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Positive solutions of the semipositone three-point boundary value problem on time scales

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Abstract: The existence of positive solutions for semipositone three-point boundary value problem on time scales was discussed, by using the Leggett-Williams fixed point theorem; the existence of at least two positive solutions was obtained. The results extend and improve some known results. As an application, an example was given to demonstrate the main result.

Key words: positive solution; semipositone; fixed point theorem; time scales

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时间尺度上半正三点边值问题的正解

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摘要: 利用Leggett-Williams 不动点定理, 研究了时间尺度上半正三点边值问题正解的存在性, 得到了至少有两个正解存在的结果. 推广和改进了一些已有结果, 并举一个例子来验证主要结论.

关键词: 正解; 半正; 定点定理; 时间尺度

0 Introduction

Let $\mathbf{T}$ be a time scale such that $0, T \in \mathbf{T}$. We study

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad t \in [0, T], \tag{0.1}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta). \tag{0.2}$$

We make the following assumptions:

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(H$_1$) $0 < \alpha < \dfrac{T}{\eta}, 0 < \beta < \dfrac{T-\alpha\eta}{T-\eta}, \eta \in (0,T)$;

(H$_2$) $f : \mathbf{C}_{ld}([0,T] \times [0,+\infty), \mathbf{R})$.

(H$_3$) There exists a constant $M > 0$ such that $f(t,u) \geqslant -M$ for $(t,u) \in [0,T] \times [0,\infty)$.

Recently, there is much attention paid to the questions of positive solutions of the boundary value problems on time scales (see [1-8] and the references therein). However, very few works can be found for that with the semipositone boundary.

In [1], Luo and Ma showed the existence of positive solutions to the problem

$$u^{\Delta\nabla}(t) + a(t)f(t,u(t)) = 0, \quad t \in [0,T], \tag{0.3}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta), \tag{0.4}$$

on time scales by Guo-Krasnoselskii's fixed point theorem and Leggett-Williams fixed-point theorem, where $f : [0,T] \times [0,+\infty) \to [0,+\infty)$. In [9], Chen studied the existence of positive solutions for semipositone m-point boundary value problems

$$u''(t) + f(t,u) = 0, \quad t \in (0,1),$$

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i).$$

By using the Leggett-Williams fixed-point theorem, some sufficient conditions guaranteed the existence of at least two positive solutions were established under weaker conditions $f : [0,1] \times [0,+\infty) \to \mathbf{R}$ was continuous and there existed a constant $M > 0$ such that $f(t,u) > -M$ for $(t,u) \in [0,1] \times [0,\infty)$.

Motivated by the ideas in [1,9], the purpose of this paper is to study the existence of positive solutions for semipositone three-point boundary value problem on time scales. Different from the previous papers, for example [1], in this paper the nonlinearity term $f(t,u)$ is allowed to be negative and may tend to infinity. Our result extend and improve some known results.The result is new even for the special case of difference equations.

1 Preliminaries

We present some basic definitions which can be found in Atici and Guseinov[8], Bohner and Peterson[10,11], Hilger[12].

In order to prove our main theorem, we will need the following lemmas, whose proofs can be found in Luo[1]. Consider the linear boundary value problem,

$$u^{\Delta\nabla}(t) + y(t) = 0, \quad t \in [0,T], \tag{1.1}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta). \tag{1.2}$$

Lemma 1.1 Let $\beta \neq \dfrac{T-\alpha\eta}{T-\eta}$. Then for $y \in \mathbf{C}_{ld}([0,T], \mathbf{R})$, the problem (1.1), (1.2) has

a unique solution

$$u(t) = -\int_0^t (t-s)y(s)\,\nabla s + \frac{(\beta-\alpha)t-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)y(s)\,\nabla s$$
$$+\frac{(1-\beta)t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)y(s)\,\nabla s. \tag{1.3}$$

Lemma 1.2 Let $0<\alpha<\dfrac{T}{\eta}$, $0\leqslant\beta<\dfrac{T-\alpha\eta}{T-\eta}$. If $y\in \mathbf{C}_{ld}([0,T],[0,\infty))$, then the unique solution $u(t)$ of the problem (1.1), (1.2) satisfies $u(t)\geqslant 0,\quad t\in[0,T]$.

Lemma 1.3 Let $0<\alpha<\dfrac{T}{\eta}$, $0<\beta<\dfrac{T-\alpha\eta}{T-\eta}$. If $y\in \mathbf{C}_{ld}([0,T],[0,\infty))$, then the unique solution to the problem (1.1), (1.2) satisfies $\min\limits_{t\in[0,T]} u(t)\geqslant r\| u\|$, where $r=\min\left\{\dfrac{\alpha(T-\eta)}{T-\alpha\eta},\quad \dfrac{\alpha\eta}{T},\quad \dfrac{\beta(T-\eta)}{T},\quad \dfrac{\beta\eta}{T}\right\}$, and $0<r<1$.

Lemma 1.4 Let $\beta\neq\dfrac{T-\alpha\eta}{T-\eta}$, $\omega(t)$ is the unique solution of the problem

$$u^{\Delta\nabla}(t)+1=0,\quad t\in[0,T],$$
$$u(0)=\beta u(\eta),\quad u(T)=\alpha u(\eta),$$

then $\omega(t)\leqslant Lr$, where r is given as in lemma 1.3 and

$$L=\frac{A}{r},\quad A=\frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s.$$

Proof From Lemma 1.1, we have

$$\begin{aligned}\omega(t) =& -\int_0^t (t-s)\,\nabla s+\frac{(\beta-\alpha)t-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)\,\nabla s\\ &+\frac{(1-\beta)t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s\\ \leqslant&\frac{(\beta-\alpha)t-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)\,\nabla s\\ &+\frac{(1-\beta)t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s\\ \leqslant&\frac{\beta t}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)\,\nabla s\\ &+\frac{t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s\\ \leqslant&\frac{\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)\,\nabla s\\ &+\frac{T+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s\\ \leqslant&\frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s=A.\end{aligned}$$

Lemma 1.5 Let $x(t) = M\omega(t)$, then (0.1), (0.2) has a positive solution $u(t)$ if and only if $y = u + x$ is a solution of

$$u^{\Delta\nabla}(t) + h(t, u(t) - x(t)) = 0, \quad t \in [0, T], \tag{1.4}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta) \tag{1.5}$$

and $y(t) > x(t), t \in [0, T]$, where $h : [0, T] \times \mathbf{R} \to \mathbf{R}^+$ is defined by

$$h(t, z) = \begin{cases} f(t, z) + M, & (t, z) \in [0, T] \times [0, \infty), \\ f(t, 0) + M, & (t, z) \in [0, T] \times (-\infty, 0). \end{cases}$$

We need the following Leggett-Williams fixed-point theorem to prove our result.

Let $0 < a < b$ be given and let φ be a nonnegative continuous concave functional on the cone P. Define the convex sets $P_r, \overline{P_r}$ and $P(\varphi, a, b)$ by $P_r = \{y \in P \mid \|y\| < r\}$, $\overline{P_r} = \{y \in P \mid \|y\| \leqslant r\}$ and $P(\varphi, a, b) = \{y \in P \mid a \leqslant \varphi(y), \|y\| \leqslant b\}$.

Lemma 1.6 (Leggett-Williams fixed-point theorem) Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let φ be a nonnegative continuous concave functional on P such that $\varphi(y) \leqslant \|y\|$ for all $y \in \overline{P_c}$. Suppose that there exist $0 < a < b < d \leqslant c$ such that

(i) $\{y \in P(\varphi, b, d) \mid \varphi(y) > b\} \neq 0$ and $\varphi(Ty) > b$ for $y \in P(\varphi, b, d)$;

(ii) $\|Ty\| < a$ for $\|y\| \leqslant a$;

(iii) $\varphi(Ty) > b$ for $y \in P(\varphi, b, d)$ with $\|Ty\| > d$.

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P_c}$ satisfying $\|y_1\| < a, \varphi(y_2) > b, \|y_3\| > a, \varphi(y_3) < b$.

2 Main result

Let $\mathbf{E} = \mathbf{C}_{ld}[0, T]$ with norm $\|u\| = \max\limits_{t \in [0,T]} |u(t)|$. Define a cone $P \subset \mathbf{E}$, by

$$P = \{u \in \mathbf{E} |\ u(t) \geqslant 0, \quad \min_{t \in [0,T]} u(t) \geqslant r \| u \|, \text{on } t \in [0, T]\}.$$

For convenience, we denote

$$h = \frac{T + \beta(T + \eta)}{(T - \alpha\eta) - \beta(T - \eta)} \int_0^T (T - s)\, \nabla s;$$

$$l = \min\Big\{ \frac{\beta\eta}{(T - \alpha\eta) - \beta(T - \eta)} \int_\eta^T (T - s)\, \nabla s, \quad \frac{\alpha\eta}{(T - \alpha\eta) - \beta(T - \eta)} \int_\eta^T (T - s)\, \nabla s \Big\}.$$

Theorem 2.1 Assume that (H_1)–(H_3) hold and there exist constants a, b, c with $LM < a < a + LM < b < rc$, such that

(A_1) $f(t, u) < \dfrac{a}{h} - M$ for $0 \leqslant t \leqslant T, 0 \leqslant u \leqslant a$;

(A_2) $f(t, u) \geqslant \dfrac{b}{l} - M$ for $0 \leqslant t \leqslant T, b - LM \leqslant u \leqslant \dfrac{b}{r}$;

(A_3) $f(t, u) \leqslant \dfrac{c}{h} - M$ for $0 \leqslant t \leqslant T, 0 \leqslant u \leqslant c$.

Then the problem (0.1), (0.2) has at least two positive solutions.

Proof For $v \in P$, define the continuous operator $T : P \to \mathbf{E}$ by

$$Tv(t) = -\int_0^t (t-s)h(s, v-x)\,\nabla s + \frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)h(s, v-x)\,\nabla s$$
$$+ \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s.$$

Then the fixed points of T just are the solutions of the boundary value problem (1.4), (1.5) from Lemma 1.1. From Lemmas 1.2, 1.3, we know that $T(P) \subset P$. It is clear to see that $T : P \to P$ is completely continuous by the Arzela-Ascoli theorem.

In the following, we show that all the conditions of Lemma 1.6 are satisfied. First, let the nonnegative continuous concave functional $\varphi : P \to [0, +\infty)$ be defined by $\varphi(u) = \min\limits_{t\in[0,T]} u(t),\ u \in P$. Evidently, for each $u \in P,\ \varphi(u) \leqslant \|u\|$. Now, if $v \in \overline{P_c}$, then $\|v\| \leqslant c$. When $v(t) \geqslant x(t)$, we have $0 \leqslant v(t) - x(t) \leqslant v(t) \leqslant c$, and thus $h(s, v-x) = f(t, v-x) + M \geqslant 0$. By (A_3), we obtain $h(t, v-x) \leqslant \dfrac{c}{h}$ for $t \in [0, T]$. When $v(t) < x(t)$, we have $v(t) - x(t) < 0$ and then $h(t, v(t)-x(t)) = f(t, 0) + M > 0$. Again by (A_3), we have $h(t, v-x) \leqslant \dfrac{c}{h}$ for $t \in [0, T]$. In one word, if $v \in \overline{P_c}$, we have $h(t, v-x) \leqslant \dfrac{c}{h}$ for $t \in [0, T]$. Consequently,

$$\begin{aligned}
\|Tv(t)\| = \max_{t\in[0,T]} |Tv(t)| = \max_{t\in[0,T]} |&-\int_0^t (t-s)h(s, v-x)\,\nabla s \\
&+ \frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)h(s, v-x)\,\nabla s \\
&+ \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s| \\
\leqslant |&\frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)h(s, v-x)\,\nabla s \\
&+ \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s| \\
\leqslant &\frac{\beta t}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)h(s, v-x)\,\nabla s \\
&+ \frac{t + \beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s \\
\leqslant &\frac{\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta (\eta-s)h(s, v-x)\,\nabla s \\
&+ \frac{T + \beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s \\
\leqslant &\frac{T + \beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)h(s, v-x)\,\nabla s \\
\leqslant &\frac{c}{h}\frac{T + \beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)\,\nabla s = c.
\end{aligned}$$

Hence, $T:\overline{P_c}\to\overline{P_c}$. In the same way, if $v\in\overline{P_a}$, then (A_1) yield $h(t,v-x)\leqslant\dfrac{a}{h}$ for $t\in[0,T]$. As in the same argument, we can obtain that $\overline{P_a}\to P_a$. Therefore, condition (ii) of lemma 1.6 is satisfied.

To check condition (i) of Lemma 1.6, choose $v(t)=\dfrac{b}{r}$, then $v\in P\Big(\varphi,b,\dfrac{b}{r}\Big)$, $\varphi(v)=\dfrac{b}{r}>b$. So $\Big\{v\in P\Big(\varphi,b,\dfrac{b}{r}\Big)\mid\varphi(v)>b\Big\}\neq\emptyset$. Moreover, if $v\in P\Big(\varphi,b,\dfrac{b}{r}\Big)$, then $b\leqslant v(t)\leqslant\dfrac{b}{r},\quad t\in[0,T]$. From Lemma 1.4, $x(t)=M\omega(t)\leqslant MLr\leqslant ar<br$. Thus $b-MLr\leqslant v(t)-x(t)\leqslant v(t)\leqslant\dfrac{b}{r}$. From $(A_2), h(t,v-x)\geqslant\dfrac{b}{l}$ for $t\in[0,T]$. By the definition of φ, there are two cases that either $\varphi(Tv(t))=Tv(0)$ or $\varphi(Tv(t))=Tv(T)$. As the former holds, we have

$$\begin{aligned}
\varphi(Tv)=Tv(0)=&-\frac{\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}(\eta-s)h(s,v-x)\,\nabla s\\
&+\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{T}(T-s)h(s,v-x)\,\nabla s\\
=&\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}Th(s,v-x)\,\nabla s+\frac{\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}sh(s,v-x)\,\nabla s\\
&-\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{T}sh(s,v-x)\,\nabla s\\
>&\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}Th(s,v-x)\,\nabla s-\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}sh(s,v-x)\,\nabla s\\
=&\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}(T-s)h(s,v-x)\,\nabla s\\
\geqslant&\frac{b}{l}\frac{\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}(T-s)\,\nabla s\geqslant b.
\end{aligned}$$

As the later holds, we have

$$\begin{aligned}
\varphi(Tv)=Tv(T)=&-\int_0^{T}(T-s)h(s,v-x)\,\nabla s+\frac{(\beta-\alpha)T-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}(\eta-s)h(s,v-x)\,\nabla s\\
&+\frac{(1-\beta)T+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{T}(T-s)h(s,v-x)\,\nabla s\\
=&\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}Th(s,v-x)\,\nabla s-\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{T}sh(s,v-x)\,\nabla s\\
&+\frac{\alpha T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}sh(s,v-x)\,\nabla s\\
>&\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}Th(s,v-x)\,\nabla s-\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}sh(s,v-x)\,\nabla s\\
=&\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}(T-s)h(s,v-x)\,\nabla s\\
\geqslant&\frac{b}{l}\frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_\eta^{T}(T-s)\,\nabla s\geqslant b.
\end{aligned}$$

So, $\varphi(Tv)>b,\ \forall v\in P(\varphi,b,\frac{b}{r})$. Therefore, condition (i) of Lemma 1.6 is satisfied.

Finally, we show that condition (iii) of Lemma 1.6 also holds. Assume that $v \in P(\varphi, b, c)$ with $\|Tv\| > \dfrac{b}{r}$, then by Lemma 1.3, we have $\varphi(Tv) = \min\limits_{t\in[0,T]} Tv(t) \geqslant r\|Tv\| > b$.

So, all the conditions of Lemma 1.6 are satisfied. Therefore, T has at least three positive fixed points $\|y_1\| < a,\ \varphi(y_2) > b,\ \|y_3\| > a,\ \varphi(y_3) < b$. Further, $u_i = y_i - x_i,\ (i = 1, 2, 3)$ are the solutions of (0.1), (0.2). Moreover,

$$y_2(t) \geqslant r\|y_2\| \geqslant r\varphi(y_2) \geqslant rb > rML \geqslant x(t), \quad t \in [0, T],$$

$$y_3(t) \geqslant r\|y_3\| > ra > rML \geqslant x(t), \quad t \in [0, T].$$

So, $u_2 = y_2 - x, u_3 = y_3 - x$ are the positive solutions of (0.1), (0.2). This completes the proof.

3 Example

Example 3.1 Now, we give a simple example to illustrate the possible application of our main result. Consider three-point BVP

$$u^{\Delta\nabla}(t) + \Big(t + \frac{2}{3}\Big)\varphi(u(t)) - \frac{1}{640} = 0 \quad t \in (0, 1) \tag{3.1}$$

$$u(0) = \frac{1}{3}u\Big(\frac{1}{2}\Big), \quad u(1) = u\Big(\frac{1}{2}\Big), \tag{3.2}$$

where

$$\varphi(u) = \begin{cases} \dfrac{1}{5}u, & 0 \leqslant u \leqslant \dfrac{7}{20}; \\ \dfrac{50\ 169}{41\ 500} - \dfrac{6\ 752}{2\ 075}u, & \dfrac{7}{20} \leqslant u \leqslant \dfrac{33}{1\ 280}; \\ \dfrac{9}{8}, & \dfrac{33}{1280} \leqslant u \leqslant \dfrac{9}{32}; \\ u - \dfrac{11}{15}, & \dfrac{9}{32} \leqslant u \leqslant 1; \\ \dfrac{4}{15}u, & u \geqslant 1. \end{cases}$$

Taking $T = 1,\ \eta = \dfrac{1}{2},\ \alpha = 1,\ \beta = \dfrac{1}{3},\ M = \dfrac{1}{640}$. It is clear, $0 < \alpha\eta = \dfrac{1}{2} < T,\ 0 < \beta(T - \eta) = \dfrac{1}{6} < \dfrac{1}{2} = T - \alpha\eta,\ r = \dfrac{1}{6},\ l = \dfrac{1}{16},\ h = \dfrac{9}{4}\ ML = \dfrac{27}{1\ 280}$.

Choosing $a = \dfrac{7}{20},\ b = \dfrac{3}{64},\ c = 1$, then $ML < a < a + ML < b < rc$. Moreover,

(1) for $t \in [0, 1],\ 0 \leqslant u \leqslant a$, we have

$$f(t, u) = \Big(t + \frac{2}{3}\Big)\varphi(u) - M \leqslant \frac{5}{3} \cdot \frac{1}{5} \cdot \frac{7}{20} - M \leqslant \frac{1}{3}a - M < \frac{4}{9}a - M = \frac{a}{h} - M;$$

(2) for $t \in [0, 1],\ b - ML = \dfrac{33}{1\ 280} \leqslant u \leqslant \dfrac{b}{r} = \dfrac{9}{32}$, we have

$$f(t, u) = \Big(t + \frac{2}{3}\Big)\varphi(u) - M \geqslant \frac{2}{3} \cdot \frac{9}{8} - M = \frac{3}{4} - M \geqslant \frac{b}{l} - M;$$

(3) for $t \in [0,1]$, $0 \leqslant u \leqslant 1$, we have

$$f(t,u) = \Big(t + \frac{2}{3}\Big)\varphi(u) - M \leqslant \frac{5}{3}\Big(1 - \frac{11}{15}\Big) = \frac{4}{9} - M = \frac{c}{h} - M.$$

Hence, all the conditions of Theorem 2.1 are satisfied. It implies that the problems (3.1), (3.2) has at least two positive solutions.

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