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On modular representations of finite-dimensional Lie superalgebras

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Abstract: In this paper, we studied representations of finite-dimensional Lie superalgebras over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. It was shown that simple modules of a finite-dimensional Lie superalgebra over $\mathbb{F}$ are finite-dimensional, and there exists an upper bound on the dimensions of simple modules. Moreover, a finite-dimensional Lie superalgebra can be embedded into a finite-dimensional restricted Lie superalgebra. We gave a criterion on simplicity of modules over a finite-dimensional restricted Lie superalgebra $\mathfrak{g}$, and defined a restricted Lie super subalgebra, then obtained a bijection between the isomorphism classes of simple modules of $\mathfrak{g}$ and those of this restricted subalgebra. These results are generalization of the corresponding ones in Lie algebras of prime characteristic.

Key words: Lie superalgebra; representation; p-envelope; p-character

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有限维李超代数的模表示

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摘要: 研究了特征大于2的代数闭域上有限维李超代数的表示. 证明了有限维李超代数的单模都是有限维的, 并且所有单模的维数有上界. 进一步,一个有限维李超代数可以嵌入到一个有限维限制李超代数. 给出了有限维限制李超代数 $\mathfrak{g}$ 上单模的判定准则, 定义了 $\mathfrak{g}$ 的一个限制李超子代数, 得到了该子代数的单模同构类和 $\mathfrak{g}$ 的单模同构类之间的一个双射. 这些结果是素特征域上李代数相关理论的推广.

关键词: 李超代数; 表示; p-包络; p-特征

0 Introduction

Recall that the finite-dimensional simple Lie superalgebras over the field of complex numbers were classified by Kac in the 1970s (cf.[1]). Furthermore, their representation theory was developed extensively.

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In recent years, there has been an increasing interest in modular representation theory of restricted Lie superalgebras. A systematical research on modular representation theory was initiated and developed in [2-6] for Lie superalgebras of classical type, and in [7-15] for Lie superalgebras of Cartan type, respectively. W. Wang and L. Zhao[3] proved a super version of the celebrated Kac-Weisfeiler Property for the classical Lie superalgebras, which by definition admit an even non-degenerate supersymmetric bilinear form and whose even subalgebras are reductive. In [7-15], all simple restricted and some simple non-restricted modules of Lie superalgebras of Cartan type were classified. Moreover, character formulas for these simple modules were given.

In this paper, we study the modular representations of finite-dimensional Lie superalgebras. This research is largely motivated by [3, 16, 17]. We briefly introduce the structure of this paper. We collect the general notations and elementary preliminaries on Lie (associative) superalgebras in Section 1. Then Section 2 is devoted to developing general representation theory for a finite-dimensional Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ over an algebraically closed field $\mathbb{F}$ of characteristic $p>2$. We show that each simple $\mathfrak{g}$-module is of finite-dimensional, and there exists an upper bound on the dimensions of simple modules. Moreover, $\mathfrak{g}$ has a finite-dimensional p-envelope which is a restricted Lie superalgebra. In some sense, this helps us to reduce representations of finite-dimensional Lie superalgebras to those of restricted ones. We then study irreducible representations of finite-dimensional restricted Lie superalgebras in Section 3. We give a criterion for simplicity of an induced module of a finite-dimensional restricted Lie superalgebra $\mathfrak{g}$, and obtain a bijection between the isomorphism classes of simple modules of $\mathfrak{g}$ and those of some restricted subalgebra (cf. Theorem 3.12). This reduces simple $\mathfrak{g}$-modules to those simple modules of a certain restricted subalgebra.

1 Notations and preliminaries

In this paper, we always assume that the ground field $\mathbb{F}$ is algebraically closed and of prime characteristic $p>2$. We exclude the case $p=2$, since in this case, Lie superalgebras coincide with $\mathbb{Z}_2$-graded Lie algebras.

1.1 Basic definitions

A superspace is a $\mathbb{Z}_2$-graded vector space $V=V_{\bar{0}}\oplus V_{\bar{1}}$, in which we call elements in $V_{\bar{0}}$ and $V_{\bar{1}}$ even and odd, respectively. We usually write $|v|\in\mathbb{Z}_2$ for the parity (or degree) of $v\in V$, which is implicitely assumed to be $\mathbb{Z}_2$-homogeneous. A superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{A}=\mathfrak{A}_{\bar{0}}\oplus\mathfrak{A}_{\bar{1}}$ endowed with an algebra structure "$\cdot$" such that $\mathfrak{A}_\alpha\cdot\mathfrak{A}_\beta\subseteq\mathfrak{A}_{\alpha+\beta}$ for any $\alpha,\beta\in\mathbb{Z}_2$. A superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ with an algebra structure $[-,-]$ is called a Lie superalgebra if for any homogeneous elements x,y,z in $\mathfrak{g}$, the following conditions hold.

(i) $[x,y]=-(-1)^{|x||y|}[y,x]$;

(ii) $[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]]$.

Homomorphisms of superalgebras (Lie superalgebras) are those linear mappings which reserve the $\mathbb{Z}_2$-grading and the superalgebra (Lie superalgebra) structure.

For a Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$, it follows from the definition that the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action. Let $(\mathfrak{A},\cdot)$ be an associative superalgebra, we denote $[x,y]:=x\cdot y-(-1)^{|x||y|}y\cdot x$ for any homogeneous elements $x,y\in\mathfrak{A}$. Then $(\mathfrak{A},[-,-])$ is a Lie superalgebra.

Example 1.1 Let $V=V_{\bar{0}}\oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$ with $\dim V_{\bar{0}}=m$ and $\dim V_{\bar{1}}=n$. Then the algebra $\mathrm{End}_{\mathbb{F}}(V)$ consisting of $\mathbb{F}$-linear transformation of V is an associative superalgebra with

$$\mathrm{End}_{\mathbb{F}}(V)_\alpha:=\{A\in\mathrm{End}_{\mathbb{F}}(V)\mid A(V_\beta)\subseteq V_{\alpha+\beta},\,\forall\,\beta\in\mathbb{Z}_2\},\ \alpha\in\mathbb{Z}_2.$$

Moreover, for any homogeneous elements $A,B\in\mathrm{End}_{\mathbb{F}}(V)$, we define a new multiplication $[-,-]$ by

$$[A,B]:=AB-(-1)^{|A||B|}BA.$$

Then $(\mathrm{End}_{\mathbb{F}}(V),[-,-])$ is the so-called general linear Lie superalgebra, denoted by $\mathfrak{gl}(V)=\mathfrak{gl}(V)_{\bar{0}}\oplus\mathfrak{gl}(V)_{\bar{1}}$ or $\mathfrak{gl}(m|n)=\mathfrak{gl}(m|n)_{\bar{0}}\oplus\mathfrak{gl}(m|n)_{\bar{1}}$. More precisely,

$$\mathfrak{gl}(m|n)_{\bar{0}}=\left\{\left.\begin{pmatrix}A&0\\0&D\end{pmatrix}\right| A\in\mathrm{Mat}_{m\times m},D\in\mathrm{Mat}_{n\times n}\right\},$$
$$\mathfrak{gl}(m|n)_{\bar{1}}=\left\{\left.\begin{pmatrix}0&B\\C&0\end{pmatrix}\right| B\in\mathrm{Mat}_{m\times n},C\in\mathrm{Mat}_{n\times m}\right\},$$

where $\mathrm{Mat}_{i\times j}$ denotes the set of all $i\times j$ matrices for $i,j\in\mathbb{N}\backslash\{0\}$.

Definition 1.2 Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. A $\mathbb{Z}_2$-graded vector space $V=V_{\bar{0}}\oplus V_{\bar{1}}$ is called a $\mathfrak{g}$-module if there exists a Lie superalgebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$.

Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. Denote by $U(\mathfrak{g})$ the universal enveloping superalgebra of $\mathfrak{g}$, which is the quotient of the tensor superalgebra $T(\mathfrak{g})$ by the ideal generated by $[x,y]-xy+(-1)^{|x||y|}yx$ for any $x,y\in\mathfrak{g}_{\bar{0}}\cup\mathfrak{g}_{\bar{1}}$. Set $Z(\mathfrak{g})=\{u\in U(\mathfrak{g})_{\bar{0}}\mid uv=vu,\,\forall\,v\in U(\mathfrak{g})\}$ which is called the even center of $U(\mathfrak{g})$.

In this paper, all Lie superalgebras are assumed to be finite-dimensional. By vector spaces, subalgebras, ideals, submodules etc., we mean in the super sense unless otherwise stated.

1.2 Key lemmas

In this subsection, we present several lemmas for later use. Let $\mathfrak{A}=\mathfrak{A}_{\bar{0}}\oplus\mathfrak{A}_{\bar{1}}$ be a superalgebra. For elements $y,z_1,\cdots,z_n$ in $\mathfrak{A}$ and

$$\mathbf{s}=(s_1,\cdots,s_n),\mathbf{t}=(t_1,\cdots,t_n)\in\mathbb{N}^n,$$

set

$$\mathbf{z}:=(z_1,\cdots,z_n)\in\mathfrak{A}^n,\mathbf{z}^{\mathbf{s}}:=z_1^{s_1}\cdots z_n^{s_n}\in\mathfrak{A},$$

and

$$\{y,\mathbf{z};\mathbf{t}\} := [\cdots[\cdots[\cdots[[\cdots[y,\underbrace{z_1],\cdots,z_1]}_{t_1\text{times}},\underbrace{z_2],\cdots,z_2]}_{t_2\text{ times}},\cdots,\underbrace{z_n],\cdots,z_n]}_{t_n\text{ times}} \in \mathfrak{A}$$

with the convention that $\{y,\mathbf{z};\mathbf{0}\}=y$. Let

$$|\mathbf{s}| := \sum_{i=1}^{n} s_i,\ \binom{\mathbf{s}}{\mathbf{t}} := \prod_{i=1}^{n}\binom{s_i}{t_i}.$$

We define a partial order "$\preccurlyeq$" on $\mathbb{N}^n$ as follows.

$$\mathbf{t}\preccurlyeq\mathbf{s} \text{ if and only if } t_i\leqslant s_i,\ \forall\, 1\leqslant i\leqslant n.$$

Lemma 1.3 Assume $\mathfrak{A}=\mathfrak{A}_{\bar 0}\oplus\mathfrak{A}_{\bar 1}$ is an associative superalgebra. Let $y\in\mathfrak{A}_{\bar 0}\cup\mathfrak{A}_{\bar 1}$, $z_1,\cdots,z_m\in\mathfrak{A}_{\bar 0}$ and $z_{m+1},\cdots,z_n\in\mathfrak{A}_{\bar 1}$. Let $\mathbf{s}=(s_1,\cdots,s_n)\in\mathbb{N}^n$ with $s_i\in\{0,1\}$ for $m+1\leqslant i\leqslant n$. Then

$$y\mathbf{z}^{\mathbf{s}} = \sum_{\mathbf{0}\preccurlyeq\mathbf{t}\preccurlyeq\mathbf{s}} \pm\binom{\mathbf{s}}{\mathbf{t}}\mathbf{z}^{\mathbf{s}-\mathbf{t}}\{y,\mathbf{z};\mathbf{t}\}. \tag{1.1}$$

Proof It is trivial for $\mathbf{s}=\mathbf{0}$. In the following, we assume $\mathbf{s}\neq\mathbf{0}$. For any $x\in\mathfrak{A}_{\bar 0}$, let L_x, R_x denote the left and right multiplications by x in $\mathfrak{A}$, respectively. Then $R_x=L_x-\mathrm{ad}x$, and L_x commutes with $\mathrm{ad}x$. We divide the proof into three cases.

Case 1 $s_{m+1}=s_{m+2}=\cdots=s_n=0$.

In this case, we proceed by induction on m. The case $m=1$ follows from the following computation.

$$\begin{aligned} yz_1^{s_1} &= \underbrace{R_{z_1}\circ R_{z_1}\circ\cdots\circ R_{z_1}}_{s_1\text{ times}}(y)\\ &= (L_{z_1}-\mathrm{ad}z_1)^{s_1}(y)\\ &= \sum_{0\leqslant t_1\leqslant s_1}(-1)^{t_1}\binom{s_1}{t_1}L_{z_1}^{s_1-t_1}(\mathrm{ad}z_1)^{t_1}(y)\\ &= \sum_{0\leqslant t_1\leqslant s_1}\binom{s_1}{t_1}z_1^{s_1-t_1}[\cdots[y,\underbrace{z_1],\cdots,z_1]}_{t_1\text{ times}}. \end{aligned}$$

Assume that $m>1$. Let $y':=yz_1^{s_1}\cdots z_{m-1}^{s_{m-1}}, \mathbf{s}'=(s_1,\cdots,s_{m-1})$. The induction hypothesis yields

$$\begin{aligned}
y\mathbf{s}^{\mathbf{s}} &= y' z_m^{s_m} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'}\{y,\mathbf{z};\mathbf{t}'\} z_m^{s_m} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'} \sum_{0\leqslant t_m\leqslant s_m} \pm \binom{s_m}{t_m} z_m^{s_m-t_m}\{\{y,\mathbf{z};\mathbf{t}'\}, z_m; t_m\} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \sum_{0\leqslant t_m\leqslant s_m} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \binom{s_m}{t_m} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'} z_m^{s_m-t_m}\{y,\mathbf{z};(\mathbf{t}',t_m)\} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}\preccurlyeq\mathbf{s}} \pm \binom{\mathbf{s}}{\mathbf{t}} \mathbf{z}^{\mathbf{s}-\mathbf{t}}\{y,\mathbf{z};\mathbf{t}\}.
\end{aligned}$$

Hence, (1.1) holds in this case.

Case 2 $s_1 = s_2 = \cdots = s_m = 0$.

In this case, we proceed by induction on $n-m$. If $n-m=1$, then

$$yz_{m+1} = [y, z_{m+1}] + (-1)^{|y|} z_{m+1} y.$$

Hence, (1.1) holds in this case. Assume that $n-m>1$. Put $y' = yz_{m+1}^{s_{m+1}} \cdots z_{n-1}^{s_{n-1}}$ and

$$\mathbf{s}' = (\underbrace{0, 0, \cdots, 0}_{m\,\text{times}}, s_{m+1}, s_{m+2}, \cdots, s_{n-1}).$$

The induction hypothesis yields

$$\begin{aligned}
y\mathbf{z}^{\mathbf{s}} &= y' z_n \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'}\{y,\mathbf{z};\mathbf{t}'\} z_n \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'}[\{y,\mathbf{z};\mathbf{t}'\}, z_n] + \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'} z_n\{y,\mathbf{z};\mathbf{t}'\} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'}\{y,\mathbf{z};(\mathbf{t}',1)\} + \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm \binom{\mathbf{s}'}{\mathbf{t}'} \mathbf{z}^{\mathbf{s}'-\mathbf{t}'} z_n\{y,\mathbf{z};(\mathbf{t}',0)\} \\
&= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}\preccurlyeq\mathbf{s}} \pm \binom{\mathbf{s}}{\mathbf{t}} \mathbf{z}^{\mathbf{s}-\mathbf{t}}\{y,\mathbf{z};\mathbf{t}\}.
\end{aligned}$$

Hence, (1.1) holds in this case.

Case 3 $(s_1, \cdots, s_m) \neq \mathbf{0}$ and $(s_{m+1}, \cdots, s_n) \neq \mathbf{0}$.

Let $\mathbf{s}' = (s_1, \cdots, s_m, 0, \cdots, 0)$, $\mathbf{s}'' = (0, \cdots, 0, s_{m+1}, \cdots, s_n)$. It follows from Case 1 and

Case 2 that

$$\begin{aligned} y\mathbf{z}^{\mathbf{s}} &= y\mathbf{z}^{\mathbf{s}'}\mathbf{z}^{\mathbf{s}''} \\ &= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm\binom{\mathbf{s}'}{\mathbf{t}'}\mathbf{z}^{\mathbf{s}'-\mathbf{t}'}\{y,\mathbf{z};\mathbf{t}'\}\mathbf{z}^{\mathbf{s}''} \\ &= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}'} \pm\binom{\mathbf{s}'}{\mathbf{t}'}\mathbf{z}^{\mathbf{s}'-\mathbf{t}'} \sum_{\mathbf{0}\preccurlyeq\mathbf{t}''\preccurlyeq\mathbf{s}''} \pm\binom{\mathbf{s}''}{\mathbf{t}''}\mathbf{z}^{\mathbf{s}''-\mathbf{t}''}\{\{y,\mathbf{z};\mathbf{t}'\},\mathbf{z};\mathbf{t}''\} \\ &= \sum_{\substack{\mathbf{0}\preccurlyeq\mathbf{t}'\preccurlyeq\mathbf{s}' \\ \mathbf{0}\preccurlyeq\mathbf{t}''\preccurlyeq\mathbf{s}''}} \pm\binom{\mathbf{s}'}{\mathbf{t}'}\binom{\mathbf{s}''}{\mathbf{t}''}\mathbf{z}^{\mathbf{s}'-\mathbf{t}'}\mathbf{z}^{\mathbf{s}''-\mathbf{t}''}\{\{y,\mathbf{z};\mathbf{t}'\},\mathbf{z};\mathbf{t}''\} \\ &= \sum_{\mathbf{0}\preccurlyeq\mathbf{t}\preccurlyeq\mathbf{s}} \pm\binom{\mathbf{s}}{\mathbf{t}}\mathbf{z}^{\mathbf{s}-\mathbf{t}}\{y,\mathbf{z};\mathbf{t}\}. \end{aligned}$$

Hence, (1.1) holds in this case.

In conclusion, we finish the proof by the three cases above.

We have the following super version of Engel's Theorem in Lie algebras.

Lemma 1.4 Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie super subalgebra. Moreover, assume that $\mathfrak{g}$ consists of nilpotent transformations. Then there exists a nonzero element $v \in V_{\bar{0}} \cup V_{\bar{1}}$ such that $xv = 0$ for any $x \in \mathfrak{g}$.

Proof Let $m = \dim\mathfrak{g}_{\bar{1}}$. We proceed by induction on m. For the case $m = 0$, the assertion follows from Engel's Theorem (see [18]). Assume that $m = 1$ and $\mathfrak{g}_{\bar{1}} = \mathrm{span}_{\mathbb{F}}\{y\}$. Since y is nilpotent, $W_1 := \{v \in V \mid yv = 0\}$ is a nonzero $\mathbb{Z}_2$-graded subspace. Moreover, it is easy to check that W_1 is a $\mathfrak{g}_{\bar{0}}$-submodule. By Engel's Theorem again, $W_2 := \{v \in W_1 \mid xv = 0, \forall\, x \in \mathfrak{g}_{\bar{0}}\}$ is a nonzero $\mathbb{Z}_2$-graded subspace. Consequently, any nonzero homogeneous vector v in W_2 satisfies the desired requirement.

Assume that $n > 1$ and the assertion holds for any $m < n$. We will show that it also holds for $m = n$. For that, regard $\mathfrak{g}_{\bar{1}}$ as a $\mathfrak{g}_{\bar{0}}$-module via adjoint action. Since

$$(\mathrm{ad}x)^{p^r}(y) = x^{p^r}y - yx^{p^r} = 0,\ \forall\, x \in \mathfrak{g}_{\bar{0}}, y \in \mathfrak{g}_{\bar{1}}, r > 0,$$

it follows that $\dim_{\mathbb{F}}[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] < \dim_{\mathbb{F}}\mathfrak{g}_{\bar{1}} = n$ by applying Engel's Theorem to $\mathfrak{g}_{\bar{0}}$ and its adjoint module $\mathfrak{g}_{\bar{1}}$. This implies that the odd part of the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ has dimension strictly less than n. According to the induction hypothesis, $W_3 := \{v \in V \mid xv = 0, \forall\, x \in [\mathfrak{g}, \mathfrak{g}]\}$ is a nonzero $\mathbb{Z}_2$-graded subspace.

Let $\{x_1, \cdots, x_l\}$ be a homogeneous basis of $\mathfrak{g}$. Since x_1 is nilpotent, and W_3 is invariant under the action of x_1, it follows that $W_3^{x_1} := \{v \in W_3 \mid x_1 v = 0\}$ is a nonzero $\mathbb{Z}_2$-graded subspace. For $2 \leqslant i \leqslant l$, define $W_3^{x_1,\cdots,x_i} := \{v \in W_3^{x_1,\cdots,x_{i-1}} \mid x_i v = 0\}$ inductively. These are nonzero $\mathbb{Z}_2$-graded subspaces by a similar argument. Then any nonzero homogeneous vector v in $W_3^{x_1,\cdots,x_l}$ satisfies the requirement of the assertion.

As a consequence of Lemma 1.4, we get the following preliminary result on representations of Lie superalgebras.

Lemma 1.5 Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and $\mathfrak{g} \subseteq L \subseteq \mathfrak{gl}(V)$ be Lie super subalgebras. Then the following statements hold.

(1) If $[\mathfrak{g},\mathfrak{g}]$ consists of nilpotent transformations and $\mathbb{F}$ contains all eigenvalues of elements in $\mathfrak{g}$, then there exists nonzero $v\in V_{\bar{0}}\cup V_{\bar{1}}$ and $\lambda\in\mathfrak{g}^*$ such that $xv=\lambda(x)v$, $\forall\, x\in\mathfrak{g}$.

(2) Let $\lambda:\ \mathfrak{g}\longrightarrow\mathbb{F}$ be an eigenvalue function, i.e., $x-\lambda(x)\,\mathrm{id}_V$ is nilpotent for any $x\in\mathfrak{g}$. Suppose that $\lambda(x)=0$ for any $x\in[\mathfrak{g},\mathfrak{g}]$. Then λ is linear.

(3) Keep assumptions as in (1). Moreover, assume that $\mathfrak{g}$ is an ideal of L and V is an irreducible L-module. Then $[\mathfrak{g},\mathfrak{g}]=0$, and any $x\in\mathfrak{g}$ has a unique eigenvalue $\lambda(x)$ on V, and $\lambda:\ \mathfrak{g}\longrightarrow\mathbb{F}$ is linear.

Proof (1) According to Lemma 1.4,

$$W_1:=\{v\in V\mid xv=0,\ \forall\, x\in[\mathfrak{g},\mathfrak{g}]\}\neq 0.$$

Take any $x\in\mathfrak{g}_{\bar{1}}$ and $y,z\in\mathfrak{g}_{\bar{0}}\cup\mathfrak{g}_{\bar{1}}$, then

$$[y,z]xv=[[y,z],x]v+(-1)^{|y|+|z|}x[y,z]v=0.$$

Hence, W_1 is invariant under the action of $\mathfrak{g}_{\bar{1}}$. Moreover, since

$$x^2v=\frac{1}{2}[x,x]v=0 \text{ and } xyv=-yxv,\ \forall\, x,y\in\mathfrak{g}_{\bar{1}},v\in W_1,$$

it follows that

$$W_2:=\{v\in W_1\mid xv=0,\ \forall\, x\in\mathfrak{g}_{\bar{1}}\}\neq 0.$$

Furthermore, W_2 is a $\mathfrak{g}_{\bar{0}}$-submodule with $xyw=yxw$, $\forall\, x,y\in\mathfrak{g}_{\bar{0}},w\in W_2$, so that we can find a nonzero homogeneous element v in W_2 and $\lambda\in\mathfrak{g}^*$ such that $xv=\lambda(x)v$, $\forall\, x\in\mathfrak{g}$.

(2) By (1), there exists $v\in V_{\bar{0}}\cup V_{\bar{1}}$ such that $xv=\lambda(x)v$. Since the left hand side is linear in x, so is the right hand side.

(3) By (1), there exists $v\in V_{\bar{0}}\cup V_{\bar{1}}$ such that $xv=0$, $\forall\, x\in[\mathfrak{g},\mathfrak{g}]$. Since V is an irreducible L-module, $V=U(L)v$. Consequently, $[\mathfrak{g},\mathfrak{g}]$ acts trivially on V, since $\mathfrak{g}$ is an ideal of L. This means that $[\mathfrak{g},\mathfrak{g}]=0$. Let $x\in\mathfrak{g}_{\bar{0}}$ and $\lambda(x)$ be an eigenvalue of x, then $[x^p,L]=(\mathrm{ad}x)^pL\subset[\mathfrak{g},\mathfrak{g}]=0$, and $\mathcal{V}:=\{v\in V\mid x^pv=\lambda(x)^pv\}$ is a nonzero L-submodule. The irreducibility of V as an L-module implies that $\mathcal{V}$ coincides with V, i.e., $x-\lambda(x)\,\mathrm{id}_V$ is nilpotent. Hence, $\lambda(x)$ is the unique eigenvalue of x. On the other hand, for any $x\in\mathfrak{g}_{\bar{1}}$,

$$x^2=\frac{1}{2}[x,x]\in[\mathfrak{g},\mathfrak{g}]=0.$$

This implies that any element $x\in\mathfrak{g}_{\bar{1}}$ is nilpotent, and 0 is the unique eigenvalue. The assertion that λ is linear follows from (2).

1.3 Restricted Lie superalgebras

The following definition is a generalization of the notion of restricted Lie algebras[17,19] to the case of Lie superalgebras.

Definition 1.6[20] A Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ is called a restricted one if $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module under the adjoint action. This is equivalent to saying that there exists a so-called p-mapping $[p]$ on $\mathfrak{g}_{\bar{0}}$ such that the following properties hold:

(i) $(\mathrm{ad}x)^p = \mathrm{ad}(x^{[p]})$ for all $x \in \mathfrak{g}_{\bar{0}}$;

(ii) $(ax)^{[p]} = a^p x^{[p]}$ for all $a \in \mathbb{F}, x \in \mathfrak{g}_{\bar{0}}$;

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y)$ for all $x, y \in \mathfrak{g}_{\bar{0}}$, where those $s_i(x,y) \in \mathfrak{g}_{\bar{0}}$ $(1 \leqslant i \leqslant p-1)$ are defined via the following formula:

$$\mathrm{ad}(tx+y)^{p-1}(x) = \sum_{i=1}^{p-1} i s_i(x,y) t^{i-1} \text{ for all } x, y \in \mathfrak{g}_{\bar{0}}.$$

Here t is an indeterminate.

Remark 1.7 Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra. Set $\xi(x) := x^p - x^{[p]} \in U(\mathfrak{g})$ for $x \in \mathfrak{g}_{\bar{0}}$. According to Definition 1.6(i), $\xi(x) \in Z(\mathfrak{g})$ for any $x \in \mathfrak{g}_{\bar{0}}$. Moreover, $\xi : \mathfrak{g}_{\bar{0}} \longrightarrow Z(\mathfrak{g})$ is p-semilinear, i.e., $\xi(ax+by) = a^p \xi(x) + b^p \xi(y)$, $\forall\, x, y \in \mathfrak{g}_{\bar{0}}, a, b \in \mathbb{F}$.

Example 1.8 Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ be the general linear Lie superalgebra. Let

$$\begin{aligned}[p]: \quad & \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}_{\bar{0}} \\ & x \longmapsto x^p,\end{aligned}$$

where $x^p = \underbrace{x \cdot x \cdot \cdots \cdot x}_{p\,\text{times}}$. Then $(\mathfrak{g}, [p])$ is a restricted Lie superalgebra. More generally, Lie superalgebras of algebraic supergroups are restricted Lie superalgebras (see [2]).

Proposition 1.9 Let $\mathfrak{g}$ be a restricted subalgebra of a restricted Lie superalgebra $(G, [p])$. Let $[p]' : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}_{\bar{0}}$ be a mapping. Then the following statements are equivalent.

(1) $[p]'$ is a p-mapping on $\mathfrak{g}$.

(2) There exists a p-semilinear mapping $f : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{z}_{G_{\bar{0}}}(\mathfrak{g})$ such that $[p]' = [p]|_{\mathfrak{g}_{\bar{0}}} + f$, where $\mathfrak{z}_{G_{\bar{0}}}(\mathfrak{g}) = \{x \in G_{\bar{0}} \mid [x,y] = 0, \forall\, y \in \mathfrak{g}\}$.

Proof (1) $\Longrightarrow$ (2). Set

$$\begin{aligned} f: \quad & \mathfrak{g}_{\bar{0}} \longrightarrow G_{\bar{0}} \\ & x \longmapsto x^{[p]'} - x^{[p]}.\end{aligned}$$

Since

$$[f(x), y] = [x^{[p]'} - x^{[p]}, y] = (\mathrm{ad}x)^p(y) - (\mathrm{ad}x)^p(y) = 0, \ \forall\, x \in \mathfrak{g}_{\bar{0}}, y \in \mathfrak{g},$$

f actually maps $\mathfrak{g}_{\bar{0}}$ into $\mathfrak{z}_{G_{\bar{0}}}(\mathfrak{g})$. For any $x, y \in \mathfrak{g}_{\bar{0}}$ and $a, b \in \mathbb{F}$, we have

$$\begin{aligned} f(ax+by) &= a^p x^{[p]'} + b^p y^{[p]'} + \sum_{i=1}^{p-1} s_i(ax, by) - \Big(a^p x^{[p]} + b^p y^{[p]} + \sum_{i=1}^{p-1} s_i(ax, by)\Big) \\ &= a^p f(x) + b^p f(y). \end{aligned}$$

Consequently, f is p-semilinear.

(2) $\Longrightarrow$ (1). We need to show that the three conditions in Definition 1.6 hold for $[p]'$.

(i) $\mathrm{ad}(x^{[p]'}) = \mathrm{ad}(x^{[p]} + f(x)) = \mathrm{ad}(x^{[p]}) = (\mathrm{ad}x)^p$, $\forall\, x \in \mathfrak{g}_{\bar{0}}$.

(ii) $(\lambda x)^{[p]'} = (\lambda x)^{[p]} + f(\lambda x) = \lambda^p x^{[p]} + \lambda^p f(x) = \lambda^p x^{[p]'}$, $\forall\, x \in \mathfrak{g}_{\bar{0}}, \lambda \in \mathbb{F}$.

(iii) For any $x, y \in \mathfrak{g}_{\bar{0}}$,

$$\begin{aligned}(x+y)^{[p]'} &= (x+y)^{[p]} + f(x+y) = x^{[p]} + f(x) + y^{[p]} + f(y) + \sum_{i=1}^{p-1} s_i(x,y) \\ &= x^{[p]'} + y^{[p]'} + \sum_{i=1}^{p-1} s_i(x,y).\end{aligned}$$

The proof is completed.

Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$. Let $Z_0(\mathfrak{g})$ be the $\mathbb{F}$-algebra generated by $x^p - x^{[p]}$ for $x \in \mathfrak{g}_{\bar{0}}$. Let $I_0(\mathfrak{g})$ be the ideal in $U(\mathfrak{g})$ generated by $x^p - x^{[p]}$ for $x \in \mathfrak{g}_{\bar{0}}$, and $u(\mathfrak{g}) = U(\mathfrak{g})/I_0(\mathfrak{g})$ which is usually called the restricted enveloping superalgebra. Suppose that $\{x_1, \cdots, x_n\}$ is a basis of $\mathfrak{g}_{\bar{0}}$, and $\{y_1, \cdots, y_m\}$ is a basis of $\mathfrak{g}_{\bar{1}}$. It follows from the semilinearity of ξ that $Z_0(\mathfrak{g})$ is generated by $\xi(x_1), \cdots, \xi(x_n)$. Moreover, by PBW Theorem, we have

Proposition 1.10　Keep notations as above, then the following statements hold.

(1) The elements $\xi(x_1), \cdots, \xi(x_n)$ are algebraically independent generators for $Z_0(\mathfrak{g})$, i.e., $Z_0(\mathfrak{g}) = \mathbb{F}[\xi(x_1), \cdots, \xi(x_n)]$ is a polynomial algebra of n indeterminates.

(2) The universal enveloping superalgebra $U(\mathfrak{g})$ is free over $Z_0(\mathfrak{g})$ with basis

$$\{x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_m^{b_m} \mid 0 \leqslant a_i \leqslant p-1, b_j = 0, 1, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}.$$

(3) The restricted enveloping superalgebra $u(\mathfrak{g})$ is finite-dimensional, and has a basis

$$\{\bar{x}_1^{a_1} \cdots \bar{x}_n^{a_n} \bar{y}_1^{b_1} \cdots \bar{y}_m^{b_m} \mid 0 \leqslant a_i \leqslant p-1, b_j = 0, 1, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}.$$

2　General representation theory

In this section, we always assume that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a finite-dimensional Lie superalgebra over an algebraically closed field of characteristic $p > 2$. We will show that each simple $\mathfrak{g}$-module is finite-dimensional, and the dimensions of simple $\mathfrak{g}$-modules have an upper bound. Moreover, each finite-dimensional Lie superalgebra can be embedded into a finite-dimensional restricted Lie superalgebra.

Proposition 2.1　Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. Then the universal enveloping superalgebra $U(\mathfrak{g})$ is a finitely generated $Z(\mathfrak{g})$-module, and $Z(\mathfrak{g})$ is a finitely generated $\mathbb{F}$-algebra.

Proof　(1) Let $\{x_1, \cdots, x_n\}$ be a basis of $\mathfrak{g}_{\bar{0}}$ and $\{y_1, \cdots, y_m\}$ be a basis of $\mathfrak{g}_{\bar{1}}$. Consider

$$\{(\mathrm{ad}x_i)^{p^j} \mid 1 \leqslant i \leqslant n, j = 0, 1, \cdots\}$$

as elements in $\mathrm{End}_{\mathbb{F}}(\mathfrak{g})$. Since $\mathfrak{g}$ is finite-dimensional, there exists $d_i \in \mathbb{N}\,(1 \leqslant i \leqslant n)$ such that

$$(\mathrm{ad}x_i)^{p^{d_i}} = \sum_{0 \leqslant j < d_i} a_{ij} (\mathrm{ad}x_i)^{p^j}, \forall\, 1 \leqslant i \leqslant n.$$

Consequently, $z_i := x_i^{p^{d_i}} - \sum\limits_{0\leqslant j<d_i} a_{ij}x_i^{p^j} \in Z(\mathfrak{g}), 1\leqslant i\leqslant n$. Let $\mathcal{O}$ be the subalgebra of $Z(\mathfrak{g})$ generated by $z_i, 1\leqslant i\leqslant n$. By PBW Theorem, $\mathcal{O}$ is a polynomial algebra of n indeterminates, and as an $\mathcal{O}$-module, $U(\mathfrak{g})$ is spanned by

$$\{x_1^{i_1}\cdots x_n^{i_n}y_1^{j_1}\cdots y_m^{j_m} \mid 0\leqslant i_k < p^{d_k}, j_s = 0,1, 1\leqslant k\leqslant n, 1\leqslant s\leqslant m\}. \tag{2.1}$$

In particular, as a $Z(\mathfrak{g})$-module, $U(\mathfrak{g})$ is spanned by those elements in (2.1).

(2) By (1), $U(\mathfrak{g})$ is a Noetherian $\mathcal{O}$-module. Hence, as a submodule, $Z(\mathfrak{g})$ is also a Noetherian $\mathcal{O}$-module. Consequently, $Z(\mathfrak{g})$ is a finitely generated $\mathcal{O}$-module. Since $\mathcal{O}$ is finitely generated, it follows that $Z(\mathfrak{g})$ is also finitely generated.

Theorem 2.2 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>2$. Then the following statements hold.

(1) Each irreducible representation of $\mathfrak{g}$ is finite-dimensional.

(2) There exists a positive integer $M(\mathfrak{g})$ such that every irreducible representation of $\mathfrak{g}$ has dimension less than $M(\mathfrak{g})$.

Proof By Proposition 2.1, we can assume that $U(\mathfrak{g}) = \sum\limits_{i=1}^{r} Z(\mathfrak{g})u_i$. Let V be a simple $\mathfrak{g}$-module. Take a nonzero homogeneous element v in V, then

$$V = U(\mathfrak{g})v = \sum_{i=1}^{r} Z(\mathfrak{g})u_i v.$$

Hence, the module V is finitely generated over $Z(\mathfrak{g})$. Since $Z(\mathfrak{g})$ is Noetherian, there exists a maximal $Z(\mathfrak{g})$-submodule $V'\subset V$. Consequently, $V/V'\cong Z(\mathfrak{g})/\mathfrak{m}$ as $Z(\mathfrak{g})$-modules for some maximal ideal $\mathfrak{m}$ of $Z(\mathfrak{g})$. Hence, $\mathfrak{m}V\subseteq V'\subsetneqq V$. Since $\mathfrak{m}V$ is a $U(\mathfrak{g})$-submodule of V and V is irreducible, it follows that $\mathfrak{m}V = 0$. Therefore, $Z(\mathfrak{g})$ acts on V as $Z(\mathfrak{g})/\mathfrak{m}\cong\mathbb{F}$. Part (1) is proved. Moreover, by the discussion above, $r+1$ is an upper bound $M(\mathfrak{g})$.

Remark 2.3 When $\mathfrak{g}$ is a restricted Lie superalgebra, the results in Theorem 2.2 were asserted in [3].

Example 2.4 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a subalgebra of $\mathfrak{gl}(2|1)$ with $\mathfrak{g}_{\bar{0}} = \mathrm{span}_{\mathbb{F}}\{h,x\}$, $\mathfrak{g}_{\bar{1}} = \mathrm{span}_{\mathbb{F}}\{y\}$, where

$$h = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, x = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Note that

$$[h,x] = x,\ [h,y] = y,\ [x,y] = [y,y] = 0$$

and

$$h^{[p]} = h,\ x^{[p]} = 0.$$

Hence, x^p and $h^p - h$ are contained in $Z(\mathfrak{g})$. Consequently,

$$U(\mathfrak{g}) = \sum_{\substack{0\leqslant i,j<p \\ 0\leqslant k\leqslant 1}} Z(\mathfrak{g})h^i x^j y^k.$$

It is easy to check that

$$hx^i = x^i(h+i),\ xh^j = (h-1)^j x,\ \forall\, 1 \leqslant i,j \leqslant p-1. \tag{2.2}$$

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be an irreducible $\mathfrak{g}$-module. By Theorem 2.2, M is finite-dimensional and x^p acts as a scalar on M, saying a^p. Hence, $(x-a)^p \cdot M = (x^p - a^p) \cdot M = 0$.

Case 1 $a = 0$.

Let $\mathfrak{g}' = \mathrm{span}_{\mathbb{F}}\{x, y\}$. Then $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$. According to Lemma 1.4, $M' := \{m \in M \mid z \cdot m = 0, \forall\, z \in \mathfrak{g}'\}$ is a nonzero $\mathbb{Z}_2$-graded subspace. Moreover, M' is a $\mathfrak{g}$-submodule of M, so that $M = M'$ by the irreducibility of M as a $\mathfrak{g}$-module. Hence, M is a simple module for the commutative Lie algebra $\mathfrak{g}/\mathfrak{g}' \cong \mathbb{F}h$. Therefore, $\dim_{\mathbb{F}} M = 1$ and h acts as a scalar on M, while x, y act trivially. Conversely, given any scalar $b \in \mathbb{F}$, we get a one-dimensional simple $\mathfrak{g}$-module, denoted by $\mathcal{M}_b$, in which h acts as multiplication by b, and $\mathfrak{g}'$ acts trivially.

Case 2 $a \neq 0$.

In this case, there exists $0 \neq v_0 \in M_{\bar{0}} \cup M_{\bar{1}}$ such that $x \cdot v_0 = av_0$. Since M is finite-dimensional and $h^p - h \in Z(\mathfrak{g})$, there exists $b \in \mathbb{F}$ such that

$$(h^p - h) \cdot v = h^p \cdot v - h \cdot v = b^p v,\ \forall\, v \in M.$$

Set $v_i := h^i \cdot v_0$ for $1 \leqslant i \leqslant p$. Then $v_p = b^p v_0 + v_1$. By (2.2), for $1 \leqslant i \leqslant p-1$, we have

$$x \cdot v_i = (h-1)^i a v_0 = a \sum_{j=0}^{i} (-1)^j \binom{i}{j} v_{i-j}. \tag{2.3}$$

It follows that $M'' := \mathrm{span}_{\mathbb{F}}\{v_0, v_1, \cdots, v_{p-1}\}$ is stable under x and h. We claim that $v_0, \cdots, v_{p-1}$ are linearly independent. Suppose the contrary, then there exists some $j < p-1$ such that $M'' = \mathrm{span}_{\mathbb{F}}\{v_0, v_1, \cdots, v_j\}$. It follows from (2.3) that $\mathrm{tr}(x|_{M''}) = (j+1)a$. On the other hand, since $[h, x] = x$, we have $\mathrm{tr}(x|_{M''}) = 0$. This implies that $(j+1)a = 0$, i.e., $j+1 \equiv 0 \pmod p$, a contradiction. Therefore, $v_0, v_1, \cdots, v_{p-1}$ are linearly independent. Moreover, M'' is an irreducible $\mathfrak{g}_{\bar{0}}$-submodule, since up to scalars, v_0 is the unique eigenvector of x on M''. We have the following natural epimorphism of $\mathfrak{g}$-modules:

$$\pi:\ U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M'' \longrightarrow M,$$

which is surjective by the simplicity of M as a $\mathfrak{g}$-module. It is easy to check that $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M''$ has a unique maximal submodule $y \otimes M''$. Consequently, $M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M''/y \otimes M''$, and $\dim_{\mathbb{F}} M = p$. Conversely, given $a, b \in \mathbb{F}$ with $a \neq 0$, we have a simple $\mathfrak{g}$-module M of dimension p with basis $v_0, \cdots, v_{p-1}$ such that y acts trivially, and the actions of h and x are given as above. We denote this simple $\mathfrak{g}$-module by $\mathcal{M}_{(a,b)}$.

In conclusion, $\{\mathcal{M}_b, \mathcal{M}_{(a,b)} \mid a \in \mathbb{F}^{\times}, b \in \mathbb{F}\}$ exhausts all non-isomorphic irreducible $\mathfrak{g}$-modules.

In the following, we study the connection of restricted and ordinary Lie superalgebras.

Definition 2.5 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra.

(1) A triple $(G, [p], \iota)$ consisting of a restricted Lie superalgebra $(G, [p])$ and a Lie superalgebra homomorphism $\iota : \mathfrak{g} \longrightarrow G$ is called a p-envelope of $\mathfrak{g}$ if ι is injective and $G = \iota(\mathfrak{g})_p$, where $\iota(\mathfrak{g})_p$ denotes the restricted subalgebra generated by $\iota(\mathfrak{g})$.

(2) A p-envelope $(G, [p], \iota)$ of $\mathfrak{g}$ is called universal, if it satisfies the following universal property: For any restricted Lie superalgebra $(H, [p]')$ and any homomorphism $f : \mathfrak{g} \longrightarrow H$, there exists a unique restricted homomorphism $g : (G, [p]) \longrightarrow (H, [p]')$ such that $g \circ \iota = f$.

The following result asserts that the universal p-envelope of a Lie superalgebra always exists and is unique.

Proposition 2.6 Every Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ has a unique universal p-envelope $\hat{\mathfrak{g}}$.

Proof Let $\widehat{\mathfrak{g}}$ be the restricted subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{g}$. Let H be a restricted Lie superalgebra and $f : \mathfrak{g} \longrightarrow H$ be a homomorphism. Recall that H canonically embedded into $u(H)$. The universal property of $U(\mathfrak{g})$ gives rise to an associative homomorphism $\bar{f} : U(\mathfrak{g}) \longrightarrow u(H)$ and $\mathfrak{g} \subset \bar{f}^{-1}(H)$. Let $x \in \mathfrak{g}_{\bar{0}} \subset \bar{f}^{-1}(H_{\bar{0}})$, then $\bar{f}(x) \in H_{\bar{0}}$ and $\bar{f}(x^p) = \bar{f}(x)^p = \bar{f}(x)^{[p]} \in H$. So, $x^p \in \bar{f}^{-1}(H)$. Therefore, $\bar{f} : \widehat{\mathfrak{g}} \longrightarrow H$ is an extension of f. Since $\widehat{\mathfrak{g}}$ is generated by $\mathfrak{g}$ and the p-th powers, this extension is unique. The uniqueness of $\widehat{\mathfrak{g}}$ follows from the definition of the universal p-envelope.

Proposition 2.7 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. Then the following statements hold.

(1) If $\mathfrak{g}$ is finite-dimensional, and $(\bar{\mathfrak{g}}, [p], \iota)$ is a p-envelope of $\mathfrak{g}$, then $\bar{\mathfrak{g}}/C(\bar{\mathfrak{g}})$ is finite-dimensional.

(2) If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{g}$ possesses a finite-dimensional p-envelope.

(3) Each homomorphism of Lie superalgebras $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ can be extended to a restricted homomorphism $\widehat{f} : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{h}}$. Moreover, if f is injective or surjective, so is $\widehat{f}$.

Proof (1) Recall that $\bar{\mathfrak{g}} = \iota(\mathfrak{g})_p$, the restricted subalgebra generated by $\iota(\mathfrak{g})$. Hence, $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \iota(\mathfrak{g})$, and $\iota(\mathfrak{g})$ is an ideal of $\bar{\mathfrak{g}}$. Let

$$\begin{aligned} \varphi : \bar{\mathfrak{g}} &\longrightarrow \mathrm{Der}_{\mathbb{F}}(\iota(\mathfrak{g})) \\ x &\longmapsto \mathrm{ad}x|_{\iota(\mathfrak{g})}. \end{aligned}$$

It is easy to check that $\mathrm{Ker}\varphi = \mathfrak{z}_{\bar{\mathfrak{g}}}(\iota(\mathfrak{g})) = \mathfrak{z}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{g}}) = C(\bar{\mathfrak{g}})$. Consequently,

$$\dim_{\mathbb{F}} \bar{\mathfrak{g}}/C(\bar{\mathfrak{g}}) \leqslant \dim_{\mathbb{F}} \mathrm{Der}_{\mathbb{F}}(\iota(\mathfrak{g})) \leqslant \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}}(\iota(\mathfrak{g})) = (\dim_{\mathbb{F}} \mathfrak{g})^2 < +\infty.$$

(2) Choose a $\mathbb{Z}_2$-graded subspace $V \subset C(\widehat{\mathfrak{g}})$ such that $C(\widehat{\mathfrak{g}}) = V \oplus (C(\widehat{\mathfrak{g}}) \cap \mathfrak{g})$. Then by Proposition 1.9, we can endow a $[p]$-structure on $\widehat{\mathfrak{g}}/V$ which contains $\mathfrak{g}$ isomorphically. Moreover,

$$\dim_{\mathbb{F}} \widehat{\mathfrak{g}}/V = \dim_{\mathbb{F}} \widehat{\mathfrak{g}}/C(\widehat{\mathfrak{g}}) + \dim_{\mathbb{F}} C(\widehat{\mathfrak{g}}) \cap \mathfrak{g} < +\infty.$$

Then the restricted subalgebra generated by $\mathfrak{g}$ in $\widehat{\mathfrak{g}}/V$ is the desired p-envelope of $\mathfrak{g}$.

(3) Since $\mathfrak{g} \longrightarrow \mathfrak{h} \hookrightarrow \widehat{\mathfrak{h}}$, the universal property of $\widehat{\mathfrak{g}}$ yields the existence of $\widehat{f}$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "f"'] & \widehat{\mathfrak{g}} \arrow[d, "\widehat{f}"] \\ \mathfrak{h} \arrow[r] & \widehat{\mathfrak{h}} \end{tikzcd}$$

If f is onto, then $\widehat{f}(\widehat{\mathfrak{g}}) \supset f(\mathfrak{g})_p = \mathfrak{h}_p = \widehat{\mathfrak{h}}$, i.e., $\widehat{f}(\widehat{\mathfrak{g}}) = \widehat{\mathfrak{h}}$. If f is injective, it extends to an injective homomorphism $U(\mathfrak{g}) \hookrightarrow U(\mathfrak{h})$. Hence, its restriction $\widehat{f}$ to $\widehat{\mathfrak{g}}$ is injective.

The following result is a superversion of Iwasawa's Theorem in the case of Lie algebras.

Theorem 2.8 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra. Then $\mathfrak{g}$ admits a finite-dimensional faithful representation ρ. Moreover, assume $x \in \mathfrak{g}_{\bar{0}}$, then $\rho(x)$ is nilpotent if and only if $\mathrm{ad}x$ is nilpotent.

Proof We first assume that $\mathfrak{g}$ is restricted with the p-mapping $[p]$. Without loss of generality, according to Proposition 1.9, we can assume that $[p]|_{\mathfrak{z}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{g})} = 0$. This implies that $\mathrm{ad}\, x$ is nilpotent if and only if x is $[p]$-nilpotent for $x \in \mathfrak{g}_{\bar{0}}$. Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(u(\mathfrak{g}))$ be the left multiplication in the restricted enveloping superalgebra $u(\mathfrak{g})$. Then ρ is a faithful representation of $\mathfrak{g}$, and x is $[p]$-nilpotent if and only if $\rho(x)$ is nilpotnet. Consequently, $\mathrm{ad}x$ is nilpotent if and only if $\rho(x)$ is nilpotent.

In general, according to Proposition 2.7, there exists a finite-dimensional p-envelope of $\mathfrak{g}$, denoted by G. By the discussion above, G admits a finite-dimensional faithful representation $\varrho : G \longrightarrow \mathfrak{gl}(V)$ with the desired property. Since $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent if and only if $\mathrm{ad}_G(x)$ is nilpotent for $x \in \mathfrak{g}_{\bar{0}}$. Thus, $\rho := \varrho|_{\mathfrak{g}}$ satisfies the required property.

We have the following close connection between representations of a Lie superalgebra and its p-envelope.

Theorem 2.9 Let G be a p-envelope of a finite-dimensional Lie superalgebra $\mathfrak{g}$ and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then there exists a representation $\bar{\rho} : G \longrightarrow \mathfrak{gl}(V)$ extending ρ, and each $\mathfrak{g}$-submodule of V is a G-submodule.

Proof The statement obviously holds for $G = \widehat{\mathfrak{g}}$, the universal p-envelope of $\mathfrak{g}$. In general, by Definition 2.5, there exists an embedding $\iota : \mathfrak{g} \hookrightarrow G$ and a restricted homomorphism $f : \widehat{\mathfrak{g}} \longrightarrow G$. Then

$$G = \iota(\mathfrak{g})_p = f(\mathfrak{g})_p = f(\mathfrak{g}_p) = f(\widehat{\mathfrak{g}}),$$

i.e., f is surjective. We can find a subspace W of $\widehat{\mathfrak{g}}$ containing $\mathfrak{g}$ such that $f|_W : W \longrightarrow G$ is an isomorphism (W is indeed a subalgebra). Then $\bar{\rho} := \tilde{\rho} \circ f|_W^{-1}$ is the desired representation of G, where $\tilde{\rho} : \widehat{\mathfrak{g}} \longrightarrow \mathfrak{gl}(V)$ is the restriction of the representation $U(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$ to $\widehat{\mathfrak{g}}$.

According to Proposition 2.7, any finite-dimensional Lie superalgebra can be embedded into a finite-dimensional restricted Lie superalgebra. In the next section, we will study representations of restricted Lie superalgebras over a field of prime characteristic.

3 Representations of restricted Lie superalgebras

In this section, we always assume that the base field $\mathbb{F}$ is algebraically closed of characteristic $p > 2$, and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$ with the p-mapping $[p]$.

Let M be a simple $\mathfrak{g}$-module. Then M is finite-dimensional by Theorem 2.2. According to Schur Lemma, $\xi(x) = x^p - x^{[p]}$ acts on M by a scalar for any $x \in \mathfrak{g}_{\bar{0}}$. We write this scalar as $\chi_{_M}(x)^p$ for some $\chi_{_M}(x) \in \mathbb{F}$. The semilinearity of ξ implies that $\chi_{_M} \in \mathfrak{g}_{\bar{0}}^*$.

Theorem 3.1 The function $\chi_{_M}$ is called the p-character of M. More generally, if V is a $\mathfrak{g}$-module and $\chi \in \mathfrak{g}_{\bar{0}}^*$, then we say V has a p-character χ if

$$\underbrace{x \cdots x}_{p\text{times}} \cdot v - x^{[p]} \cdot v = \chi(x)^p v, \ \forall\, x \in \mathfrak{g}_{\bar{0}}, v \in V.$$

In the following, when we write $\chi \in \mathfrak{g}^*$, we always make convention that $\chi|_{\mathfrak{g}_{\bar{1}}} = 0$. We also refer $\chi \in \mathfrak{g}_{\bar{0}}^*$ as a linear function on $\mathfrak{g}$ with $\chi(\mathfrak{g}_{\bar{1}}) = 0$.

Remark 3.2 If M has a p-character χ and M' has a p-character χ', then M^* has a p-character $-\chi$ and $M \otimes M'$ has a p-character $\chi + \chi'$.

The $\mathfrak{g}$-modules with p-character 0 are called restricted modules. They correspond to Lie superalgebra homomorphisms $\rho:\ \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ with $\rho(x)^p = \rho(x^{[p]})$, $\forall\, x \in \mathfrak{g}_{\bar{0}}$.

For $\chi \in \mathfrak{g}_{\bar{0}}^*$, define $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}})$, where $(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}})$ denotes the ideal of $U(\mathfrak{g})$ generated by $x^p - x^{[p]} - \chi(x)^p$ for $x \in \mathfrak{g}_{\bar{0}}$. Each $U_\chi(\mathfrak{g})$ is called a χ-reduced enveloping superalgebra of $\mathfrak{g}$. For $\chi = 0$, $U_0(\mathfrak{g})$ is just the restricted enveloping superalgebra $u(\mathfrak{g})$. We have a one-to-one correspondence between $\mathfrak{g}$-modules with p-character χ and $U_\chi(\mathfrak{g})$-modules. By PBW Theorem, we have

Proposition 3.3 Let $\chi \in \mathfrak{g}_{\bar{0}}^*$. If $\{x_1, \cdots, x_n\}$ is a basis of $\mathfrak{g}_{\bar{0}}$ and $\{y_1, \cdots, y_m\}$ is a basis of $\mathfrak{g}_{\bar{1}}$, then the superalgebra $U_\chi(\mathfrak{g})$ has the following basis

$$\{\bar{x}_1^{a_1} \cdots \bar{x}_n^{a_n} \bar{y}_1^{b_1} \cdots \bar{y}_m^{b_m} \mid 0 \leqslant a_i < p, b_j = 0, 1, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}.$$

In particular, $\dim_{\mathbb{F}} U_\chi(\mathfrak{g}) = 2^{\dim_{\mathbb{F}} \mathfrak{g}_{\bar{1}}} p^{\dim_{\mathbb{F}} \mathfrak{g}_{\bar{0}}}$.

The following result asserts that the composition factors of a finite-dimensional indecomposable $\mathfrak{g}$-module have the same p-character.

Proposition 3.4 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field $\mathbb{F}$, and M a finite-dimensional indecomposable $\mathfrak{g}$-module. Then there exists a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ such that each simple composition factor of M has the p-character χ.

Proof Let $d = \dim_{\mathbb{F}} M$. Take a basis $\{x_1, \cdots, x_n\}$ of $\mathfrak{g}_{\bar{0}}$. Consider $x_1^p - x_1^{[p]}$ as a linear transformation on M. We can decompose M as a direct sum of $\mathbb{Z}_2$-graded vector subspaces:

$$M = M_{\lambda_{11}} \oplus M_{\lambda_{12}} \oplus \cdots \oplus M_{\lambda_{1s}},$$

where

$$M_{\lambda_{1i}} = \{v \in M | (x_1^p - x_1^{[p]} - \lambda_{1i})^d v = 0\},\ 1 \leqslant i \leqslant s.$$

Since $x_1^p - x_1^{[p]} \in Z(\mathfrak{g})$, each $M_{\lambda_{1i}}$ is a $\mathfrak{g}$-submodule. The indecomposability of M as a $\mathfrak{g}$-module implies that $s = 1$, i.e., $(x_1^p - x_1^{[p]} - \lambda_{11})^d v = 0$, $\forall\, v \in M$.

Applying similar arguments, there exist unique $\lambda_{21}, \cdots, \lambda_{n1} \in \mathbb{F}$ such that

$$(x_i^p - x_i^{[p]} - \lambda_{i1})^d v = 0,\ \forall\, i\,(2 \leqslant i \leqslant n),\ v \in M.$$

Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ with $\chi(x_i)^p = \lambda_{i1}$ for $1 \leqslant i \leqslant n$. Then

$$(x^p - x^{[p]} - \chi(x)^p)^d v = 0,\ \forall\, x \in \mathfrak{g}_{\bar{0}},\ v \in M.$$

Consequently, each simple composition factor of M admits the p-character χ.

As a direct consequence, we have

Corollary 3.5 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field, and V a finite-dimensional $\mathfrak{g}$-module. Then V can be decomposed into direct sum of submodules: $V = \bigoplus\limits_{i=1}^{t} V_i$, where the composition factors of each V_i have the same p-character $\chi_i \in \mathfrak{g}_{\bar{0}}^*$ for $1 \leqslant i \leqslant t$. Those χ_i $(1 \leqslant i \leqslant t)$ are called the generalized p-characters of V.

In the following, we always assume that I is an ideal of a finite-dimensional restricted Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, and $\lambda \in I_{\bar{0}}^*$ with $\lambda([I, I]) = 0$. Let

$$\mathfrak{g}^\lambda := \{x \in \mathfrak{g} \mid \lambda([x, y]) = 0,\ \forall\, y \in I\},$$

which is a restricted subalgebra of $\mathfrak{g}$. Moreover, I is also an ideal of $\mathfrak{g}^\lambda$.

Let $\{z_1, \cdots, z_l, z_{l+1}, \cdots, z_r\}$ be a cobasis of $\mathfrak{g}^\lambda$ in $\mathfrak{g}$, where $z_i \in \mathfrak{g}_{\bar{0}}$, $z_j \in \mathfrak{g}_{\bar{1}}$ for $1 \leqslant i \leqslant l < j \leqslant r$. For a given $\chi \in \mathfrak{g}^*$ (recall the convention that $\chi|_{\mathfrak{g}_{\bar{1}}} = 0$) and a finite-dimensional $\mathfrak{g}^\lambda$-module M with the p-character $\chi|_{\mathfrak{g}^\lambda}$ and

$$x \cdot v = \lambda(x) v,\ \forall\, x \in I,\ v \in M,$$

let

$$V := \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^\lambda)} M$$

be the induced $U_\chi(\mathfrak{g})$-module. As a vector space, we have

$$V = \sum_{\mathbf{0} \preccurlyeq \mathbf{s} \preccurlyeq \tau} \mathbb{F}\mathbf{z}^{\mathbf{s}} \otimes M,$$

where $\mathbf{z}^{\mathbf{s}} = z_1^{s_1} \cdots z_r^{s_r}$ for $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{N}^r$ and

$$\tau = (\underbrace{p-1, \cdots, p-1}_{l\,\text{times}}, \underbrace{1, \cdots, 1}_{r-l\,\text{times}}).$$

For $j \in \mathbb{N}$, set

$$V_{(j)} = \sum_{\substack{\mathbf{0} \preccurlyeq \mathbf{s} \preccurlyeq \tau \\ |\mathbf{s}| \leqslant j}} \mathbb{F}\mathbf{z}^{\mathbf{s}} \otimes M.$$

We then have a filtration

$$0 \subset V_{(0)} \subset V_{(1)} \subset \cdots \subset V_{((p-2)l+r)} = V.$$

We need the following lemma for later use.

Lemma 3.6 Keep notations as above.

(1) There exist $y_1, \cdots, y_l \in I_{\bar{0}}$ and $y_{l+1}, \cdots, y_r \in I_{\bar{1}}$ such that $\lambda([y_i, z_j]) = \delta_{ij}$ for $1 \leqslant i, j \leqslant r$.

(2) For any $v \in M$, $\mathbf{s} \in \mathbb{N}^r$ with $\mathbf{s} \preccurlyeq \tau$, we have

$$(y_i - \lambda(y_i)) \cdot \mathbf{z}^{\mathbf{s}} \otimes v \equiv \pm s_i \mathbf{z}^{\mathbf{s}-\varepsilon_i} \otimes v \pmod{V_{|\mathbf{s}|-2}}.$$

Proof (1) Set $C = \sum\limits_{i=1}^{r} \mathbb{F} z_i$. Define $B_\lambda(z, y) = \lambda([z, y])$ for $z \in \mathfrak{g}$ and $y \in I$. We then get a linear map

$$\begin{aligned} \phi : C &\longrightarrow I^* \\ x &\longmapsto B_\lambda(x, -). \end{aligned}$$

Then ϕ is injective. Consequently, $\phi(z_1), \cdots, \phi(z_r)$ are linearly independent. Hence, there exists $y_1, \cdots, y_r \in I$ such that

$$\phi(z_i)(y_j) = \lambda([z_i, y_j]) = \delta_{ij},\ 1 \leqslant i, j \leqslant r.$$

Since $\lambda(\mathfrak{g}_{\bar{1}}) = 0$, we can choose $y_{_1}, \cdots, y_{_l} \in I_{\bar{0}}$ and $y_{_{l+1}}, \cdots, y_r \in I_{\bar{1}}$.

(2) According to Lemma 1.3,

$$(y_i - \lambda(y_i))\mathbf{z}^{\mathbf{s}} = \sum_{\mathbf{0} \preccurlyeq \mathbf{t} \preccurlyeq \mathbf{s}} \pm \binom{\mathbf{s}}{\mathbf{t}} \mathbf{z}^{\mathbf{s}-\mathbf{t}} \{y_i - \lambda(y_i), \mathbf{z}; \mathbf{t}\}.$$

For $\mathbf{t} \neq \mathbf{0}$, we have $\{y_i - \lambda(y_i), \mathbf{z}; \mathbf{t}\} = \{y_i, \mathbf{z}; \mathbf{t}\} \in I$, and $\{y_i, \mathbf{z}; \mathbf{t}\} \otimes v \in 1 \otimes M = V_{(0)}$. Consequently,

$$\begin{aligned} (y_i - \lambda(y_i)) \cdot \mathbf{z}^{\mathbf{s}} \otimes v &\equiv \sum_{|\mathbf{t}| \leqslant 1} \pm \binom{\mathbf{s}}{\mathbf{t}} \mathbf{z}^{\mathbf{s}-\mathbf{t}} \{y_i - \lambda(y_i), \mathbf{z}; \mathbf{t}\} \otimes v \\ &\equiv \pm \mathbf{z}^{\mathbf{s}}(y_i - \lambda(y_i)) \otimes v \pm \sum_{j=1}^{r} s_j \mathbf{z}^{\mathbf{s}-\varepsilon_j} [y_i, z_j] \otimes v \\ &\equiv \pm s_i \mathbf{z}^{\mathbf{s}-\varepsilon_i} \otimes v \,(\mathrm{mod} V_{|\mathbf{s}|-2}). \end{aligned}$$

With aid of Lemma 3.6, we get the following result describing the submodule structure of the induced module $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$.

Proposition 3.7 Let W be a $\mathfrak{g}$-submodule of $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$. Then there exists a $\mathfrak{g}^\lambda$-submodule M' of M such that $W \cap (1 \otimes M) = 1 \otimes M'$ and $W = \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M', \chi)$.

Proof Let $M' := \{v \in M \mid 1 \otimes v \in W\}$. Then M' is a $\mathfrak{g}^\lambda$-submodule of M. Moreover, $W \cap (1 \otimes M) = 1 \otimes M'$. For $j \in \mathbb{N}$, set

$$W_{(j)} := \sum_{\substack{\mathbf{0} \preccurlyeq \mathbf{s} \preccurlyeq \tau \\ |\mathbf{s}| \leqslant j}} \mathbb{F} \mathbf{z}^{\mathbf{s}} \otimes M' \subset \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi).$$

Then $W_{(0)} = W \cap V_{(0)}$. We will show that $W \cap V_{(j)} \subseteq W_{(j)}$ by induction on j. Let $j \geqslant 1$ and suppose that $W \cap V_{(j-1)} \subseteq W_{(j-1)}$. Let $v \in W \cap V_{(j)}$. Choose a cobasis $\{v_1, \cdots, v_t\}$ of M' in M. Without loss of generality, we can assume that

$$v = \sum_{k=1}^{t} \sum_{\substack{\mathbf{s} \preccurlyeq \tau \\ |\mathbf{s}| \leqslant j}} a_{\mathbf{s},k} \mathbf{z}^{\mathbf{s}} \otimes v_k,$$

where $a_{\mathbf{s},k} \in \mathbb{F}$ for $\mathbf{s} \preccurlyeq \tau, |\mathbf{s}| \leqslant j$ and $1 \leqslant k \leqslant t$. According to Lemma 3.6, we have

$$(y_i - \lambda(y_i)) \cdot v = \sum_{k=1}^{t} \sum_{|\mathbf{s}|=j} a_{\mathbf{s},k} s_i \mathbf{z}^{\mathbf{s}-\varepsilon_i} \otimes v_k \pmod{V_{(j-2)}},\ 1 \leqslant i \leqslant r.$$

Hence, $(y_i - \lambda(y_i)) \cdot v \in W \cap V_{(j-1)} \subset W_{(j-1)}$. It follows from the definition of $W_{(j-1)}$ that $s_i a_{\mathbf{s},k} = 0$ for $|\mathbf{s}| = j$ and $1 \leqslant i \leqslant r, 1 \leqslant k \leqslant t$. Consequently, $v = 0$. This implies that $W \cap V_{(j)} \subseteq W_{(j)}$. On the other hand, it is obvious that $W_{(j)} \subseteq W \cap V_{(j)}$, so that $W \cap V_{(j)} = W_{(j)}, \forall\, j \geqslant 0$. Hence, $W = W \cap V = W \cap V_{(p-2)l+r} = W_{(p-2)l+r} = \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M', \chi)$.

As a direct consequence, we have the following criterion on irreducibility of the induced module $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$.

Theorem 3.8 The induced $U_\chi(\mathfrak{g})$-module $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$ is irreducible if and only if M is irreducible.

Proof The sufficient implication is obvious. It suffices to show the necessary implication. Suppose that M is irreducible. Let W be a $\mathfrak{g}$-submodule of $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$. By Proposition 3.7, there exists a $\mathfrak{g}^\lambda$-submodule M' of M such that $W = \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M', \chi)$. Consequently, $W = 0$ or $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(M, \chi)$ corresponding to $M' = 0$ or $M' = M$.

For a $\mathfrak{g}$-module V, set $V^\lambda := \{v \in V \mid y \cdot v = \lambda(y)v, \forall\, y \in I\}$, which is a $\mathfrak{g}^\lambda$-submodule of V by a straightforward computation.

Theorem 3.9 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field. Let V be an irreducible $\mathfrak{g}$-module, and I be an ideal of $\mathfrak{g}$. Then the following statements hold.

(1) If V has a p-character $\chi \in \mathfrak{g}^*$ and there is $\lambda \in I^*$ with $\lambda([I, I]) = 0$ and $V^\lambda \neq 0$, then $V \cong \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(V^\lambda, \chi)$ and V^λ is an irreducible $\mathfrak{g}^\lambda$-module.

(2) If $[I, I]$ operates nilpotently on V, then there exists $\chi \in \mathfrak{g}^*$, $\lambda \in I^*$ with $\lambda([I, I]) = 0$ such that $V \cong \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(V^\lambda, \chi)$.

Proof (1) Since V is irreducible, there exists $\chi \in \mathfrak{g}^*$ such that V is a finite-dimensional $U_\chi(\mathfrak{g})$-module, and we have the following surjective homomorphism

$$\begin{aligned} \Psi:\ \mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(V^\lambda, \chi) &\longrightarrow V \\ u \otimes v &\longmapsto u \cdot v. \end{aligned}$$

Note that $\mathrm{Ker}\Psi$ is a $\mathfrak{g}$-submodule of $\mathrm{Ind}_{\mathfrak{g}^\lambda}^{\mathfrak{g}}(V^\lambda, \chi)$ which intersects $1 \otimes V^\lambda$ trivially. This implies that $\mathrm{Ker}\Psi = 0$ by Proposition 3.7. Hence, Ψ is an isomorphism and V^λ is irreducible by Theorem 3.8.

(2) follows from Lemma 1.5 and the statement (1).

Remark 3.10 If $I \lhd \mathfrak{g}$ is an abelian ideal, then Theorem 3.9(2) applies.

Definition 3.11 Let V be a $\mathfrak{g}$-module and $I \lhd \mathfrak{g}$ be an ideal. We say $\lambda \in I^*$ a good eigenvalue function for V if $\lambda([I,I]) = 0$ and $V^{\lambda} \neq 0$.

Let $\chi \in \mathfrak{g}^*$ and $I \lhd \mathfrak{g}$. Let $\lambda \in I^*$ with $\lambda([I,I]) = 0$. We denote by $\mathfrak{C}_{\chi,\lambda}$ (resp. $\mathfrak{D}_{\chi,\lambda}$) the set of isomorphism classes of irreducible $\mathfrak{g}$ (resp. $\mathfrak{g}^{\lambda}$) modules with p-character χ (resp. $\chi|_{\mathfrak{g}^{\lambda}}$) and a good eigenvalue function λ.

Theorem 3.12 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field. Let $\chi \in \mathfrak{g}^*$. Let $I \lhd \mathfrak{g}$ be an ideal and $\lambda \in I^*$ with $\lambda([I,I]) = 0$. Then the following map

$$\begin{aligned}\Upsilon : \mathfrak{C}_{\chi,\lambda} &\longrightarrow \mathfrak{D}_{\chi,\lambda}\\ V &\longmapsto V^{\lambda}\end{aligned}$$

is bijective.

Proof By Theorem 3.9, Υ is well-defined. Let

$$\begin{aligned}\Gamma : \mathfrak{D}_{\chi,\lambda} &\longrightarrow \mathfrak{C}_{\chi,\lambda}\\ M &\longmapsto \mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(M,\chi)\end{aligned}$$

which is well-defined by Theorem 3.8.

Let M be an irreducible $\mathfrak{g}^{\lambda}$-module with p-character $\chi|_{\mathfrak{g}^{\lambda}}$ and a good eigenvalue function λ. Set $V := \mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(M,\chi)$. Then V is irreducible by Theorem 3.8. Moreover, $1 \otimes M \subseteq V^{\lambda}$ by Lemma 1.5(3). Thanks to Theorem 3.9, $V \cong \mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(V^{\lambda},\chi)$. Consequently, $V^{\lambda} = 1 \otimes M$ by comparing their dimensions, i.e., $\Upsilon \circ \Gamma(M) \cong M$.

Conversely, let V be an irreducible $\mathfrak{g}$-module with p-character χ and a good eigenvalue function λ. Then $V \cong \mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(V^{\lambda},\chi)$ by Theorem 3.9, i.e., $\Gamma \circ \Upsilon(V) \cong V$. Therefore, Υ is bijective, and Γ is its inverse map.

Example 3.13 Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the so-called Heisenberg Lie superalgebra with $\mathfrak{g}_{\bar{0}} = \mathrm{span}_{\mathbb{F}}\{c\}$, $\mathfrak{g}_{\bar{1}} = \mathrm{span}_{\mathbb{F}}\{x_i, y_j \mid 1 \leqslant i,j \leqslant n\}$, and the p-mapping $[p]$ and the Lie bracket subject to the following rules:

$$c^{[p]} = c,\ [x_i, y_j] = \delta_{ij}c,\ [x_i, x_j] = [y_i, y_j] = [c, x_i] = [c, y_j] = 0,\ \forall\, 1 \leqslant i,j \leqslant n.$$

Let $0 \neq \chi \in \mathfrak{g}_{\bar{0}}^*$ and $\Lambda_{\chi} := \{\mu \in \mathbb{F} \mid \mu^p - \mu = \chi(c)^p\}$. Let $I = \mathrm{span}_{\mathbb{F}}\{c, x_i \mid 1 \leqslant i \leqslant n\}$ which is an abelian ideal of $\mathfrak{g}$. Let $\lambda \in I^*$ with $\lambda(c) \in \Lambda_{\chi}$, and $\lambda(x_i) = 0$, $1 \leqslant i \leqslant n$. Then $\mathfrak{g}^{\lambda} = I$ by a direct computation. By Theorem 3.8 and Theorem 3.9, each simple $\mathfrak{g}$-module with p-character χ and a good eigenvalue function λ is of the form $\mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(\mathbb{F}v_{\lambda},\chi)$, where $\mathbb{F}v_{\lambda}$ is the one-dimensional I-module with $c \cdot v_{\lambda} = \lambda(c)v_{\lambda}$ and $x_i \cdot v_{\lambda} = 0$, $1 \leqslant i \leqslant n$. Moreover, for any $\chi \in \mathfrak{g}^*$, since $U_{\chi}(I)$ is a local superalgebra, any simple $U_{\chi}(I)$-module is one-dimensional, and there are totally p simple modules $\mathbb{F}v_{\lambda}$ with $c \cdot v_{\lambda} = \lambda v_{\lambda}$ and $x_i \cdot v_{\lambda} = 0$ $(1 \leqslant i \leqslant n)$, where $\lambda \in \Lambda_{\chi}$. Hence, each simple $U_{\chi}(\mathfrak{g})$-module is of the form $\mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(\mathbb{F}v_{\lambda},\chi)$ with $\lambda \in \Lambda_{\chi}$. Moreover, $\mathrm{Ind}_{\mathfrak{g}^{\lambda}}^{\mathfrak{g}}(\mathbb{F}v_{\lambda},\chi) \cong \mathrm{Ind}_{\mathfrak{g}^{\mu}}^{\mathfrak{g}}(\mathbb{F}v_{\mu},\chi)$ if and only if $\lambda = \mu$.

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