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Weighted finite difference methods for two-sided space-time fractional diffusion equations

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Abstract: A weighted finite difference scheme was proposed in order to solve initial-boundary value problems of space-time fractional diffusion equations. Their stability was analyzed by means of discrete energy method. Using mathematical induction, we proved that the scheme was convergent under the same condition. Illustrative example was included to demonstrate the validity and applicability of the scheme.

Key words: fractional diffusion equation; space-time fractional derivative; weighted difference scheme; convergence; stability

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两边空间-时间分数阶扩散方程的加权有限差分格式

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摘要: 对于空间-时间分数阶扩散方程的初边值问题提出了一种加权差分格式. 利用能量估计, 得到了差分格式的稳定性. 然后使用数学归纳法证明了在相同的条件下, 所提出的格式是收敛的. 最后通过一个例子说明了所提出的格式是可靠的、有效的.

关键词: 分数阶扩散方程; 空间-时间分数阶导数; 加权差分格式; 收敛性; 稳定性

0 Introduction

In recent years, the fractional differential equations have been widely used in diverse fields, such as in physics^[1-3], finance^[4,5], hydrology^[6], engineering^[7], mathematics^[8-10] and material science.

Fractional order partial differential equations are generalizations of classical partial differential equations. When a fractional derivative replaces the second derivative in the diffusion

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equation, it leads to enhanced diffusion (also called super-diffusion)^[11,12]. A fractional time derivative leads to sub-diffusion, where a cloud of particles spreads slower than the classical $t^{\frac{1}{2}}$ rate^[13,14].

In this paper, we consider the two sided space-time fractional diffusion equation, which can be written as

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} = -v(x) \frac{\partial u(x, t)}{\partial x} + d_+(x) \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} + d_-(x) \frac{\partial^\alpha u(x, t)}{\partial_- x^\alpha} + f(x, t),$$

$$x \in [L, R], t \in [0, T], \quad (1a)$$

$$u(L, t) = 0, u(R, t) = \varphi(t), t \in [0, T], \quad (1b)$$

$$u(x, 0) = u_0(x), x \in [L, R], \quad (1c)$$

where α and β are parameters describing the order of the fractional space and time derivatives, respectively, physical considerations restrict $0 < \beta < 1$, $1 < \alpha < 2$.

The functions $v(x)$, $d_+(x)$ and $d_-(x)$ are all non-negative and bounded.

The left-handed (+) and the right-handed (-) fractional derivatives are all the Riemann-Liouville fractional derivatives, which are defined^[15] respectively by

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} &= \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_L^x \frac{u(\xi, t)}{(x - \xi)^{\alpha+1-n}} d\xi, \\ \frac{\partial^\alpha u(x, t)}{\partial_- x^\alpha} &= \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^R \frac{u(\xi, t)}{(x - \xi)^{\alpha+1-n}} d\xi, \end{aligned}$$

where n is an integer such that $n - 1 < \alpha < n$. The time derivative $\frac{\partial^\beta u(x, t)}{\partial t^\beta}$ is given by a Caputo fractional derivative

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - \eta)^{-\beta} \frac{\partial u(x, \eta)}{\partial \eta} d\eta,$$

where $\Gamma(\cdot)$ is the gamma function.

Fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. Since fractional derivatives are non-local operators, finite difference schemes^[16] for fractional partial differential equations are more complex than partial differential equations. In [17], Ding et al. proposed a weighted finite difference scheme for space fractional partial differential equations. In this paper, a weighted finite difference for two sided space-time diffusion equation is presented.

1 Finite difference scheme

The domain $[L, R] \times [0, T]$ will be divided into a $J \times N$ mesh with spatial step size $h = (R - L)/J$ in x direction and the time step size $\tau = T/N$, respectively. Denote u_j^n be the numerical solution at (x_i, t_n) for $x_i = L + ih$, $t_n = n\tau$, $j = 0, 1, \dots, J$, $n = 0, 1, \dots, N$.

The shifted Grünwald formula is applied to discretize the left-handed fractional derivative and right-handed fractional derivative

$$\begin{aligned}\frac{\partial^\alpha u(x_i, t_n)}{\partial_+ x^\alpha} &= \frac{1}{h^\alpha} \sum_{j=0}^{i+1} g_j u(x_i - (j-1)h, t_n) + o(h), \\ \frac{\partial^\alpha u(x_i, t_n)}{\partial_- x^\alpha} &= \frac{1}{h^\alpha} \sum_{j=0}^{N-i+1} g_j u(x_i + (j-1)h, t_n) + o(h),\end{aligned}$$

where the Grünwald coefficients are defined by

$$g_0 = 1, g_j = \left(1 - \frac{\alpha+1}{j}\right) g_{j-1}, j = 1, 2, 3, \dots$$

As usual, we take the following finite difference approximation for Caputo fractional derivative

$$\frac{\partial^\beta u(x_i, t_n)}{\partial t^\beta} = \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^n \frac{u(x_i, t_{n+1-j}) - u(x_i, t_{n-j})}{\tau} [(j+1)^{1-\beta} - j^{1-\beta}] + o(\tau).$$

Now we approximate (1) using a weighted finite difference scheme (WFDMs):

$$\begin{aligned}& \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^n \frac{u_i^{n+1-j} - u_i^{n-j}}{\tau} [(j+1)^{1-\beta} - j^{1-\beta}] = -v_i \left[\theta \frac{u_i^n - u_{i-1}^n}{h} + (1-\theta) \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} \right] \\ & + \frac{d_{+i}}{h^\alpha} \left[\theta \sum_{k=0}^{i+1} g_k u_{i-k+1}^n + (1-\theta) \sum_{k=0}^{i+1} g_k u_{i-k+1}^{n+1} \right] + \frac{d_{-i}}{h^\alpha} \left[\theta \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^n \right. \\ & \left. + (1-\theta) \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^{n+1} \right] + \theta f_i^n + (1-\theta) f_i^{n+1},\end{aligned}\quad (2)$$

for $i = 1, 2, \dots, J-1$, $n = 0, 1, \dots, N-1$, where θ is the weighting parameter subjected to $0 \leq \theta \leq 1$. When $\theta = 0$, we have the implicit difference scheme. When $\theta = 1$, we get the explicit difference scheme. When $\theta = \frac{1}{2}$, we get the space-time fractional Crank-Nicolson difference scheme. Reordering (2), we obtain for $n = 0$,

$$\begin{aligned}& - (1-\theta)(\xi_i + \eta_i g_2 + \zeta_i) u_{i-1}^1 + [1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] u_i^n \\ & - (1-\theta)(\eta_i + \zeta_i g_2) u_{i+1}^n - (1-\theta) \eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^1 - (1-\theta) \zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^1 \\ & = \theta(\xi_i + \eta_i g_2 + \zeta_i) u_{i-1}^0 + [1 - \theta(\xi_i - \eta_i g_1 - \zeta_i g_1)] u_i^0 + \theta(\eta_i + \zeta_i g_2) u \\ & + \theta \eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^0 + \theta \zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^0 + \Gamma(1-\beta) \tau^\beta (\theta f_i^n + (2-\theta) f_i^{n+1}).\end{aligned}\quad (3)$$

And for $n > 0$

$$\begin{aligned}
 & - (1 - \theta)(\xi_i + \eta_i g_2 + \zeta_i) u_{i-1}^{n+1} + [1 + (1 - \theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] u_i^{n+1} \\
 & - (1 - \theta)(\eta_i + \zeta_i g_2) u_{i+1}^{n+1} - (1 - \theta) \eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^{n+1} - (1 - \theta) \zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^{n+1} \\
 & = \theta(\xi_i + \eta_i g_2 + \zeta_i) u_{i-1}^n + [2 - 2^{1-\beta} - \theta(\xi_i - \eta_i g_1 - \zeta_i g_1)] u_i^n \\
 & + \theta(\eta_i + \zeta_i g_2) u_{i+1}^n + \theta \eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^n + \theta \zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^n + \sum_{j=1}^{n-1} d_j u_i^{n-j} \\
 & + u_i^0 [(n+1)^{1-\beta} - n^{1-\beta}] + \Gamma(1-\beta) \tau^\beta (\theta f_i^n + (2-\theta) f_i^{n+1}), \tag{4}
 \end{aligned}$$

with boundary conditions $u_0^n = 0, u_J^n = \varphi(t_n), n = 1, 2, \dots, N-1$, and initial conditions

$$u_i^0 = u_0(x_i), i = 0, 1, \dots, J,$$

where $\xi_i = \frac{v_i t^\beta \Gamma(2-\beta)}{h}$, $\eta_i = \frac{d_{+i} \tau^\beta \Gamma(2-\beta)}{h^\alpha}$, $\zeta_i = \frac{d_{-i} \tau^\beta \Gamma(2-\beta)}{h^\alpha}$ and $d_j = 2(j+1)^{1-\beta} - (j+2)^{1-\beta} - j^{1-\beta}$, $j = 1, 2, \dots, n-1$. Eqs (3) and (4) can also be written in matrix form:

$$AU^1 = B_0 U^0 + Q^0, \quad AU^{n+1} = BU^n + d_1 U^{n-1} + \dots + d_{n-1} U^1 + [(n+1)^{1-\beta} - n^{1-\beta}] U^0 + Q^n,$$

Where

$$\begin{aligned}
 U^n &= (u_1^n, u_2^n, \dots, u_{J-1}^n)^T, \quad U^0 = (u_0(x_1), u_0(x_2), \dots, u_0(x_{J-1}))^T, \\
 b &= (\eta_{J-1} + \zeta_{J-1} g_2) [(1-\theta) u_J^{n+1} + \theta u_J^n], \quad F^n = (f_1^n, f_2^n, \dots, f_{J-1}^n + b)^T, \\
 E &= (\zeta_1 g_J, \zeta_2 g_{J-1}, \dots, \zeta_{J-1} g_2)^T, \\
 Q^n &= \Gamma(2-\beta) \tau^\beta (\theta F^n + (1-\theta) F^{n+1}) + (1-\theta) U_J^{n+1} E + \theta U_J^n E,
 \end{aligned}$$

and matrix $A = (A_{ij})_{(J-1)(J-1)}$ is defined as follows:

$$A_{ij} = \begin{cases} -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i), & j = i-1, \\ 1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1), & j = i \\ -(1-\theta)(\eta_i + \zeta_i g_2), & j = n+1 \\ -(1-\theta)\eta_i g_{i+1-j}, & j = 1, 2, \dots, i-2 \\ -(1-\theta)\zeta_i g_{j+1-i}, & j = i+2, i+3, \dots, J-1. \end{cases}$$

It is obvious that matrix A is strictly dominant, the system defined by (3) and (4) has unique solution.

2 Stability and convergence

Theorem 1 The weighted finite difference scheme (2) for solving equation (1) is stable under the condition

$$\theta \left[\frac{\tau^\beta}{h} \max_{x \in [L, R]} v(x) + \frac{\alpha \tau^\beta}{h^\alpha} \max_{x \in [L, R]} (d_+(x) + d_-(x)) \right] \leq \frac{2 - 2^{1-\beta}}{\Gamma(2-\beta)}. \tag{5}$$

Proof Let $u_i^n, \tilde{u}_i^n (i = 1, 2, \dots, J, n = 0, 1, 2, \dots, N-1)$ be the solution of difference scheme (2) with initial value u_i^0 and \tilde{u}_i^0 , respectively. Let $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$, for $n-1$, we have

$$\begin{aligned}
& - (1-\theta)(\xi_i + \eta_i g_2 + \zeta_i) \varepsilon_{i-1}^1 + [1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] \varepsilon_i^1 \\
& - (1-\theta)(\eta_i + \zeta_i g_2) \varepsilon_{i+1}^1 - (1-\theta) \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^1 - (1-\theta) \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^1 \\
& = \theta(\xi_i + \eta_i g_2 + \zeta_i) \varepsilon_{i-1}^0 + [1 - \theta(\xi_i - \eta_i g_1 - \zeta_i g_1)] \varepsilon_i^0 + \theta(\eta_i + \zeta_i g_2) \varepsilon_{i+1}^0 \\
& + \theta \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^0 + \theta \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^0
\end{aligned} \tag{6}$$

For $n > 0$,

$$\begin{aligned}
& - (1-\theta)(\xi_i + \eta_i g_2 + \zeta_i) \varepsilon_{i-1}^{n+1} + [1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] \varepsilon_i^{n+1} \\
& - (1-\theta)(\eta_i + \zeta_i g_2) \varepsilon_{i+1}^{n+1} - (1-\theta) \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^{n+1} - (1-\theta) \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^{n+1} \\
& = \theta(\xi_i + \eta_i g_2 + \zeta_i) \varepsilon_{i-1}^n + [2 - 2^{1-\beta} - \theta(\xi_i - \eta_i g_1 - \zeta_i g_1)] \varepsilon_i^n + \theta(\eta_i + \zeta_i) \\
& + \theta \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^n + \theta \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^n + \sum_{j=1}^{n-1} d_j \varepsilon_i^{n-j} + [(n+1)^{1-\beta} - n^{1-\beta}] \varepsilon_i^0.
\end{aligned} \tag{7}$$

Then Eq. (6) and Eq. (7) can be written in the following matrix form

$$AE^1 = B_0 E^0, AE^{n+1} = BE^n + d_1 E^{n-1} + \dots + d_{n-1} E^1 + [(n+1)^{1-\beta} - n^{1-\beta}] E^0,$$

where $E^n = (\varepsilon_1^n, \varepsilon_2^n, \dots, \varepsilon_{J-}^n)$. We will use mathematical induction method to prove

$$\|E^n\|_\infty \leq \|E^0\|_\infty, n = 0, 1, \dots, N-1.$$

In fact, if $n = 1$, suppose $|\varepsilon_l^1| = \max_{1 \leq i \leq J-1} |\varepsilon_i^1|$, note that $\xi_i, \eta_i, \zeta_i > 0$ and for any integer number m , $\sum_{j=0}^m g_j < 0$, from (5), we get

$$\begin{aligned}
\|E^1\|_\infty & = |\varepsilon_l^1| \leq -(1-\theta) \eta_l \sum_{k=0}^{l+1} g_k |\varepsilon_l^1| + |\varepsilon_l^1| - (1-\theta) \zeta_l \sum_{k=0}^{J-l+1} |\varepsilon_l^1| \\
& \leq - (1-\theta)(\xi_l + \eta_l g_2 + \zeta_l) |\varepsilon_{l-1}^1| + [1 + (1-\theta)(\xi_l - \eta_l g_1 - \zeta_l g_1)] |\varepsilon_l^1| \\
& - (1-\theta)(\xi_l + \eta_l g_2) |\varepsilon_{l+1}^1| - (1-\theta) \eta_l \sum_{k=0}^{l+1} g_k |\varepsilon_{l-k+1}^1| - (1-\theta) \zeta_l \sum_{k=0}^{J-l+1} g_l |\varepsilon_{l+k-1}^1| \\
& \leq \|E^0\|_\infty.
\end{aligned}$$

Suppose that $\|E^n\|_\infty \leq \|E^0\|_\infty$, $n = 1, 2, \dots, s$, then when $n = s + 1$, let $|\varepsilon_l^{s+1}| = \max_{1 \leq i \leq J-1} |\varepsilon_i^{s+1}|$. Similar to former estimate, we obtain

$$\begin{aligned} \|E^{s+1}\|_\infty &\leq -(1-\theta)(\xi_l + \eta_l g_2 + \zeta_l) \varepsilon_{l-1}^{s+1} + [1 + (1-\theta)(\xi_l - \eta_l g_1 - \zeta_l g_1)] \varepsilon_l^{s+1} \\ &\quad - (1-\theta)(\eta_l + \zeta_l g_2) \varepsilon_{l+1}^{s+1} - (1-\theta) \eta_l \sum_{k=3}^{l+1} g_k \varepsilon_{l-k+1}^{s+1} - (1-\theta) \zeta_l \sum_{k=3}^{J-l+1} g_k \varepsilon_{l+k-1}^{s+1} \\ &\leq (2-2^{1-\beta}) \|E^0\|_\infty + \theta \eta_l \sum_{k=0}^{l+1} g_k \|E^0\|_\infty + \theta \zeta_l \sum_{k=0}^{J-l+1} g_k \|E^0\|_\infty + \sum_{j=0}^{s-1} d_j \|E^0\|_\infty \\ &\quad + [(s+1)^{1-\beta} - s^{1-\beta}] \|E^0\|_\infty \leq \|E^0\|_\infty. \end{aligned}$$

Therefore $\|E^{s+1}\|_\infty \leq \|E^0\|_\infty$. This completes the proof.

Theorem 2 Suppose that $u(x, t)$ is the sufficiently smooth solution of (1) and u_i^k is the difference solution of difference scheme (2). If τ and h satisfy (5), then there exists positive constant $M > 0$, such that

$$\|u(x_i, t_n) - u_i^n\|_\infty \leq M \sigma_{n-1}^{-1} (\tau^{1+\beta} + \tau^\beta h).$$

Proof Denote $e_i^n = u(x_i, t_n) - u_i^n$ and $e^n = (e_1^n, e_2^n, \dots, e_{J-1}^n)$. Notice that $e_j^0 = 0$, we have: if $n = 0$,

$$\begin{aligned} R_i^1 &= -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^1 + [1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] e_i^1 \\ &\quad - (1-\theta)(\eta_i + \zeta_i g_2) e_{i+1}^1 - (1-\theta) \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^1 - (1-\theta) \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^1; \end{aligned}$$

if $n > 0$,

$$\begin{aligned} R_i^{n+1} &= -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^{n+1} + [1 + (1-\theta)(\xi_i - \eta_i g_1 - \zeta_i g_1)] e_i^{n+1} \\ &\quad - (1-\theta)(\eta_i + \zeta_i g_2) e_{i+1}^{n+1} - (1-\theta) \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^{n+1} - (1-\theta) \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^{n+1} \\ &\quad - \theta(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^n - 2 - 2^{1-\beta} - \theta(\xi_i - \eta_i g_1 - \zeta_i g_1) e_i^n - \theta(\eta_i + \zeta_i g_2) e_{i+1}^n \\ &\quad - \theta \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^n - \theta \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^n - \sum_{j=3}^{n-1} d_j e_i^{n-j}. \end{aligned}$$

Where R_i^{n+1} is the truncation error of difference scheme (2). Furthermore, there exists a positive constant M independent of step sizes such that $|R_i^{n+1}| \leq M(\tau^{1+\beta} + \tau^\beta h)$.

We will prove by inductive method. If $k = 1$, let $|e_l^1| = \max_{1 \leq i \leq J-1} |e_i^1|$ and

$\sigma_j = (j+1)^{1-\beta} - j^{1-\beta}$, we have

$$\begin{aligned}
\|e^1\|_\infty &\leq -(1-\theta)(\xi_l + \eta_l g_2 + \zeta_l) |e_{l-1}^1| + [1 + (1-\theta)(\xi_l - \eta_l g_1 - \zeta_l g_1) |e_l^1| \\
&\quad - (1-\theta)(\eta_l + \zeta_l g_2) |e_{l+1}^1| - (1-\theta)\eta_l \sum_{k=3}^{l+1} g_k |e_{l-k+1}^1| - (1-\theta)\zeta_l \sum_{k=3}^{J-l+1} g_k |e_{l+k-1}^1| \\
&\leq -(1-\theta)(\xi_l + \eta_l g_2 + \zeta_l) e_{l-1}^1 + [1 + (1-\theta)(\xi_l - \eta_l g_1 - \zeta_l g_1)] e_l^1 \\
&\quad - (1-\theta)(\eta_l + \zeta_l g_2) e_{l+1}^1 - (1-\theta)\eta_l \sum_{k=3}^{l+1} g_k e_{l-k+1}^1 - (1-\theta)\zeta_l \sum_{k=3}^{J-l+1} g_k e_{l+k-1}^1 \\
&= |R_l^1| \leq M(\tau^{1+\beta} + \tau^\beta h) = \sigma_0^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}$$

Assume that $\|e^n\|_\infty \leq M\sigma_{n-1}^{-1}(\tau^{1+\beta} + \tau^\beta h)$, $n = 1, 2, \dots, s$, then when $n = 1 + s$, let $|e_l^{s+1}| = \max_{1 \leq i \leq J-1} |e_i^{s+1}|$, notice that $\sigma_j^{-1} < \sigma_k^{-1}$, $j = 0, 1, \dots, k-1$. Similarly, we obtain

$$\begin{aligned}
\|e^{s+1}\|_\infty &\leq d_1 \|e^s\|_\infty + \sum_{j=1}^{n-1} d_j \|e^{s-j}\|_\infty + M(\tau^{1+\beta} + \tau^\beta h) \\
&\leq (d_1 \sigma_{s-1}^{-1} + d_2 \sigma_{s-1}^{-1} + \dots + d_s \sigma_0^{-1} + 1) M(\tau^{1+\beta} + \tau^\beta h) \\
&\leq \sigma_s^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}$$

Therefore Theorem 2 is proved.

Since

$$\lim_{n \rightarrow \infty} \frac{\sigma_n^{-1}}{n^\beta} = \lim_{n \rightarrow \infty} \frac{n^{-1}}{(1-\beta)n^{-1}} = \frac{1}{1-\beta}.$$

there is a constant C_1 for which $\|e^n\|_\infty \leq C_1 n^\beta (\tau^{1+\beta} + \tau^\beta h)$. When $n\tau \leq T$ is finite, we obtain the following theorem.

Theorem 3 Under the conditions of Theorem 2, then there exists positive constant $C > 0$, such that

$$\|u(x_i, t_n) - u_i^n\| \leq C(\tau + h), i = 1, 2, \dots, J-1; n = 1, 2, \dots, N.$$

3 Numerical experiment

We consider the following two-sided space-time fractional diffusive equations.

$$\frac{\partial^{0.6} u(x, t)}{\partial t^{0.6}} = -\frac{\partial u(x, t)}{\partial x} + d_+(x) \frac{\partial^{1.6} u(x, t)}{\partial_+ x^{1.6}} + d_-(x) \frac{\partial^{1.6} u(x, t)}{\partial_- x^{1.6}} + f(x, t),$$

$$(x, t) \in [0, 1] \times [0, 1], u(0, t) = 0, u(1, t) = 1 + 4t^2, t \in [0, 1], u(x, 0) = x^2, x \in [0, 1],$$

where $d_+(x) = \frac{2}{5}\Gamma(0.4)x^{0.6}$, $d_-(x) = 5\Gamma(0.4)(1-x)^{1.6}$, And $f(x, t) = \frac{100}{7\Gamma(0.4)}x^2 t^{1.4} + (1 + 4t^2)(-25x^2 + 40x - 12)$. The exact solution is $u(x, t) = (1 + 4t^2)x^2$.

Tab. 1 shows the maximum absolute numerical error between the exact solution and the numerical solution obtained by WFDMs with $\theta = 1$. From Tab. 1, it can be seen that our scheme is conditionally stable and yields convergence with $o(\tau + h)$. Tab. 2 and Tab. 3 show the error between the exact solution and the numerical solution obtained by WFDMs with $\theta = \frac{1}{2}$ and $\theta = 0$, respectively. Tab. 2 and Tab. 3 show the second-order convergence in l_∞ norm of the scheme. From the above three tables, it can be seen that the numerical tests are in excellent agreement with theoretical analysis.

Tab. 1 The error $\max |u_i^k - u(x_i, t_k)|$ for the WFDMs with $\theta = 1$

N	J	State	Maximum error
10	10	Divergence	1.553 0e+019
100	10	Divergence	2.037 5e+164
1 000	10	Divergence	Infinity
30 000	10	Convergence	0.7667 1

Tab. 2 The error and convergence rate for the scheme with $\theta = \frac{1}{2}$

N	J	Maximum error	Convergence rate
200	200	0.040 8	—
400	400	0.021 1	1.933 6
800	800	0.010 8	1.953 7
1 600	1 600	0.005 5	1.963 6

Tab. 3 The error and convergence rate for the scheme with $\theta = 0$

N	J	Maximum error	Convergence rate
200	200	0.041 5	—
400	400	0.021 4	1.939 3
800	800	0.011 0	1.945 5
1 600	1 600	0.005 6	1.964 3

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显然, G 在 \mathbb{C} 上有无穷多零点 ζ_m . 当 $m \rightarrow \infty$ 时, $\zeta_m \rightarrow \infty$. 由式 (24) 知,

$$|G^{(k)}(\zeta_m)| = |\zeta_m + B \exp(A\zeta_m)| \leq |\zeta_m^l|,$$

故存在 $M > 0$, 对每个 m , $\left| \frac{\exp(A\zeta_m)}{\zeta_m^l} \right| \leq M$. 但是, 当 $m \rightarrow \infty$ 时,

$$\left| \frac{G(\zeta_m)}{\zeta_m^l} \right| = \left| \frac{l!}{(k+l)!} \zeta_m^k + a_{k-1} \zeta_m^{k-1-l} + \cdots + a_0 \zeta_m^{-l} + \frac{BA^{-k} \exp(A\zeta_m)}{\zeta_m^l} \right| \rightarrow \infty,$$

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