Step-stress accelerated degradation test planning based on Wiener process with correlation

Lei He, Rong-Xian Yue & Daojiang He

Taylor & Francis
Taylor & Francis Group

Check for updates

Step-stress accelerated degradation test planning based on Wiener process with correlation

Lei He[a], Rong-Xian Yue[a,b] and Daojiang He[c]

[a]Department of Mathematics, Shanghai Normal University, Shanghai, People's Republic of China; [b]Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, People's Republic of China; [c]Department of Statistics, Anhui Normal University, Wuhu, People's Republic of China

ABSTRACT

To assess the lifetime distribution of highly reliable or expensive product, one of the most commonly used strategies is to construct step-stress accelerated degradation test (SSADT) which can curtail the test duration and reduce the test cost. In reality, it is not unusual for a unit with a higher degradation rate which exhibits a more volatile degradation path. Recently, Ye, Chen, and Shen [(2015). A new class of Wiener process models for degradation analysis. *Reliability Engineering and System Safety*, *139*, 58–67] proposed a Wiener process to capture the positive correlation between the drift rate and the volatility. In this paper, an optimal SSADT plan is developed under the assumption that the underlying degradation path follows the Wiener process with correlation. Firstly, the stochastic diffusion process is introduced to model a typical SSADT problem. Then the design variables, including the sample size, the measurement frequency and the numbers of measurements under each stress level, are optimised by minimising the asymptotic variance of the estimated p-percentile of the product's lifetime distribution subject to the total experimental cost not exceeding a pre-specified budget. Finally, a numerical example is presented to illustrate the proposed method.

1. Introduction

With the rapid development of modern technology, the reliability of newly designed products has been greatly improved. Accordingly, reliability assessment has become a critical and challenging task for such highly reliable products. The traditional accelerated life test is a commonly used approach that records only time-to-failure data under harsher conditions (such as higher temperature, voltage, pressure and vibration). For highly reliable products, however, a failure phenomenon is difficult to appear during a short period of time. In such restraints, if there are some quality characteristics of the product, which can reflect the variety of product properties and can be easily observed within the product degradation process. Then collecting these degradation data at higher stress level is an alternative approach to predict the product's lifetime information under normal operating conditions. Such an experiment is called as an accelerated degradation test (ADT). In this respect, it is worth mentioning the monographs by Nelson (1990) and Meeker and Escobar (1998).

Research on statistical inference for data from ADT or the optimal design of an ADT experiment has been done by numerous authors. Among these, a famous degradation model is the Wiener process with positive drifts, which means the degradation increment

is independent and normally distributed due to the law of large numbers. Doksum and Normand (1995) used the Wiener process to describe a biomarker series, and they assumed that the drifts are the function of stress levels (also called covariates). Similar assumptions to incorporate covariates in the Wiener process can be found in Park and Padgett (2006), Pan and Balakrishnan (2010), Lim and Yum (2011), Lim (2012) and Guan et al. (2015). In addition, it is to be remarked that the Gamma process and the inverse Gaussian process have also wide applications in degradation data analysis. See Ye and Chen (2014), Ye et al. (2014), Guida et al. (2018) and among others, for some recent references.

According to different stress loadings, ADTs can be classified into constant-stress ADT (CSADT), step-stress ADT (SSADT) and progressive-stress ADT . Here, CSADT is the most popular method in practical applications. However, it usually needs a number of products to proceed the experiment. For instance, we consider a 3-level CSADT with 25 test units allocated at each stress level, then 3 accelerated loading devices and 75 test units are needed to complete the experiment. In general, it may not be possible to obtain so many test units because of the constraint of the manufacturing costs. To overcome this problem, Tseng and Wen (2000) proposed the SSADT, which is a special type of stress

loading in which all units are tested together and the stress level is increased step by step until the experiment is completed. Clearly, the advantage of the SSADT is that only a few test units are needed to collect degradation data.

It is well known that a careful choice of design can improve the quality of statistical analysis substantially, and therefore, the optimal design problem of an SSADT experiment received considerable attention in the literature. For example, Liao and Tseng (2006) designed an optimal SSADT by minimising the variance of the estimated p-percentile subject to constraint on the total cost. Tseng et al. (2009) provided an optimal SSADT plan by minimising the asymptotic variance of the estimated mean-time-to-failure (MTTF) under the same cost constraint. A Bayesian methodology for designing SSADT has been presented by Li et al. (2015) using Kullback–Leibler divergence as the optimality criterion. For more recent developments on this topic, one may refer to Pan and Sun (2014), Hamada (2015), Hu et al. (2015), Sung and Yum (2016), Wang et al. (2016) and Kim and Sung (2017).

For many products, it is not uncommon to see that a unit with a higher degradation rate would possess a larger degradation variation (see, e.g. Ye et al., 2015). In general, the Gamma process model and the inverse Gaussian process model can be used to show the positive correlation between the drift rate and the volatility. However, the existing Wiener process models fail to capture such an important relation. In order to fill the gap, Ye et al. (2015) proposed a new class of Wiener process to describe the degradation path. The degradation characteristic $Y(t)$ with $Y(0) = 0$ at time t can be expressed as

$$M_0 : Y(t) = \eta \Lambda(t) + \sigma \eta \mathcal{B}(\Lambda(t)), \qquad (1)$$

where η is the drift parameter reflecting the effect of stress level in the ADT model, which determines the degradation rate of the products, $\Lambda(t)$ is a monotone increasing function with $\Lambda(0) = 0$, σ is an unknown parameter and $\mathcal{B}(\cdot)$ is a standard Brownian motion. It is easily seen that model (1) is slightly different from the typical Wiener process (see Whitmore and Schenkelberg, 1997) given by

$$M_1 : Y(t) = \eta \Lambda(t) + \sigma_B \mathcal{B}(\Lambda(t)), \qquad (2)$$

and implies the positive correlation by setting the volatility parameter σ_B as $\sigma \eta$.

To the best of our knowledge, no research focuses on the modelling and design problems of the SSADT for the Wiener process with correlation. The purpose of the present paper is to extend the recent result of Ye et al. (2015) to model a typical SSADT problem using the Wiener process model (1), and an optimal SSADT plan for the degradation data based on Wiener process model (1) is developed. Several design variables, including the sample size, the measurement frequency and the numbers of measurements under each stress level, are to be determined by minimising the asymptotic variance of the estimated p-percentile of the product's lifetime distribution under the total cost constraints.

The rest of this paper is structured as follows. In Section 2, the SSADT model based on the Wiener process with correlation is described and several assumptions are given. In Section 3, the optimisation problem for the SSADT is constructed and the detailed algorithm is developed to solve this problem. In Section 4, a pilot study is carried out to show the performance of the proposed method. Some concluding remarks are given in Section 5.

2. The stochastic SSADT model

Suppose that the degradation follows Wiener process model (1), and the time-to-failure is defined as the moment that the corresponding degradation process $Y(t)$ first reaches a pre-specified failure threshold $\omega > 0$, say τ, then

$$\tau = \inf\{t \geq 0 \mid Y(t) \geq \omega\}.$$

It is readily shown that $\Lambda(\tau)$ conforms to the inverse Gaussian distribution. But the p-quantile of the inverse Gaussian distribution does not have an analytic form. By Padgett and Tomlinson (2004) and Ye et al. (2015), when the lifetime is large enough the approximate form of the p-quantile of τ can be expressed as

$$\xi_p = \Lambda^{-1}\left(\tfrac{1}{4}\left(z_p \sigma + \sqrt{z_p^2 \sigma^2 + 4\omega/\eta}\right)^2\right). \qquad (3)$$

To assess the reliability of the product, we want to design an efficient SSADT experiment under some constraints, so that the p-quantile of the lifetime distribution of the product can be predicted as precisely as possible. Let $S_1, S_2, \ldots, S_m$ denote m higher accelerating stress levels such that

$$S_0 < S_1 < \cdots < S_m,$$

where S_0 denotes the normal usage stress level. Then an SSADT experiment can be expressed as follows: firstly, we assume that there are n units available for a degradation test that uses stress level S_1, and the measurement frequency and the number of measurements under the stress S_1 are f and l_1, respectively. Thus, the duration time under the stress S_1 is $t_1 = l_1 f$. Then we increase the stress level to S_2, and the duration time is up to $t_2 = l_2 f$. Continue the process until the stress is up to S_m, and the experiment is terminated at $t_m = l_m f$. Consequently, the testing stress S of an SSADT experiment

can be described as

$$S = \begin{cases} S_1, & 0 \le t < t_1, \\ S_2, & t_1 \le t < t_2, \\ \vdots & \vdots \\ S_m, & t_{m-1} \le t < t_m. \end{cases}$$

It is well known that the accelerating stress S can affect the degradation process of the product. Given S_i, then we have

$$Y(t \mid S_i) = \eta_i \Lambda(t) + \sigma \eta_i \mathcal{B}(\Lambda(t)), \quad i = 0, 1, \ldots, m. \tag{4}$$

Furthermore, the following assumptions $A1$ and $A2$ will be considered in this paper.

$A1$: The degradation $Y(t \mid S_i)$ under each stress S_i follows a normal distribution with mean $\eta_i \Lambda(t)$ and variance $\sigma^2 \eta_i^2 \Lambda(t)$. For simplicity, it is assumed that $\Lambda(t) = t^\beta$, where $\beta > 0$ is a constant.

$A2$: The relationship between η_i and the stress level S_i, $i = 0, 1, \ldots, m$, is log-linear. Then, after standardising the stress levels, we have

$$\log(\eta_i) = a + bs_i, \quad i = 0, 1, \ldots, m, \tag{5}$$

where a and $b > 0$ are unknown parameters to be estimated, and $s_i \in [0, 1]$. Usually, the typical inverse power law relation, the Arrhenius relation and the exponential relation are included in the assumption $A2$. For more details, see Xu and Tang (2015) and Ye et al. (2014).

Let $Y_{ss}(t)$ be the degradation path of an SSADT based on Wiener process model (1), and v_i, $i = 1, 2, \ldots, m$, the equivalent starting time of the degradation process under stress level S_i with $v_1 = 0$. Then, the relationship between $Y_{ss}(t)$ and $Y(t \mid S_i)$, $i = 1, 2, \ldots, m$, can be expressed as

$$Y_{ss}(t) = \begin{cases} Y(t \mid S_1), & 0 \le t < t_1, \\ Y(v_2 + t - t_1 \mid S_2), & t_1 \le t < t_2, \\ \vdots & \vdots \\ Y(v_m + t - t_{m-1} \mid S_m), & t_{m-1} \le t < t_m. \end{cases}$$

It is easy to see that the equivalent degradation time v_i under stress S_i contains the accumulated amount of degradation from $t = 0$ to the end of degradation test under stress S_{i-1}, which satisfies

$$Y(v_i \mid S_i) = Y(v_{i-1} + t_{i-1} - t_{i-2} \mid S_{i-1}),$$
$$i = 2, 3, \ldots, m.$$

According to Equations (4) and (5), we have

$$e^{a+bs_i} v_i^\beta = e^{a+bs_{i-1}} (v_{i-1} + t_{i-1} - t_{i-2})^\beta,$$
$$i = 2, 3, \ldots, m, \tag{6}$$

which yields

$$v_i = e^{-b(s_i - s_{i-1})/\beta} (v_{i-1} + t_{i-1} - t_{i-2}),$$
$$i = 2, 3, \ldots, m. \tag{7}$$

It follows from Equation (7) that

$$\begin{aligned} v_i &= t_1 e^{-b(s_i - s_1)/\beta} + (t_2 - t_1) e^{-b(s_i - s_2)/\beta} \\ &\quad + \cdots + (t_{i-1} - t_{i-2}) e^{-b(s_i - s_{i-1})/\beta} \\ &= \sum_{\kappa=1}^{i-1} (t_\kappa - t_{\kappa-1}) e^{-b(s_i - s_\kappa)/\beta}, \quad i = 2, 3, \ldots, m. \end{aligned} \tag{8}$$

Thus, the distribution of the degradation $Y_{ss}(t)$ at time $t_{i-1} \le t < t_i$ can be written in the following unified form:

$$\begin{aligned} Y_{ss}(t) &= Y(v_i + t - t_{i-1} \mid S_i) \\ &\sim N(e^{a+bs_i} \rho_i^\beta, \sigma^2 e^{2a+2bs_i} \rho_i^\beta), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \rho_i &= v_i + t - t_{i-1} = t - t_{i-1} \\ &\quad + \sum_{\kappa=1}^{i-1} (t_\kappa - t_{\kappa-1}) e^{-b(s_i - s_\kappa)/\beta}, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{10}$$

3. Optimal test plan

Based on the SSADT modelling above, we now consider the optimisation problem of determining the sample size n, the measurement frequency f and the measurement times l_i within the stress level S_i, $i = 1, 2, \ldots, m$, by minimising the asymptotic variance of the estimated ξ_p under the usage stress S_0 subject to a pre-specified total cost. Therefore, the framework for accomplishing the optimisation problem includes the following three main parts: (i) the estimation of the model parameters $\theta = (a, b, \sigma^2, \beta)$ by using the maximum-likelihood estimator (MLE) method; (ii) the computation of the asymptotic variance of $\widehat{\xi}_p$ under the usage stress S_0 and (iii) the definition of the total cost function.

3.1. Parameters estimation

Let $\{Y_{ss}^{(k)}(t_j)\}$ be a set of observed SSADT paths from Wiener process model (1), where $1 \le k \le n$, $t_{i-1} \le t_j < t_i$, $1 \le i \le m$, $\zeta_{i-1} + 1 \le j \le \zeta_i$ and $\zeta_i = l_1 + \cdots + l_i$ with $\zeta_0 = 0$. According to Equation (9), we obtain that

$$\begin{aligned} Y_{ss}^{(k)}(t_j) &= Y^{(k)}(v_i + t_j - t_{i-1} \mid S_i) \\ &\sim N(e^{a+bs_i} \rho_{ij}^\beta, \sigma^2 e^{2a+2bs_i} \rho_{ij}^\beta), \end{aligned}$$

where

$$\rho_{ij} = v_i + t_j - t_{i-1} = t_j - t_{i-1}$$
$$+ \sum_{\kappa=1}^{i-1} (t_\kappa - t_{\kappa-1}) \, e^{-b(s_i - s_\kappa)/\beta}. \quad (11)$$

Because the degradation of the product follows Wiener process model (1) under the different stress S_i, then each independent increment $y_{ijk} = Y_{ss}^{(k)}(t_j) - Y_{ss}^{(k)}(t_{j-1}) = Y^{(k)}(\rho_{ij} \mid S_i) - Y^{(k)}(\rho_{i,j-1} \mid S_i)$ conforms to the normal distribution, which is

$$y_{ijk} \sim N(e^{a+bs_i}\lambda_{ij}, \sigma^2 \, e^{2a+2bs_i}\lambda_{ij}),$$

where $\lambda_{ij} = \rho_{ij}^\beta - \rho_{i,j-1}^\beta$, Moreover, y_{ijk}'s are mutually independent. Thus, the likelihood function of θ is given by

$$L(\theta) = \prod_{k=1}^n \prod_{i=1}^m \prod_{j=\zeta_{i-1}+1}^{\zeta_i} \frac{1}{\sqrt{2\pi\sigma^2 \, e^{2a+2bs_i}\lambda_{ij}}}$$
$$\times \exp\left\{ -\frac{(y_{ijk} - e^{a+bs_i}\lambda_{ij})^2}{2\sigma^2 \, e^{2a+2bs_i}\lambda_{ij}} \right\}. \quad (12)$$

Then, the MLE $\hat\theta = (\hat a, \hat b, \hat\sigma^2, \hat\beta)$ of θ can be obtained by a numerical optimisation approach after logarithm transformation. Substituting the MLE into Equations (3) and (5), we can obtain the estimator $\widehat{\xi_p}$ of ξ_p under the normal operating stress S_0.

3.2. Computation of AVar($\widehat{\xi_p}$)

Based on the assumption $A2$ in Section 2, the approximated p-quantile of the lifetime distribution of the product under the usage stress S_0 can be readily obtained, i.e.,

$$\xi_p = \left[\frac{1}{2} \left(z_p\sigma + \sqrt{z_p^2\sigma^2 + 4\omega \, e^{-a}} \right) \right]^{2/\beta}. \quad (13)$$

By the delta method, the asymptotic variance of $\widehat{\xi_p}$ can be expressed as

$$\text{AVar}(\widehat{\xi_p}) = (\nabla\xi_p)^{\mathrm{T}} I^{-1}(\theta)(\nabla\xi_p),$$

where $\nabla\xi_p$ is the gradient vector of ξ_p with respect to $\theta = (a, b, \sigma^2, \beta)$, i.e.,

$$\nabla\xi_p = \left(\frac{\partial\xi_p}{\partial a}, \frac{\partial\xi_p}{\partial b}, \frac{\partial\xi_p}{\partial\sigma^2}, \frac{\partial\xi_p}{\partial\beta} \right)^{\mathrm{T}}, \quad (14)$$

and $I(\theta)$ is the Fisher information matrix of θ, i.e.,

$$I(\theta) = \begin{pmatrix} \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial a^2}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial a\partial b}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial b\partial a}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial b^2}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\sigma^2\partial a}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\sigma^2\partial b}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\beta\partial a}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\beta\partial b}\right) \end{pmatrix}$$
$$\times \begin{pmatrix} \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial a\partial\sigma^2}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial a\partial\beta}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial b\partial\sigma^2}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial b\partial\beta}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial(\sigma^2)^2}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\sigma^2\partial\beta}\right) \\ \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\beta\partial\sigma^2}\right) & \mathrm{E}\left(-\frac{\partial^2\mathcal{L}(\theta)}{\partial\beta^2}\right) \end{pmatrix},$$

where $\mathcal{L}(\theta)$ denotes the log-likelihood function of θ, up to a constant, which is given by

$$\mathcal{L}(\theta) = \sum_{k=1}^n \sum_{i=1}^m \sum_{j=\zeta_{i-1}+1}^{\zeta_i} \left[-\frac{1}{2}\log(\sigma^2) - a - bs_i \right.$$
$$\left. -\frac{1}{2}\log(\lambda_{ij}) - \frac{(y_{ijk} - e^{a+bs_i}\lambda_{ij})^2}{2\sigma^2 \, e^{2a+2bs_i}\lambda_{ij}} \right].$$

The detailed expressions for all the elements of $\nabla\xi_p$ and $I(\theta)$ are listed in the appendix.

3.3. The cost function

Similar to Liao and Tseng (2006), we define the total cost of conducting the SSADT experiment as

$$TC(n, f, l_1, \ldots, l_m) = C_{op}f \sum_{i=1}^m l_i$$
$$+ C_{mea}n \sum_{i=1}^m l_i + C_{it}n, \quad (15)$$

which includes three principal forms of the experimental cost as follows: (i) the cost of conducting the experiment $C_{op}f \sum_{i=1}^m l_i$, where C_{op} denotes the unit cost of operation per time; (ii) the cost of measurement $C_{mea}n \sum_{i=1}^m l_i$, where C_{mea} denotes the unit cost of measurement and (iii) the cost of testing the devices $C_{it}n$, where C_{it} denotes the unit cost for each test device.

3.4. Optimisation model and algorithm

Let C_b denotes the total experimental budget, then the constraint in the SSADT plan is that the total experimental cost TC does not exceed the pre-specified budget C_b. Therefore, the optimisation problem can be written as

Minimise: $\text{AVar}(\widehat{\xi}_p \mid n, f, l_1, \ldots, l_m)$,
Subject to:

$$C_{op}f \sum_{i=1}^{m} l_i + C_{mea}n \sum_{i=1}^{m} l_i + C_{it}n \leqslant C_b,$$

where $n, f, l_1, \ldots, l_m \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

In fact, due to the complexity of the objective function, the solution of the above optimisation problem does not have an analytic expression. However, with the simplicity in the structure of the constraint, and the integer restriction on the decision variables, the optimal solution $(n^*, f^*, l_1^*, \ldots, l_m^*)$ can be obtained easily by using the following algorithm.

Step 1: Set $n_{\max} = \lfloor (C_b - mC_{op})/(mC_{mea} + C_{it}) \rfloor$, where $\lfloor x \rfloor$ is the largest integer that is not greater than x, and $n_{\max}$ is the largest possible number for n when $f = 1$, and $l_i = 1$ for $1 \leqslant i \leqslant m$.

Step 2: Let $n = 1$.

Step 3: Set $f_{max} = \lfloor (C_b - mnC_{mea} - nC_{it})/ (mC_{op}) \rfloor$, where f_{max} is the largest possible number for f when $l_i = 1$ for $1 \leqslant i \leqslant m$, under fixed n.

Step 4: Let $f = 1$.

Step 5: Find $l_1, \ldots, l_m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} l_i \leqslant \frac{C_b - nC_{it}}{fC_{op} + nC_{mea}}.$$

Step 6: Compute $\text{AVar}(\widehat{\xi}_p \mid n, f, l_1, \ldots, l_m)$ by $(n, f, l_1, \ldots, l_m)$.

Step 7: Set $f = f+1$, and repeat steps 5 and 6 until $f = f_{max}$.

Step 8: Set $n = n+1$, and repeat steps 3 and 7 until $n = n_{max}$.

Step 9: The optimal solution $(n^*, f^*, l_1^*, \ldots, l_m^*)$ can then be obtained by minimising $\text{AVar}(\widehat{\xi}_p \mid n, f, l_1, \ldots, l_m)$.

It should be pointed out that the asymptotic variance $\text{AVar}(\widehat{\xi}_p \mid n, f, l_1, \ldots, l_m)$ is the function of the parameters θ given the design variables $(n, f, l_1, \ldots, l_m)$. Therefore, it is necessary to give an initial value of the parameter in order to determine the optimal plan. In practical applications, initial values could be obtained from the previous experience or engineering judgement. For illustrative purpose, we adopt a pilot study to evaluate the performance of the proposed SSADT optimal plan in the next section.

4. Illustrative examples

In this section, we illustrate the proposed procedure with the degradation data of light-emitting diodes (LEDs) in Hamada et al. (2008, p. 290). In the experiment, 25 LEDs were observed at 25°C, 65°C and 105°C, respectively, and the luminosity data of each LED product were collected at 29 inspection times. It was assumed that the standard operating temperature was 20°C. An LED fails when the LED relative luminosity drops to 0.5, i.e., 50% of initial luminosity. In order to capture the degradation path of the LED products by the Wiener process with correlation, a logarithm transformation for degradation data is considered, then the threshold value ω is equivalent to $-\log(0.5)$. Furthermore, let $\Lambda(t) = t^\beta$, and the Arrhenius relationship is assumed between the parameter η and the temperature.

To assess the goodness of fit between the Wiener process with correlation M_0 and the general Wiener process M_1 for the data from the CSADT, the AIC is employed, where the AIC is defined to be $\text{AIC} = 2k - 2\log(\ell)$ with k the number of parameters and ℓ the maximised value of the likelihood function of the model. Then the MLEs of the model parameters and the corresponding values of the AIC are calculated and given in Table 1. From Table 1, we can see that the degradation model based on the Wiener process with correlation fits the data better.

In the following, we use the MLEs of the parameters in model M_0 as the true values, then a pilot study is developed, and a corresponding SSADT experiment is arranged to estimate the model parameters. By applying the Monte Carlo method, a set of SSADT data are generated based on two stress levels $S_1 = 25°C$ and $S_2 = 65°C$. Also we set $n_0 = 30$, $f_0 = 10$, unit time is 4 (in hours), and the numbers of measurements for S_1 and S_2 are $l_1^0 = 100$ and $l_2^0 = 100$, respectively. Then, the MLE of θ can be obtained directly from Equation (12), which is

$$\hat{\theta} = (\hat{a}, \hat{b}, \hat{\sigma}^2, \hat{\beta})$$
$$= (-12.9156, 1.3097, 12764.23, 1.4906).$$

It can be observed that the estimation $\hat{\theta}$ and the true values are nearly the same. Now, we use the estimation $\hat{\theta}$ to compute the asymptotic variance of $\widehat{\xi}_p$ in the optimisation problem accordingly.

For illustrative purposes, we assume that the cost configurations of C_{op}, C_{mea} and C_{it} are

$$C_{op} = 1.6 \text{ dollars/unit time},$$
$$C_{mea} = 0.8 \text{ dollars/measurement}$$
$$\text{and } C_{it} = 85 \text{ dollars/unit},$$

respectively. Under a pre-specified budget C_b, an optimal SSADT plan with m higher stress levels can be

Table 1. MLEs and AIC of two different Wiener process models for the LEDs' data.

	a	b	σ^2	σ_B^2	β	$\log(\ell)$	AIC
M_0	−12.9173	1.3035	12,819.21	–	1.4916	1594.93	−3181.87
M_1	−13.3374	2.3262	–	7.9463×10^{-7}	1.4445	1502.81	−2997.64

Table 2. Optimal two-level SSADT plans under different C_b.

C_b	n^*	f^*	l_1^*	l_2^*	$std(\hat{\xi}_{0.1})$	Total cost
1000	4	10	19	15	699.49	992.8
1500	6	11	25	19	435.58	1495.6
2000	8	11	31	24	324.72	2000.0
2500	10	13	32	25	261.53	2491.6
3000	12	13	36	29	219.20	2996.0

Table 3. Optimal three-level SSADT plans under different C_b.

C_b	n^*	f^*	l_1^*	l_2^*	l_3^*	$std(\hat{\xi}_{0.1})$	Total cost
1000	4	10	21	1	12	648.63	992.8
1500	6	12	26	1	14	417.92	1494
2000	8	11	34	1	14	310.55	2000
2500	10	16	30	1	18	246.76	2496.4
3000	11	16	37	1	22	205.86	2999

determined by using the algorithm given in Section 3. In the following, we will mainly discuss the optimal design for the cases of $m = 2$ and $m = 3$.

4.1. Optimal SSADT plan for $m = 2$

Suppose that two higher stress levels are set as $S_1 = 25°C$ and $S_2 = 65°C$. Then the design variables are (n, f, l_1, l_2), which can be determined under different budget constraint C_b. The results are summarized in Table 2. For instance, the optimal test plan is $(n^*, f^*, l_1^*, l_2^*) = (6, 11, 25, 19)$ when $C_b = 1500$, which means that the optimal sample size is 6, and the optimal measurement frequency is $11 \times 4 = 44$ hours. The corresponding optimal numbers of measurements for $25°C$ and $65°C$ are 25 and 19, respectively. Under such a test plan, the total cost is \$1495.6, and the corresponding approximate standard deviation of $\hat{\xi}_{0.1}$ is 435.58.

According to Table 2, it can be seen that the optimal sample size, the optimal measurement frequency and the total measurement times are increasing when the budget C_b increases. However, the approximate standard deviation of $\hat{\xi}_{0.1}$ is gradually decreasing with the budget C_b. The results indicate that the precision of the estimated $\hat{\xi}_{0.1}$ becomes higher with the increase of the total budget C_b.

4.2. Optimal SSADT plan for $m = 3$

In this subsection, we consider a three-level SSADT plan and set the three levels as $S_1 = 25$, $S_2 = 65$, $S_3 = 105$, where S_3 is the highest allowable stress condition. Also suppose that the true value of the model parameters is $\hat{\theta}$, and $(C_{op}, C_{mea}, C_{it}) = (1.6, 0.8, 85)$. Then, the optimal design variables (n, f, l_1, l_2, l_3), under various constraints of pre-fixed budget C_b, can be determined by solving the optimisation problem addressed in Section 3. The optimal test plans are shown in Table 3.

In this case, we also find that the optimal sample size, the optimal measurement frequency and the total measurement times increase with the budget C_b, and the

approximate standard deviation of $\hat{\xi}_{0.1}$ decreases with C_b. However, it should be noted that only one measurement time is assigned to the second stress level for each budget constraint. That is, the optimal experiment is mainly determined by the lowest and highest stress level. Comparing the results in Tables 2 and 3, we can see that the test plans with $m = 2$ and $m = 3$ are nearly the same in terms of the sample size, the measurement frequency and the total experiment cost, while the standard deviation of $\hat{\xi}_{0.1}$ in the test plans with $m = 2$ is slightly larger than the case of $m = 3$. Under the allowed loss of the estimated precision of the parameters, we recommend using the SSADT plans with $m = 2$ in practical application, since it is convenient and effective compared with the three-stress SSADT plans.

4.3. Sensitive analysis for the parameters and stress levels

In practice, the unknown parameters (a, b, σ^2, β) could not be estimated precisely in the pilot study. Thus, it is of great importance to investigate the effect of the estimated error on the optimal test plan. Let ϵ_1, ϵ_2, ϵ_3 and ϵ_4 denote the predicted errors of the parameters a, b, σ^2 and β, respectively. Then, by setting the same cost configuration $(C_{op}, C_{mea}, C_{it}, C_b) = (1.6, 0.8, 85, 1500)$, the optimal plan, the corresponding approximate standard deviation and the estimated $\hat{\xi}_{0.1}$ can be obtained under various combinations of $((1 + \epsilon_1)a, (1 + \epsilon_2)b, (1 + \epsilon_3)\sigma^2, (1 + \epsilon_4)\beta)$, which are listed in Table 4. From these results, it is easy to see

Table 4. Optimal plans under various combination of $((1 + \epsilon_1)a, (1 + \epsilon_2)b, (1 + \epsilon_3)\sigma^2, (1 + \epsilon_4)\beta)$.

ϵ_1	ϵ_2	ϵ_3	ϵ_4	n^*	f^*	l_1^*	l_2^*	$std(\hat{\xi}_{0.1})$	$\hat{\xi}_{0.1}$
4%	2%	8%	2%	6	11	26	18	510.42	4674.32
4%	0	0	0	6	11	26	18	653.51	5565.53
4%	−2%	−8%	−2%	6	12	24	17	842.74	6675.59
0	2%	0	−2%	6	11	24	20	568.85	4467.13
0	0	−8%	2%	6	11	25	19	331.53	3236.26
0	−2%	8%	0	6	11	24	20	442.69	3749.21
−4%	2%	−8%	0	6	8	29	27	282.59	2555.69
−4%	0	8%	−2%	6	11	22	22	372.93	2942.87
−4%	−2%	0	2%	6	8	30	26	224.02	2170.88
0	0	0	0	6	11	25	19	435.58	3775.96

Table 5. Optimal plans under various combination of S_1 and S_2 with $C_b = 1500$.

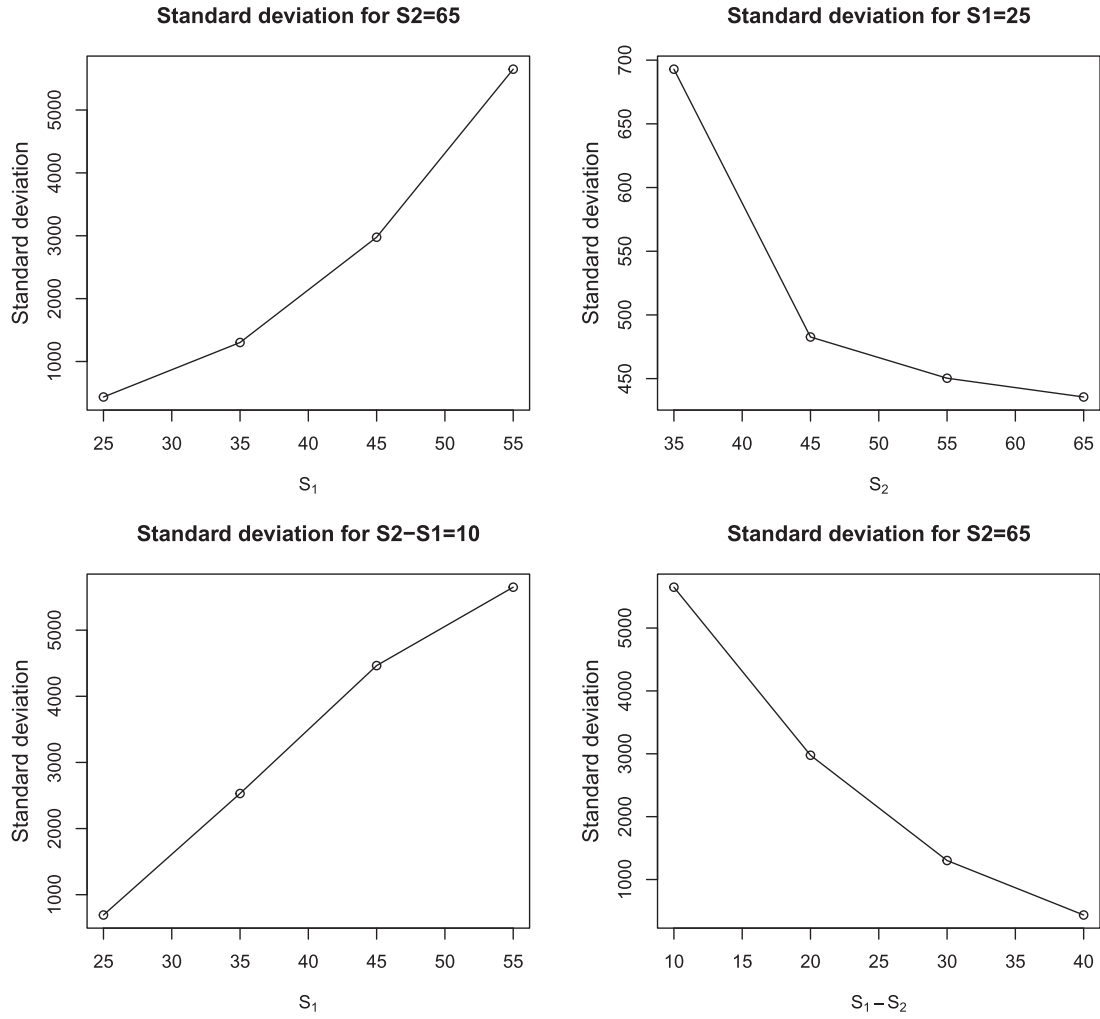| S_1	S_2	n^*	f^*	l_1^*	l_2^*	$std(\hat{\xi}_{0.1}) \times 10^2$
25°C	65°C	6	11	25	19	435.58
35°C	65°C	4	1	16	225	1302.41
45°C	65°C	5	1	8	183	2976.79
55°C	65°C	14	1	3	21	5649.51
25°C	55°C	6	11	23	21	450.32
35°C	55°C	4	1	13	228	1618.30
45°C	55°C	7	1	5	120	4464.34
25°C	45°C	6	8	24	32	482.67
35°C	45°C	5	1	9	182	2530.13
25°C	35°C	5	6	27	52	692.90

Figure 1. Plots of the trends revealed by Tabel 5.

that the optimal test plan $(n^*, f^*, l_1^* l_2^*)$ tends to be robust when the bias is not too large.

On the other hand, we find that the optimal design problem is also dependent on the choice of the stress levels. Hence, it is equally vital to investigate the sensitivity of the stress level selection for the test plan. Taking the same constraint settings above, the results of the optimal test plans under various combinations of stress levels can be obtained by solving the optimisation problem in Section 3. Then, some of the points are quite clear from Table 5:

- For the fixed S_2, the estimated precision of $\hat{\xi}_{0.1}$ becomes lower when the stress level S_1 increases, since the approximate standard deviation of $\hat{\xi}_{0.1}$ gets the maximal value at $S_1 = 55°C$. Instead, the estimated precision of $\hat{\xi}_{0.1}$ increases with S_2 when the stress level S_1 is pre-fixed, since the approximate standard deviation of $\hat{\xi}_{0.1}$ is decreasing.
- The approximate standard deviation of $\hat{\xi}_{0.1}$ is larger in the case of increasing S_1 when the range of S_2-S_1 is fixed. But if the range of S_2-S_1 is flexible, the approximate standard deviation of $\hat{\xi}_{0.1}$ decreases with the various cases of $S_2 - S_1$.

Note that these two findings can also be displayed clearly in Figure 1. Based on the sensitivity analysis for the stress levels S_1 and S_2, it can be seen that the reasonable choice of stress level is of critical importance for the optimal test plan. Since the stress level selection has substantial effect on the estimated precision of $\hat{\xi}_{0.1}$. So in order to make the test more effective, we recommend extending the difference between the stress levels S_1 and S_2 in practice.

4.4. Stability of the SSADT plan

To assess the stability of the optimal test plan (such as, $(n^*, f^*, l_1^*, l_2^*) = (12, 13, 36, 29)$) under the cost configuration $(C_{op}, C_{mea}, C_{it}, C_b) = (1.6, 0.8, 85, 3000)$, a simulation experiment is constructed to compare the case of varying (l_1, l_2) in terms of the average of the MLEs and the root-mean-squared errors (RMSE). Set $\hat{\theta}$ as the true parameters of the SSADT model. By prefixing $(n, f) = (12, 13)$, 1000 sets of SSADT degradation data are generated for each combination under the restriction $l_1 + l_2 = 65$. The simulated results are listed in Table 6. From Table 6, it is seen that the proposed

Table 6. A stability analysis for the optimal SSADT plan with $C_b = 3000$.

(n, f, l_1, l_2)	$\hat{a}$	$\hat{b}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\xi}_{0.1}$	TS[a]
(12, 13, 30, 35)	−12.9266	1.3080	123.04	1.4915	3804.08	
	(0.5194)[b]	(0.1208)	(37.19)	(0.0697)	(300.66)	224.28
(12, 13, 33, 32)	−12.9571	1.3073	120.40	1.4958	3793.05	
	(0.5237)	(0.1187)	(38.20)	(0.0702)	(295.78)	220.58
(12, 13, 36, 29)	−12.9367	1.3065	118.57	1.4916	3795.13	
	(0.4810)	(0.1125)	(33.34)	(0.0637)	(263.50)	219.20
(12, 13, 40, 25)	−12.9507	1.3103	118.67	1.4951	3793.89	
	(0.4617)	(0.1079)	(32.24)	(0.0605)	(259.48)	222.90
(12, 13, 42, 23)	−12.9534	1.3089	120.20	1.4953	3800.73	
	(0.4668)	(0.1112)	(36.43)	(0.0601)	(259.36)	223.39
True value	−12.9156	1.3097	112.98	1.4906	3775.96	

[a] TS denotes the theoretical standard deviation of $\hat{\xi}_{0.1}$.
[b] The value in parentheses is the corresponding RMSE of the estimated parameter.

optimal test plan yields the smallest RMSE of $\hat{\xi}_{0.1}$ that is near to the theoretical result by altering the test plan.

5. Concluding remarks

In this paper, we propose an optimal SSADT plan for the Wiener degradation process with correlation. By using the criterion of minimising the asymptotic variance of the estimated p-quantile of the product's lifetime distribution under the usage stress S_0 subject to the total cost not exceeding a pre-specified budget, several decision variables, including the sample size, the measurement frequency and the numbers of measurements under each stress level, are determined. An optimisation algorithm is proposed to derive the optimal test plan. Finally, a pilot study is conducted to evaluate the performance of the proposed SSADT optimal plan. Optimal two-level SSADT plans and optimal three-level SSADT plans are discussed under the same cost configuration. Two sensitivity analyses for the estimated parameters and stress levels are also considered respectively. And a Monte Carlo simulation method is employed to study the stability of the proposed optimal test plan. From the numerical results, some concluding remarks are addressed as follows:

- A two-level SSADT plan is recommended in practical applications, since it is easy to implement and has similar efficiency compared with the three-level SSADT plan.
- The sensitivity analysis for the parameters reveals that the optimal test plan is quite robust to moderate departures of the real values of the model parameters.
- The sensitivity analysis for the stress levels shows that extending the difference between the low stress and the high stress in the allowable range can improve the effectiveness of the test.
- The simulation study indicates that the optimal test plan is stable, since most of the simulation results are close to the theoretical results.

This study has considered taking the asymptotic variance of the estimated p-percentile of the product's lifetime distribution as the objective function of the optimisation problem. In fact, other objective functions, such as minimising the approximate variance of the estimated MTTF, can also be considered. And similar results can be obtained by using the same optimisation algorithm.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes on contributors

Lei He is a Ph.D candidate in the Department of Mathematics at Shanghai Normal University.

Rong-Xian Yue is a professor in Shanghai Normal University.

Daojiang He is a professor in Anhui Normal University.

References

Doksum, K. A., & Normand, S. L. T. (1995). Gaussian models for degradation processes – Part I: Methods for the analysis of biomarker data. *Lifetime Data Analysis*, *1*(2), 131–144.

Guan, Q., Tang, Y. C., & Xu, A. C. (2015). Objective Bayesian analysis accelerated degradation test based on Wiener process models. *Applied Mathematical Modelling*, *40*(4), 2743–2755.

Guida, M., Postiglione, F., & Pulcini, G. (2018). A Bayesian approach for non-homogeneous gamma degradation processes. *Communications in Statistics – Theory and Methods*. doi:10.1080/03610926.2018.1440306

Hamada, M. S. (2015). Bayesian analysis of step-stress accelerated life tests and its use in planning. *Quality Engineering*, 27(3), 276–282.

Hamada, M. S., Wilson, A., & Reese, C. S. (2008). *Bayesian reliability*. Berlin: Springer.

Hu, C. H., Lee, M. Y., & Tang, J. (2015). Optimum step-stress accelerated degradation test for Wiener degradation process under constraints. *European Journal of Operational Research*, 241, 412–421.

Kim, Y. S., & Sung, S.-I. (2017). Practical lifetime estimation strategy based on partially step-stress-accelerated degradation tests. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 231(5), 605–614.

Li, X. Y., Mohammad, R., Ge, Z. Z., Mohamed, A., & Lee, J. (2015). Bayesian optimal design of step stress accelerated degradation testing. *Journal of Systems Engineering and Electronics*, 26(3), 502–513.

Liao, C. M., & Tseng, S. T. (2006). Optimal design for step-stress accelerated degradation tests. *IEEE Transactions on Reliability*, 55(1), 59–66.

Lim, H., & Yum, B.-J. (2011). Optimal design of accelerated degradation tests based on Wiener process models. *Journal of Applied Statistics*, 38(2), 309–325.

Lim, H.-S. (2012). Optimal design of accelerated degradation tests under the constraint of total experimental cost in the case that the degradation characteristic follows a Wiener process. *Journal of the Korean Society for Quality Management*, 40(2), 117–125.

Meeker, W. Q., & Escobar, L. A. (1998). *Statistical methods for reliability data*. New York, NY: Wiley.

Nelson, W. (1990). *Accelerated testing: Statistical models, test plans, and data analyses*. New York, NY: Wiley.

Padgett, W. J., & Tomlinson, M. A. (2004). Inference from accelerated degradation and failure data based on Gaussian process models. *Lifetime Data Analysis*, 10(2), 191–206.

Pan, Z., & Balakrishnan, N. (2010). Multiple-steps step-stress accelerated degradation modeling based on Wiener and gamma processes. *Communications in Statistics – Simulation and Computation*, 39(39), 1384–1402.

Pan, Z., & Sun, Q. (2014). Optimal design for step-stress accelerated degradation test with multiple performance characteristics based on gamma processes. *Communications in Statistics – Simulation and Computation*, 43(2), 298–314.

Park, C., & Padgett, W. J. (2006). Stochastic degradation models with several accelerating variables. *IEEE Transactions on Reliability*, 55(2), 379–390.

Sung, S.-I., & Yum, B.-J. (2016). Optimal design of step-stress accelerated degradation tests based on the Wiener degradation process. *Quality Technology & Quantitative Management*, 13(4), 367–393.

Tseng, S. T., Balakrishnan, N., & Tsai, C. C. (2009). Optimal step-stress accelerated degradation test plan for gamma degradation processes. *IEEE Transactions on Reliability*, 58(4), 611–618.

Tseng, S.-T., & Wen, Z.-C. (2000). Step-stress accelerated degradation analysis of highly-reliable products. *Journal of Quality Technology*, 32, 209–216.

Wang, H., Wang, G. J., & Duan, F. J. (2016). Planning of step-stress accelerated degradation test based on the inverse Gaussian process. *Reliability Engineering and System Safety*, 154, 97–105.

Whitmore, G. A., & Schenkelberg, F. (1997). Modelling accelerated degradation data using Wiener diffusion with a time scale transformation. *Lifetime Data Analysis*, 3(1), 27–45.

Xu, A. C., & Tang, Y. C. (2015). Reference optimality criterion for planning accelerated life testing. *Journal of Statistical Planning and Inference*, 167, 14–26.

Ye, Z. S., Chen, L. P., Tang, L. C., & Xie, M. (2014). Accelerated degradation test planning using the inverse Gaussian process. *IEEE Transactions on Reliability*, 63(3), 750–763.

Ye, Z. S., & Chen, N. (2014). The inverse Gaussian process as a degradation model. *Technometrics*, 56(3), 302–311.

Ye, Z. S., Chen, N., & Shen, Y. (2015). A new class of Wiener process models for degradation analysis. *Reliability Engineering and System Safety*, 139, 58–67.

Appendix

The elements of the Fisher information matrix $I(\theta)$ are given by

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial a^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left(2+\frac{\lambda_{ij}}{\sigma^2}\right),$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial a\partial b}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left[\left(2+\frac{\lambda_{ij}}{\sigma^2}\right)s_i + \left(\frac{1}{\sigma^2}+\frac{1}{\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial b}\right)\right],$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial a\partial \sigma^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\frac{1}{\sigma^2},$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial a\partial \beta}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left(\frac{1}{\sigma^2}+\frac{1}{\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial \beta}\right),$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial b^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left[\left(\frac{1}{2\lambda_{ij}^2}+\frac{1}{\sigma^2\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial b}\right)^2 + 2s_i\left(\frac{1}{\sigma^2}+\frac{1}{\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial b}\right)+s_i^2\left(2+\frac{\lambda_{ij}}{\sigma^2}\right)\right],$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial b\partial \sigma^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left[\frac{1}{2\sigma^2\lambda_{ij}}\left(\frac{\partial\lambda_{ij}}{\partial b}\right)+\frac{s_i}{\sigma^2}\right],$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial b\partial \beta}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left[\left(\frac{1}{2\lambda_{ij}^2}+\frac{1}{\sigma^2\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial b}\right)\left(\frac{\partial\lambda_{ij}}{\partial \beta}\right)+s_i\left(\frac{1}{\sigma^2}+\frac{1}{\lambda_{ij}}\right)\left(\frac{\partial\lambda_{ij}}{\partial \beta}\right)\right],$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial (\sigma^2)^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\frac{1}{2\sigma^4},$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial \sigma^2\partial \beta}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\frac{1}{2\sigma^2\lambda_{ij}}\left(\frac{\partial\lambda_{ij}}{\partial \beta}\right),$$

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\theta)}{\partial \beta^2}\right) = \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}$$
$$\times \left(\frac{1}{2\lambda_{ij}^2} + \frac{1}{\sigma^2 \lambda_{ij}}\right)\left(\frac{\partial \lambda_{ij}}{\partial \beta}\right)^2,$$

where

$$\frac{\partial \lambda_{ij}}{\partial b} = (\rho_{ij}^{\beta-1} - \rho_{i,j-1}^{\beta-1})\sum_{\kappa=1}^{i-1}(t_\kappa - t_{\kappa-1})[s_\kappa - s_i]\,\mathrm{e}^{-b(s_i-s_\kappa)/\beta},$$

$$\frac{\partial \lambda_{ij}}{\partial \beta} = \frac{b}{\beta^2}[\rho_{ij}^{\beta}\log(\rho_{ij}) - \rho_{i,j-1}^{\beta}\log(\rho_{i,j-1})]$$
$$\times \sum_{\kappa=1}^{i-1}(t_\kappa - t_{\kappa-1})[s_i - s_\kappa]\,\mathrm{e}^{-b(s_i-s_\kappa)/\beta}$$

and ρ_{ij} is defined in Equation (11). Furthermore, the detailed expressions of $\nabla \xi_p$ in Equation (14) are derived as follows.

$$\frac{\partial \xi_p}{\partial a} = -\frac{2\omega\,\mathrm{e}^{-a}}{\beta}\left[\frac{1}{2}\left(z_p\sigma + \sqrt{z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}}\right)\right]^{2/\beta-1}$$
$$\times \left[z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}\right]^{-1/2},$$

$$\frac{\partial \xi_p}{\partial b} = 0,$$

$$\frac{\partial \xi_p}{\partial \sigma^2} = \frac{1}{2\beta}\left[\frac{1}{2}\left(z_p\sigma + \sqrt{z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}}\right)\right]^{2/\beta-1}$$
$$\times \left[\frac{z_p}{\sigma} + z_p^2\left(z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}\right)^{-1/2}\right],$$

$$\frac{\partial \xi_p}{\partial \beta} = -\frac{2}{\beta^2}\left[\frac{1}{2}\left(z_p\sigma + \sqrt{z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}}\right)\right]^{2/\beta}$$
$$\times \log\left[\frac{1}{2}\left(z_p\sigma + \sqrt{z_p^2\sigma^2 + 4\omega\mathrm{e}^{-a}}\right)\right].$$