Moreover, when $r = -1$ or 1, there are totally p non-isomorphic irreducible L-modules with character χ. When $r = 0$, there are totally $p-1$ non-isomorphic irreducible L-modules with character χ. When $1 < r < p-1$, up to isomorphism, there is only one irreducible L-module with character χ. The dimensions are given as follows.

$$\dim_k M = \begin{cases} 1, & \text{if} \quad r = -1,\ M \cong M(0), \\ p-1, & \text{if} \quad r = -1,\ M \cong M(1), \\ p, & \text{if} \quad r = -1,\ M \cong M(\lambda),\ \lambda \in \{2, 3, \cdots, p-1\}, \\ p^{s+1}, & \text{if} \quad 0 \leqslant r < p-1. \end{cases}$$

Applying Theorem 1.1 and Theorem 4.1, we obtain the following result on representations of the Witt algebra L over a finite field $\mathbb{F}_q$ and its algebraic closure $k = \overline{\mathbb{F}}_q$.

Theorem 4.2 The following statements hold.

(1) Assume M is an irreducible L-module with character χ of height less than $p-1$, and M is split over $\mathbb{F}_q$. Then $\chi(e_i) \in \mathbb{F}_q$, $\forall -1 \leqslant i \leqslant p-2$.

(2) Let $\chi \in L^*$ with $\mathrm{ht}(\chi) < p-1$ and $\chi(e_i) \in \mathbb{F}_q$, $\forall -1 \leqslant i \leqslant p-2$. Moreover, when $\mathrm{ht}(\chi) = 1$, χ additionally satisfies that $\lambda \in \mathbb{F}_q$ for any λ satisfying $\lambda^p - \lambda = \chi(e_0)^p$. Then any irreducible L-module M with character χ is split over $\mathbb{F}_q$.

Proof Assume M is an irreducible L-module with character χ, we can assume $\mathrm{ht}(\chi) > 0$ if we take Theorem 3.2 into account. With this assumption, by Theorem 4.1, if $r = \mathrm{ht}(\chi) > 1$, then $M = U(L,\chi)v_\chi$ such that $xv_\chi = \chi(x)v_\chi, \forall\, x \in L_s$, where $s = [\frac{r}{2}]$. And if $r = \mathrm{ht}(\chi) = 1$, then $M = U(L,\chi)v_\chi$ such that $xv_\chi = \chi(x)v_\chi, \forall\, x \in L_1$ and $e_0 v_\chi = \lambda v_\chi$ for some λ satisfying $\lambda^p - \lambda = \chi(e_0)^p$.

(1) If M is split over $\mathbb{F}_q$, then by Remark 3.2, there exists a module Frobenius morphism F_M on M such that $M = M_1 \bigotimes_{\mathbb{F}_q} k$, where $M_1 = M^{F_M}$ is an L^{F_0}-module. Therefore

$$0 = F_M((e_i^p - e_i^{[p]} - \chi(e_i)^p)M) = (F_0(e_i)^p - F_0(e_i)^{[p]} - \chi(e_i)^{pq})M = (e_i^p - e_i^{[p]} - \chi(e_i)^{pq})M, \forall -1 \leqslant i \leqslant p-2.$$

which implies that $\chi(e_i) \in \mathbb{F}_q$, $\forall -1 \leqslant i \leqslant p-2$.

(2) By the assumption stated, $F_0 J_\chi = J_\chi$, where $J_\chi = \langle e_i^p - e_i^{[p]} - \chi(e_i)^p \mid -1 \leqslant i \leqslant p-2\rangle$. Therefore, F_0 can be extended to be defined on the χ-reduced enveloping algebra $U(L,\chi)$.

Define

$$\begin{aligned} F_M: \qquad M = U(L,\chi)v_\chi &\longrightarrow M = U(L,\chi)v_\chi \\ \sum a_{\underline{\mathbf{i}}}\, e_{-1}^{i_1} e_0^{i_2} \cdots e_{s-1}^{i_{s-1}} v_\chi &\longmapsto \sum a_{\underline{\mathbf{i}}}^q\, e_{-1}^{i_1} e_0^{i_2} \cdots e_{s-1}^{i_{s-1}} v_\chi \end{aligned}$$

Then one can easily check that F_M is a module Frobenius morphism on M. Hence, by Theorem 1.1, M^{F_M} is an irreducible L^{F_0}-module, and $M \cong M^{F_M} \bigotimes_{\mathbb{F}_q} k$. Therefore M is split over $\mathbb{F}_q$.

Applying Theorem 1.1, Theorem 4.1 and Theorem 4.2, we immediately obtain the following corollary.

Corollary 4.1 Let $L = W(1;\mathbf{1})$ be the Witt algebra over an algebraic closed field $k = \bar{\mathbb{F}}_q$. Let F_0 be the standard Frobenius morphism on L. Set $L_{\mathbb{F}_q} = L^{F_0} = \mathbb{F}_q\text{-span}\{e_i \mid -1 \leqslant i \leqslant p-2\}$ which is the Witt algebra over a finite field $\mathbb{F}_q$. Assume $\chi \in (L_{\mathbb{F}_q})^*$ such that $r = \mathrm{ht}(\chi) < p-1$. Moreover, if $r = 1$, assume additionally that $\lambda \in \mathbb{F}_q$ for all λ satisfying $\lambda^p - \lambda = \chi(e_0)^p$. Then any irreducible $L_{\mathbb{F}_q}$-module M with character χ over a finite field $\mathbb{F}_q$ is as follows.

Case I: $r = -1$. In this case, $M \cong M(\lambda) := U(L_{\mathbb{F}_q},\chi) \bigotimes_{U_p((L_{\mathbb{F}_q})_0)} (L_{\mathbb{F}_q})_0(\lambda)$ for some $\lambda \in \{2,\cdots,p-1\}$, or $M \cong M(\mu) := \big(U(L_{\mathbb{F}_q},\chi) \bigotimes_{U_p((L_{\mathbb{F}_q})_0)} (L_{\mathbb{F}_q})_0(\mu)\big)/M'$ for some $\mu \in \{0,1\}$. Here $(L_{\mathbb{F}_q})_0(\lambda)$ with $\lambda \in \mathbb{F}_p$ represents a one-dimensional restricted $\mathfrak{h} = \mathbb{F}_q e_0$-module with the action of e_0 as multiplication by the scalar λ, which can be viewed as an $(L_{\mathbb{F}_q})_0$-module with the trivial action by $(L_{\mathbb{F}_q})_1$. And M' is the unique maximal submodule of $U(L_{\mathbb{F}_q},\chi) \bigotimes_{U_p((L_{\mathbb{F}_q})_0)} (L_{\mathbb{F}_q})_0(\mu)$.

Case II: $0 \leqslant r < p-1$. In this case, $M \cong U(L_{\mathbb{F}_q},\chi) \bigotimes_{U((L_{\mathbb{F}_q})_s,\chi)} \mathbf{1}_\chi$. Here $s = [\frac{r}{2}]$ and $\mathbf{1}_\chi$ is a one-dimensional $U((L_{\mathbb{F}_q})_s,\chi)$-module with a basis w_χ such that $xw_\chi = \chi(x)w_\chi$, $\forall\, x \in (L_{\mathbb{F}_q})_s \cap (L_{\mathbb{F}_q})_1$.

Moreover, when $r = -1$ or 1, there are totally p non-isomorphic irreducible $L_{\mathbb{F}_q}$-modules with character χ. When $r = 0$, there are totally $p-1$ non-isomorphic irreducible $L_{\mathbb{F}_q}$-modules with character χ. When $1 < r < p-1$, up to isomorphism, there is only one irreducible $L_{\mathbb{F}_q}$-module with character χ. The dimensions of irreducible modules with character χ over $\mathbb{F}_q$ are given as follows.

$$\dim_{\mathbb{F}_q} M = \begin{cases} 1, & \text{if} \quad r = -1,\ M \cong M(0), \\ p-1, & \text{if} \quad r = -1,\ M \cong M(1), \\ p, & \text{if} \quad r = -1,\ M \cong M(\lambda),\ \lambda \in \{2,3,\cdots,p-1\}, \\ p^{s+1}, & \text{if} \quad 0 \leqslant r < p-1. \end{cases}$$

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