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Objective Bayesian analysis for the accelerated degradation model using Wiener process with measurement errors

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ABSTRACT
The Wiener process as a degradation model plays an important role in the degradation analysis. In this paper, we propose an objective Bayesian analysis for an acceleration degradation Wiener model which is subjected to measurement errors. The Jeffreys prior and reference priors under different group orderings are first derived, the propriety of the posteriors is then validated. It is shown that two of the reference priors can yield proper posteriors while the others cannot. A simulation study is carried out to investigate the frequentist performance of the approach compared to the maximum likelihood method. Finally, the approach is applied to analyse a real data.

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Accelerated degradation model; objective Bayesian analysis; Wiener process; measurement errors; Jeffreys prior; reference prior

1. Introduction

In practical applications, one may want to estimate the failure time of a product, however, it is quite difficult to collect sufficient failure time data due to the high quality of the product. If quality characteristics exist, whose degradation over time can be related to reliability, an effective approach is to collect the degradation data of the product, and then use the degradation data to predict the failure time. Compared with the traditional failure time analysis in term of asymptotic efficiency, the degradation analysis has demonstrated a higher precision. General discussion of degradation models and their uses is included in Simgpurwalla (1995) and Meeker and Escobar (1998).

Moreover, it is also hard to observe enough useful degradation data under normal experiment conditions. For this, Nelson (1990) proposed an accelerated degradation test (ADT) which collects degradation data under harsher conditions and then predicts the mean-time-to-failure (MTTF) under the normal conditions. Accelerated degradation models can be divided into three classes, which are constant-stress ADT (CSADT), step-stress ADT (SSADT) and progressive-stress ADT (PSADT), respectively. Due to the attractive mathematical properties and physical interpretations, degradation models based on Wiener process have been extensively utilised to describe the accelerated degradation of products. There are some work along this topic, for example, Doksum and Hoyland (1992) introduced the conception of PSADT to assess the product's lifetime distribution. Tang, Yang, and Xie (2004) and Liao and Tseng (2006) considered the optimisation of SSADT which assumed that the degradation characteristic followed a Wiener process. Ye, Chen, and Shen (2015) studied a new ADT model by introducing a common parameter in the mean and the variance into the Wiener degradation model.

In the aforementioned work, the methods of degradation analysis are mainly from the frequentist or subjective Bayesian perspectives. The objective Bayesian method has attracted much attention in the literature, since it has many advantages in statistical analysis, one can see Berger (2006) and references therein for more details. Recently, the objective Bayesian method has been applied to the statistical analysis for degradation models. For example, Xu and Tang (2012) used the objective Bayesian method to analyse a linear degradation model; Guan, Tang, and Xu (2016) proposed an objective Bayesian analysis for a CSADT model; He, He, and Cao (2016) introduced the approach to analyse a Wiener degradation model with random effects.

On the other hand, in real applications, it is inevitable to introduce some measurement errors during the observation process. Therefore, it is better to include the measurement errors in the degradation models. In this paper, we investigate an accelerated degradation Wiener model which is subjected to measurement errors, and then use the objective Bayesian method to analyse the model.

The rest of this paper is organised as follows. In Section 2, the accelerated degradation Wiener model with measurement errors is introduced. In Section 3, the Jeffreys prior and reference priors under different group orderings are derived. In Section 4, the propriety of the posterior distributions based on the non-informative priors is validated. In Section 5, a

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simulation study is carried out to see the performance of the Bayesian estimates compared with the maximum likelihood estimates. In Section 6, the proposed approach is applied to a real data in Zhao and Elsayed (2004). Some concluding remarks are given in Section 7.

2. The accelerated degradation model

Assume that $X(t)$ is the degradation characteristic of a product measured at time t, consider the following linear degradation model based on a Wiener process with drift:

$$X(t) = \mu t + \sigma B(t), \tag{1}$$

where μ is the drift parameter, σ is the diffusion coefficient and $B(t)$ is the standard Brownian motion.

The degradation rate is affected by many factors, such as the voltage, pressure, temperature and so on, which are usually called acceleration variables or stress levels. To make the prediction of the lifetime more accurate, the degradation model should take those stress levels into account when the information is available. In this paper, we will consider a CSADT.

Let $s_0 < s_1 < \cdots < s_r$ be $r{+}1$ stress levels, where s_0 stands for the stress level under normal experiment condition, and s_r represents the stress level under the harshest experiment condition. It is assumed that the drift parameter is related to the stress levels, while the diffusion coefficient is independent of the stress levels. That is to say, the model for CSADT turns into

$$X_i(t) = \mu_i t + \sigma B(t), \tag{2}$$

where $X_i(t)$ is the degradation process under the ith stress level, and μ_i is the corresponding drift parameter, for $i = 0, 1, \ldots, r$.

In ADTs, when a measurement is taken at time t, there is usually a measurement error. Assume that $Y_i(t)$ is the observable variable, and $\epsilon_i(t)$ is the corresponding measurement error, then

$$Y_i(t) = X_i(t) + \epsilon_i(t), \tag{3}$$

where $\epsilon_i(t)$ is assumed to be independent of $X_i(t)$ and distributed as a normal distribution with the mean 0 and the variance σ_ϵ^2, for $i = 1, 2, \ldots, r$.

Suppose that there are n_i units under the ith stress level s_i in the CSADT, and let m_{ij} be the number of measurements for the jth unit under the ith stress level, for $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, n_i$. Given i and j, assume that y_{ijk}, $k = 1, 2, \ldots, m_{ij}$ are the observations at the measurement time $t_{ij1} < t_{ij2} < \cdots < t_{ijm}$. For simplicity, we now consider the increment model. Denote

$$\Delta y_{ijk} = y_{ijk} - y_{ij(k-1)}, \quad \Delta t_{ijk} = t_{ijk} - t_{ij(k-1)}, \tag{4}$$

where $y_{ij0} = 0$ and $t_{ij0} = 0$. Let $y_{ij} = (\Delta y_{ij1}, \Delta y_{ij2}, \ldots, \Delta y_{ijk})'$ and $\tau_{ij} = (\Delta t_{ij1}, \Delta t_{ij2}, \ldots, \Delta t_{ijk})'$, $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, n_i$; $k = 1, 2, \ldots, m_{ij}$. Then y_{ij} follows a multivariate normal distribution:

$$y_{ij} \sim N(\mu_i \tau_{ij}, \Sigma_{ij}), \tag{5}$$

where

$$\Sigma_{ij} = \begin{pmatrix} \sigma^2\Delta t_{ij1} + 2\sigma_\epsilon^2 & -\sigma_\epsilon^2 & 0 & \cdots & 0 \\ -\sigma_\epsilon^2 & \sigma^2\Delta t_{ij2} + 2\sigma_\epsilon^2 & -\sigma_\epsilon^2 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\sigma_\epsilon^2 & \sigma^2\Delta t_{ij(m_{ij}-1)} + 2\sigma_\epsilon^2 & -\sigma_\epsilon^2 \\ 0 & \cdots & 0 & -\sigma_\epsilon^2 & \sigma^2\Delta t_{ijm_{ij}} + 2\sigma_\epsilon^2 \end{pmatrix}. \tag{6}$$

Let ω denote the threshold value, which is often determined by the manufacturer standard. The lifetime T_i of a product under the stress level s_i is defined as

$$T_i = \inf\{t \geq 0 \,|\, x_i(t) \geq \omega\}.$$

According to Ye, Shen, and Xie (2012), T_i follows an inverse Gaussian distribution, whose probability density function is given by

$$f_{IG}\left(t_i; \frac{\omega}{\mu_i}, \frac{\omega^2}{\sigma^2}\right) = \sqrt{\frac{\omega^2}{2\pi t_i^3 \sigma^2}} \times \exp\left\{-\frac{\mu_i^2 t_i^2 - 2\mu_i t_i \omega + \omega^2}{2\sigma^2 t_i}\right\}, \quad t_i > 0. \tag{7}$$

And the MTTF under the level s_i is

$$\mathrm{MTTF}_i = E(T_i) = \frac{\omega}{\mu_i}, \quad i = 0, 1, \ldots, r. \tag{8}$$

According to Ye and Chen (2014), the functions between the parameter μ_i and the stress level s_i often have the following three forms:

(i) the Power law model: $\mu_i = p \cdot s_i^q$,
(ii) the Arrhenius model: $\mu_i = p \cdot \mathrm{e}^{-q/s_i}$,
(iii) the Exponential model: $\mu_i = p \cdot \mathrm{e}^{q s_i}$,

where p and q are unknown parameters. Note that the Arrhenius model and the Power law model are widely used when the stress level is the temperature or voltage, while the Exponential model is usually used to characterise the effect of the weathering variable.

Under this assumption, we can rewrite the parameter μ_i as follows:

$$\mu_i = \mu_0 \theta^{h_i}, \quad i = 1, 2, \ldots, r,$$

where

$$\theta = \exp\{q[g(s_1) - g(s_0)]\} = \frac{\mu_1}{\mu_0} > 1$$

is the acceleration factor from the stress level s_0 to s_1, and

$$h_i = \frac{g(s_i) - g(s_0)}{g(s_1) - g(s_0)}, \quad i = 1, 2, \ldots, r.$$

Note that $h_r > \cdots > h_1 = 1$. Moreover, $g(s_i) = -1/s_i$ for the Arrhenius model, $g(s_i) = \log(s_i)$ for the Power law model and $g(s_i) = s_i$ for the Exponential model, respectively.

By reparameterisation, let

$$\eta = \frac{\sigma_\epsilon^2}{\sigma^2},$$

then $\Sigma_{ij} = \sigma^2 Q_{ij}$, where

$$Q_{ij} = \begin{pmatrix} \Delta t_{ij1} + 2\eta & -\eta & 0 & \cdots & 0 \\ -\eta & \Delta t_{ij2} + 2\eta & -\eta & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\eta & \Delta t_{ij(m_{ij}-1)} + 2\eta & -\eta \\ 0 & \cdots & 0 & -\eta & \Delta t_{ijm_{ij}} + 2\eta \end{pmatrix}. \tag{9}$$

Consequently, the likelihood function for the new parameters $\boldsymbol{\xi} = (\sigma, \eta, \theta, \mu_0)$ is given by

$$L(\boldsymbol{y} \mid \boldsymbol{\xi}) = \prod_{i=1}^{r} \prod_{j=1}^{n_i} \sqrt{\frac{1}{(2\pi\sigma^2)^{m_{ij}} |Q_{ij}|}} \times \exp\left\{ -\frac{(y_{ij} - \mu_0 \theta^{h_i} \tau_{ij})' Q_{ij}^{-1} (y_{ij} - \mu_0 \theta^{h_i} \tau_{ij})}{2\sigma^2} \right\}, \tag{10}$$

where $\boldsymbol{y} = \{y_{ij} : i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, n_i\}$.

3. Non-informative priors

In this section, we will derive some important non-informative priors for the model (3), which include the Jeffreys prior and three reference priors under different groups.

Theorem 3.1: *The Fisher information matrix for the parameters $\boldsymbol{\xi} = (\mu_0, \theta, \sigma, \eta)$ in the model* (3) *is*

$$\boldsymbol{I}(\boldsymbol{\xi}) = \begin{pmatrix} \boldsymbol{I}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{22} \end{pmatrix}, \tag{11}$$

where

$$\boldsymbol{I}_{11} = \begin{pmatrix} \frac{1}{\sigma^2} \sum_{i=1}^{r} a_i(\eta) \theta^{2h_i} & \frac{\mu_0}{\sigma^2} \sum_{i=1}^{r} a_i(\eta) h_i \theta^{2h_i - 1} \\ \frac{\mu_0}{\sigma^2} \sum_{i=1}^{r} a_i(\eta) h_i \theta^{2h_i - 1} & \frac{\mu_0^2}{\sigma^2} \sum_{i=1}^{r} a_i(\eta) h_i^2 \theta^{2h_i - 2} \end{pmatrix},$$

$$\boldsymbol{I}_{22} = \begin{pmatrix} \frac{2m}{\sigma^2} & \frac{1}{\sigma} \sum_{i=1}^{r} \sum_{j=1}^{n_i} \mathrm{tr}(\Psi_{ij}) \\ \frac{1}{\sigma} \sum_{i=1}^{r} \sum_{j=1}^{n_i} \mathrm{tr}(\Psi_{ij}) & \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n_i} \mathrm{tr}(\Psi_{ij}^2) \end{pmatrix},$$

and

$$m = \sum_{i=1}^{r} \sum_{j=1}^{n_i} m_{ij}, \quad a_i(\eta) = \sum_{j=1}^{n_i} \tau_{ij}' Q_{ij}^{-1} \tau_{ij} \tag{12}$$

$$\Psi_{ij} = Q_{ij}^{-1} \cdot \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{m_{ij} \times m_{ij}}. \tag{13}$$

Proof: The log-likelihood function, up to a constant, is given by

$$l(\boldsymbol{y} \mid \boldsymbol{\xi}) = -m \log \sigma - \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{n_i} \log |Q_{ij}| - \frac{1}{2\sigma^2} \sum_{i=1}^{r} \sum_{j=1}^{n_i} (y_{ij} - \mu_0 \theta^{h_i} \tau_{ij})' \times Q_{ij}^{-1} (y_{ij} - \mu_0 \theta^{h_i} \tau_{ij}).$$

We just prove the right lower 2×2 corner of the Fisher information matrix, since the other entries can be obtained by direct differentiation of $l(\boldsymbol{y} \mid \boldsymbol{\xi})$.

Taking the first partial derivatives of $l(\boldsymbol{y} \mid \boldsymbol{\xi})$ with respect to η, and the second partial derivatives with respect to σ, and applying Fact 3 in Berger, Oliveira, and

Sanao (2001), we can get

$$\frac{\partial l(\boldsymbol{y}\,|\,\boldsymbol{\xi})}{\partial \sigma^2} = \frac{m}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij})' \times Q_{ij}^{-1}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij}),$$

$$\frac{\partial l(\boldsymbol{y}\,|\,\boldsymbol{\xi})}{\partial \eta} = -\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}) + \frac{1}{2\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij})'Q_{ij}^{-1} \times \frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij}).$$

Taking derivation of $\partial l(\boldsymbol{y}\,|\,\boldsymbol{\xi})/\partial\eta$ with respect to σ results in

$$\frac{\partial^2 l(\boldsymbol{y}\,|\,\boldsymbol{\xi})}{\partial \sigma \partial \eta} = -\frac{1}{\sigma^3}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij})'Q_{ij}^{-1} \times \frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij}).$$

Using Fact 4 in Berger et al. (2001), we have

$$E\left(\frac{\partial^2 l(\boldsymbol{y}|\boldsymbol{\xi})}{\partial \sigma \partial \eta}\right) = -\sum_{i=1}^{r}\sum_{j=1}^{n_i}\frac{1}{\sigma^3} \times tr\left(Q_{ij}^{-1}\frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}\sigma^2 Q_{ij}\right) = -\frac{1}{\sigma}\sum_{i=1}^{r}\sum_{j=1}^{n_i} tr(\Psi_{ij}),$$

$$E\left(\frac{\partial^2 l(\boldsymbol{y}\,|\,\boldsymbol{\xi})}{\partial \sigma^2}\right) = \frac{m}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}\left(Q_{ij}^{-1}\sigma^2 Q_{ij}\right) = -\frac{2m}{\sigma^2},$$

$$E\left(\frac{\partial l(\boldsymbol{y}|\boldsymbol{\xi})}{\partial \eta}\right) = -\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}) + \frac{1}{2\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{n_i} \times \mathrm{tr}\left(Q_{ij}^{-1}\frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}\sigma^2 Q_{ij}\right) = 0.$$

Consequently, we obtain that

$$E\left(\frac{\partial l(\boldsymbol{y}|\boldsymbol{\xi})}{\partial \eta}\right)^2 = \mathrm{Var}\left(\frac{1}{2\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{n_i}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij})' \times Q_{ij}^{-1}\frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}(y_{ij} - \mu_0\theta^{h_i}\tau_{ij})\right)$$

$$= \frac{1}{4\sigma^4}\sum_{i=1}^{r}\sum_{j=1}^{n_i} \times 2\mathrm{tr}\left(Q_{ij}^{-1}\frac{\partial Q_{ij}}{\partial \eta}Q_{ij}^{-1}\sigma^2 Q_{ij}\right)^2 = \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2).$$

Then the result of Theorem 3.1 is straightforward. ■

According to Jeffreys (1961), the Jeffreys prior is proportional to the square root of the determinant of the Fisher information matrix. Hence, the following theorem can be obtained.

Theorem 3.2: *The Jeffreys prior for* $\boldsymbol{\xi} = (\mu_0, \theta, \sigma, \eta)$ *is given by*

$$\pi_J(\boldsymbol{\xi}) \propto \frac{\mu_0}{\sigma^3}\sqrt{f_1(\eta)g_1(\eta,\theta)}, \tag{14}$$

where

$$f_1(\eta) = m\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2) - \left(\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij})\right)^2, \tag{15}$$

$$g_1(\eta,\theta) = \sum_{1\le i<j\le r} a_i(\eta)a_j(\eta)(h_i - h_j)^2\theta^{2h_i+2h_j-2}. \tag{16}$$

In addition to the Jeffreys prior, Bernardo (1979) proposed the reference prior for deriving non-informative priors which separates the parameters into several different group orderings of interest. Reference prior has become one of the most useful non-informative priors in the literature, see Berger and Bernardo (1992), Sun and Berger (1998) and Berger, Bernardo, and Sun (2009) and references therein for more details. Now, we present the reference priors for the parameters under different group orderings.

Theorem 3.3: *Consider the degradation model* (3), *then we have*

(1) *the reference prior under the group ordering* $\{\sigma, (\eta,\theta), \mu_0\}$ *is of the form*

$$\pi_{R_1} \propto \frac{1}{\sigma}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^{r}\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2)}{\sum_{i=1}^{r}a_i(\eta)\theta^{2h_i}}\right)^{1/2}, \tag{17}$$

(2) *the reference prior under the group ordering* $\{(\mu_0,\theta,\eta),\sigma\}$ *is given by*

$$\pi_{R_2} \propto \frac{\mu_0}{\sigma}\sqrt{f_1(\eta)g_1(\eta,\theta)}, \tag{18}$$

(3) *the reference prior under the group ordering* $\{\mu_0, (\theta, \eta), \sigma\}$ *has the form*

$$\pi_{R_3} \propto \frac{1}{\sigma}\left(f_1(\eta)\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2}, \tag{19}$$

where $f_1(\eta)$ *and* $g_1(\eta, \theta)$ *are defined in* (15) *and* (16), *respectively.*

Proof: We prove the result for the group ordering $\{\mu_0, (\theta, \eta), \sigma)\}$, and the proof of the others is similar. It is readily to obtain that the inverse of the Fisher information matrix in (11) is

$$\boldsymbol{H}(\boldsymbol{\xi}) = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{22} \end{pmatrix}, \tag{20}$$

where

$$\boldsymbol{\Sigma}_{11} = \begin{pmatrix} \frac{\sigma^2}{g_1(\eta,\theta)}\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2} & -\frac{\sigma^2}{\mu_0 g_1(\eta,\theta)}\sum_{i=1}^{r} a_i(\eta)h_i\theta^{2h_i-1} \\ -\frac{\sigma^2}{\mu_0 g_1(\eta,\theta)}\sum_{i=1}^{r} a_i(\eta)h_i\theta^{2h_i-1} & \frac{\sigma^2}{\mu_0^2 g_1(\eta,\theta)}\sum_{i=1}^{r} a_i(\eta)\theta^{2h_i} \end{pmatrix},$$

$$\boldsymbol{\Sigma}_{22} = \begin{pmatrix} \frac{2m}{f_1(\eta)} & -\frac{\sigma}{f_1(\eta)}\sum_{i=1}^{r}\sum_{j=1}^{n_i} tr(\Psi_{ij}) \\ -\frac{\sigma}{f_1(\eta)}\sum_{i=1}^{r}\sum_{j=1}^{n_i} tr(\Psi_{ij}) & \frac{\sigma^2}{2f_1(\eta)}\sum_{i=1}^{r}\sum_{j=1}^{n_i} tr(\Psi_{ij}^2) \end{pmatrix}.$$

Following Berger and Bernardo (1992) with a slight difference in the notation, we can obtain that

$$k_1 = \frac{g_1(\eta,\theta)}{\sigma^2\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}},$$

$$|k_2| = \frac{\mu_0^2 f_1(\eta)}{2\sigma^2 m}\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}, \quad k_3 = \frac{2m}{\sigma^2}.$$

Now we choose compact sets $\Omega_l = [c_{1l}, d_{1l}] \times [c_{2l}, d_{2l}] \times [c_{3l}, d_{3l}] \times [c_{4l}, d_{4l}]$ for $(\mu_0, \theta, \eta, \sigma)$, such that $c_{1l}, c_{3l}, c_{4l} \to 0$, $c_{2l} \to 1$ and $d_{1l}, d_{2l}, d_{3l}, d_{4l} \to \infty$, as $l \to \infty$. Then we have

$$\begin{aligned} &\pi_3^l(\sigma \mid \mu_0, (\theta, \eta)) \\ &\quad = \frac{|k_3|^{1/2}\mathbf{1}_{[c_{4l},d_{4l}]}(\sigma, \eta)}{\int_{c_{4l}}^{d_{4l}} |k_3|^{1/2}\,\mathrm{d}\sigma} \\ &\quad = \frac{1}{\sigma(\log d_{4l} - \log c_{4l})}\mathbf{1}_{[c_{4l},d_{4l}]}(\sigma), \end{aligned}$$

where $\mathbf{1}_A(\cdot)$ stands for the indicator function of a set A. Thus

$$\begin{aligned} &E_2^l[\log(|k_2|) \mid \mu_0, (\theta, \eta)] \\ &\quad = \int_{c_{4l}}^{d_{4l}} \log(|k_2|)\pi_3^l(\sigma \mid \mu_0, (\theta, \eta))\,\mathrm{d}\sigma \\ &\quad = \log\left\{\mu_0^2 f_1(\eta)\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}\right\} + C_1(l), \end{aligned}$$

where

$$C_1(l) = \int_{c_{4l}}^{d_{4l}} \frac{1}{2m\sigma^2}\pi_3^l((\sigma, \eta) \mid \mu_0, \theta)\,\mathrm{d}\sigma.$$

It follows that

$$\begin{aligned} &\pi_2^l((\theta, \eta), \sigma) \mid \mu_0) \\ &\quad = \frac{\pi_3^l(\sigma \mid \mu_0, (\theta, \eta))\exp\left\{\frac{1}{2}E_2^l[\log(|k_2|) \mid \mu_0, (\theta, \eta)]\right\}\mathbf{1}_{[c_{2l},d_{2l}]}(\theta, \eta)}{\int_{c_{3l}}^{d_{3l}}\int_{c_{2l}}^{d_{2l}} \exp\left\{\frac{1}{2}E_2^l[\log(|k_2|) \mid \mu_0, (\theta, \eta)]\right\}\mathrm{d}\theta\,\mathrm{d}\eta} \\ &\quad = \pi_3^l(\sigma|\mu_0, (\theta, \eta))\left(f_1(\eta)\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2} \\ &\quad \times C_2(l)\mathbf{1}_{[c_{2l},d_{2l}]}(\theta), \end{aligned}$$

where

$$C_2(l) = \int_{c_{3l}}^{d_{3l}}\int_{c_{2l}}^{d_{2l}}\left(f_1(\eta)\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2}\mathrm{d}\theta\,\mathrm{d}\eta.$$

Consequently,

$$\begin{aligned} &E_1^l[\log(|k_1|) \mid \mu_0] \\ &\quad = \int_{c_{4l}}^{d_{4l}}\int_{c_{3l}}^{d_{3l}}\int_{c_{2l}}^{d_{2l}} \log(|k_1|) \\ &\quad \times \pi_2^l((\theta, \eta), \sigma \mid \mu_0)\,\mathrm{d}\theta\,\mathrm{d}\sigma\,\mathrm{d}\eta \\ &\quad = \int_{c_{4l}}^{d_{4l}}\int_{c_{3l}}^{d_{3l}}\int_{c_{2l}}^{d_{2l}} \log\left\{\frac{g_1(\eta,\theta)}{\sigma^2\sum_{i=1}^{r} a_i(\eta)h_i^2\theta^{2h_i-2}}\right\} \\ &\quad \times \pi_2^l(\theta, (\eta, \sigma) \mid \mu_0)\,\mathrm{d}\theta\,\mathrm{d}\sigma\,\mathrm{d}\eta \end{aligned}$$

is a constant, and

$$\pi_1^l(\mu_0,\theta,\sigma,\eta)) = \frac{\pi_2^l(\theta,(\sigma,\eta)\mid\mu_0)\exp\left\{\frac{1}{2}E_1^l[\log(|k_1|)\mid\sigma]\right\}\mathbf{1}_{[c_{1l},d_{1l}]}(\mu_0)}{\int_{c_{1l}}^{d_{1l}}\frac{1}{2}E_1^l[\log(|k_1|)\mid\sigma]\,\mathrm{d}\mu_0} = \frac{C_2(l)}{\sigma(d_{1l}-c_{1l})\log\frac{d_{4l}}{c_{4l}}}\left(f_1(\eta)\sum_{i=1}^r a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2} \times \mathbf{1}_{\Omega_l}(\sigma,\eta,\theta,\mu_0).$$

Let $(\mu_0^*,\theta^*,\sigma^*,\eta^*)$ be an inner point of Ω_l, then the reference prior under the group ordering $\{\mu_0,(\theta,\eta),\sigma\}$ is given by

$$\pi_{R_1}(\mu_0,\theta,\sigma,\eta) = \lim_{l\to\infty}\frac{\pi_1^l(\mu_0,(\theta,\eta),\sigma)}{\pi_1^l(\mu_0^*,(\theta^*,\eta^*),\sigma^*)} \propto \frac{1}{\sigma}\left(f_1(\eta)\sum_{i=1}^r a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2}.$$

This completes the proof of Theorem 3.3. ■

4. Posterior analysis

It can be seen that the priors derived in Section 3 are all improper, so it is necessary to verify whether the posterior distributions based on these priors are proper or not.

We first prove a lemma, which plays an important role in proving the subsequent theorems.

Lemma 4.1: *The elements of Q_{ij}^{-1} are all positive.*

Proof: It is readily shown that Q_{ij} is positive definite, thus Q_{ij}^{-1} is also positive definite, which implies that the main diagonals of Q_{ij}^{-1} are all positive. By the method of mathematical induction and the theory of partition matrix, it can be shown that the off-diagonal entries of Q_{ij}^{-1} are also positive. ■

Theorem 4.1: *The posterior distributions of $(\sigma,\eta,\theta,\mu_0)$ based on the prior π_{R_1} and π_{R_2} are both proper.*

Proof: We only prove the propriety of the posterior distribution based on the prior π_{R_1}, the other one can be shown in a similar way.

From (10), we can rewrite the likelihood function as follows:

$$L(\boldsymbol{y}\mid\boldsymbol{\xi}) = (2\pi\sigma^2)^{-m/2}\prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2} \times \exp\left\{-\frac{(\boldsymbol{y}-\mu_0\tau)'\Lambda^{-1}(\boldsymbol{y}-\mu_0\tau)}{2\sigma^2}\right\}, \tag{21}$$

where

$$\tau = (\tau_1',\tau_2',\ldots,\tau_r')',\quad \tau_i = \theta^{h_i}(\tau_{i1}',\tau_{i2}',\ldots,\tau_{in_i}')',\quad i=1,2,\ldots,r, \tag{22}$$

and

$$\Lambda = \mathrm{diag}(Q_{11},Q_{12},\ldots,Q_{1n_1},\ldots,Q_{r1},Q_{r2},\ldots,Q_{rn_r}). \tag{23}$$

From the likelihood function (21) and the reference prior $\pi_{R_1}(\sigma,(\eta,\theta),\mu_0)$, the joint posterior density of $(\mu_0,\theta,\sigma,\eta)$ is given by

$$\pi_{R_1}(\sigma,\eta,\theta,\mu_0\mid\boldsymbol{y}) \propto L(\boldsymbol{y}\mid\boldsymbol{\xi})\pi_{R_1}(\sigma,\eta,\theta,\mu_0) \propto \frac{1}{\sigma^{m+1}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \times\prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2} \times\exp\left\{-\frac{(\boldsymbol{y}-\mu_0\tau)'\Lambda^{-1}(\boldsymbol{y}-\mu_0\tau)}{2\sigma^2}\right\}.$$

Let

$$\hat{\mu} = (\tau'\Lambda^{-1}\tau)^{-1}\tau'\lambda^{-1}\boldsymbol{y}, \tag{24}$$

$$\lambda = (\boldsymbol{y}-\hat{\mu}\tau)'\Lambda^{-1}(\boldsymbol{y}-\hat{\mu}\tau). \tag{25}$$

Using Lemma 4.1, it can be shown that

$$\frac{\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}y_{rj}}{\theta^{h_r}\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}\tau_{ij}} < \hat{\mu} < \frac{\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}y_{ij}}{\theta^{h_r}\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}\tau_{rj}}, \tag{26}$$

$$f_2(\eta) < \lambda < f_3(\eta), \tag{27}$$

where

$$f_2(\eta) = \boldsymbol{y}'\Lambda^{-1}\boldsymbol{y} - \frac{2\left(\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}y_{ij}\right)^2}{\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}\tau_{rj}} + \left(\frac{\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}y_{rj}}{\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}\tau_{ij}}\right)^2\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}\tau_{rj}, \tag{28}$$

$$f_3(\eta) = \boldsymbol{y}'\Lambda^{-1}\boldsymbol{y} + \left(\frac{\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}y_{ij}}{\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}\tau_{rj}}\right)^2 \times\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}\tau_{ij}. \tag{29}$$

Therefore,

$$\pi_{R_1}(\sigma,\eta,\theta,\mu_0 \mid \boldsymbol{y}) \propto \frac{1}{\sigma^{m+1}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2} \times \exp\left\{-\frac{\lambda+(\mu_0-\hat{\mu})^2\tau'\Lambda^{-1}\tau}{2\sigma^2}\right\}.$$

Now taking the integration of $\pi_{R_1}(\sigma,\eta,\theta,\mu_0 \mid \boldsymbol{y})$ with respect to μ_0, then we have

$$\begin{aligned}\pi_{R_1}(\sigma,\eta,\theta \mid \boldsymbol{y}) &\propto \frac{1}{\sigma^{m+1}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}tr(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}\exp\left\{-\frac{\lambda}{2\sigma^2}\right\} \\ &\quad\times \int_0^\infty \exp\left\{-\frac{(\mu_0-\hat{\mu})^2\tau'\Lambda^{-1}\tau}{2\sigma^2}\right\}\mathrm{d}\mu_0 \\ &\propto \frac{1}{\sigma^{m+1}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}tr(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}\exp\left\{-\frac{\lambda}{2\sigma^2}\right\}\frac{\sigma}{\sqrt{\tau'\Lambda^{-1}\tau}}.\end{aligned}$$

When integrating $\pi_{R_1}(\sigma,\eta,\theta \mid \boldsymbol{y})$ with respect to σ, we can get

$$\begin{aligned}\pi_{R_1}(\eta,\theta \mid \boldsymbol{y}) &= \int_0^\infty \pi_{R_1}(\sigma,\eta,\theta \mid \boldsymbol{y})\,\mathrm{d}\sigma \\ &\propto \frac{1}{\sqrt{\tau'\Lambda^{-1}\tau}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2} \\ &\quad\times \int_0^\infty \frac{1}{\sigma^m}\exp\left\{-\frac{\lambda}{2\sigma^2}\right\}\mathrm{d}\sigma \\ &\propto \frac{1}{\sqrt{\tau'\Lambda^{-1}\tau}}\left(\frac{g_1(\eta,\theta)\sum_{i=1}^r\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2)}{\sum_{i=1}^r a_i(\eta)\theta^{2h_i}}\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}\lambda^{-(m-1)/2}. \qquad (30)\end{aligned}$$

Noticing that

$$\theta^{2h_r}\sum_{j=1}^{n_r}\tau_{rj}'Q_{rj}^{-1}\tau_{rj} < \tau'\Lambda^{-1}\tau < \theta^{2h_r}\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}\tau_{ij}, \qquad (31)$$

and combining with (26) and (27), we can obtain that an upper bound for (30) is

$$\begin{aligned}u(\eta,\theta) &:= \frac{\theta^{h_r+h_{r-1}-1}}{a_r(\eta)\theta^{2h_r}}\left(\sum_{i=1}^r\sum_{j=1}^{n_i}tr(\Psi_{ij}^2) \times \sum_{1\le i<j\le r}a_i(\eta)a_j(\eta)(h_i-h_j)^2\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}f_2(\eta)^{-(m-1)/2} \\ &= f_4(\eta)\theta^{h_{r-1}-h_r-1},\end{aligned}$$

where

$$f_4(\eta) = \frac{1}{a_r(\eta)}\left(\sum_{i=1}^r\sum_{j=1}^{n_i}\mathrm{tr}(\Psi_{ij}^2) \times \sum_{1\le i<j\le r}a_i(\eta)a_j(\eta)(h_i-h_j)^2\right)^{1/2} \times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}f_2(\eta)^{-(m-1)/2}. \qquad (32)$$

It can be shown that as θ and η tend to ∞,

$$f_4(\eta) = O\left(\eta^{-3/2}\right). \qquad (33)$$

Consequently, we have

$$\int_0^\infty\int_1^\infty f_4(\eta)\theta^{h_{r-1}-h_r-1}\,\mathrm{d}\theta\,\mathrm{d}\eta < \infty, \qquad (34)$$

since $h_{r-1} < h_r$. This implies that the posterior distribution $\pi_{R_1}(\sigma,\eta,\theta,\mu_0 \mid \boldsymbol{y})$ is proper. ■

Theorem 4.2: *The posterior distributions of $(\sigma,\eta,\theta,\mu_0)$ based on the priors π_{R_3} and π_J are both improper.*

Proof: It is readily shown that the marginal posterior distribution of the parameters (η,θ) based on the prior π_{R_3} is given by

$$\pi_{R_3}(\eta,\theta \mid \boldsymbol{y}) \propto \frac{1}{\sqrt{\tau'\Lambda^{-1}\tau}}\left(f_1(\eta)\sum_{i=1}^r a_i(\eta)h_i^2\theta^{2h_i-2}\right)^{1/2} \times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}\lambda^{-(m-1)/2}. \qquad (35)$$

Following the proof of Theorem 4.1, we can obtain that

$$\begin{aligned}\pi_{R_3}(\eta,\theta \mid \boldsymbol{y}) &\ge \left(\frac{f_1(\eta)a_r(\eta)h_r^2\theta^{2h_r-2}}{\theta^{2h_r}\sum_{i=1}^r\sum_{j=1}^{n_i}\tau_{ij}'Q_{ij}^{-1}\tau_{ij}}\right)^{1/2} \\ &\quad\times \prod_{i=1}^r\prod_{j=1}^{n_i}|Q_{ij}|^{-1/2}f_3(\eta)^{-(m-1)/2} \\ &\propto f_5(\eta)\frac{1}{\theta},\end{aligned}$$

where

$$f_5(\eta) = \left(\frac{f_1(\eta) a_r(\eta)}{\sum_{i=1}^{r} \sum_{j=1}^{n_i} \tau'_{ij} Q_{ij}^{-1} \tau_{ij}} \right)^{1/2} \times \prod_{i=1}^{r} \prod_{j=1}^{n_i} |Q_{ij}|^{-1/2} f_3(\eta)^{-(m-1)/2}. \quad (36)$$

Note that as η tends to ∞,

$$f_5(\eta) = O\left(\eta^{-3/2}\right). \quad (37)$$

Consequently, we have

$$\int_0^{\infty} \int_1^{\infty} f_5(\eta) \frac{1}{\theta} \mathrm{d}\theta \, \mathrm{d}\eta = \infty. \quad (38)$$

This implies that $\pi_{R_3}(\eta, \theta \mid \boldsymbol{y})$ is not integrable with respect to θ and η, which follows that the posterior distribution under π_{R_3} is improper. By analogous arguments, we can show that the posterior distribution under π_J is also improper. ■

It follows from Theorems 4.1 and 4.2 that only the priors π_{R_1} and π_{R_2} enable posterior inferences. In order to generate samples from the joint posterior distribution of $(\sigma, \eta, \theta, \mu_0)$, say $\pi(\sigma, \eta, \theta, \mu_0 \mid \boldsymbol{y})$, we can use the following steps.

Step 1. The random walk Metropolis algorithm can be employed to generate samples from those marginal posterior distributions of (μ_0, θ, η), denoted by $\pi_{R_1}(\eta, \theta, \mu_0 \mid \boldsymbol{y})$ and $\pi_{R_2}(\eta, \theta, \mu_0 \mid \boldsymbol{y})$:

$$\pi_{R_1}(\eta, \theta, \mu_0 \mid \boldsymbol{y}) \propto \prod_{i=1}^{r} \prod_{j=1}^{n_i} |Q_{ij}|^{-1/2} \times \left(\frac{g_1(\eta, \theta) \sum_{i=1}^{r} \sum_{j=1}^{n_i} \operatorname{tr}(\Psi_{ij}^2)}{\sum_{i=1}^{r} a_i(\eta) \theta^{2h_i}} \right)^{1/2} \psi^{-m/2}, \quad (39)$$

$$\pi_{R_2}(\eta, \theta, \mu_0 | \boldsymbol{y}) \propto \prod_{i=1}^{r} \prod_{j=1}^{n_i} |Q_{ij}|^{-1/2} \mu_0 \sqrt{f_1(\eta) g_1(\eta, \theta)} \psi^{-m/2}, \quad (40)$$

where

$$\psi = \frac{1}{2} (\boldsymbol{y} - \mu_0 x)' \Lambda^{-1} (\boldsymbol{y} - \mu_0 x). \quad (41)$$

Step 2. Then draw samples from the conditional posterior distribution of σ given (η, θ, μ_0), denoted by $\pi_{R_1}(\sigma^2 \mid \mu_0, \theta, \eta, \boldsymbol{y})$ and $\pi_{R_2}(\sigma^2 \mid \mu_0, \theta, \eta, \boldsymbol{y})$. For the priors π_{R_1} and π_{R_2}, it can be shown that

$$\pi_{R_1}(\sigma^2 \mid \mu_0, \theta, \eta, \boldsymbol{y}) \sim \mathrm{IG}\left(\frac{m}{2}, \psi\right), \quad (42)$$

$$\pi_{R_2}(\sigma^2 \mid \mu_0, \theta, \eta, \boldsymbol{y}) \sim \mathrm{IG}\left(\frac{m}{2}, \psi\right), \quad (43)$$

where $\mathrm{IG}(a, b)$ refers to an inverse Gamma distribution with the shape parameter a and the scale parameter b.

Step 3. The samples of σ_ϵ^2 can be obtained from $\sigma_\epsilon^2 = \eta\sigma^2$.

5. Simulation study

In this section, we will investigate the performance of the Bayesian estimators based on the priors π_{R_1} and π_{R_2}. In the simulation experiment, three stress levels are used to observe the degradation process. The true values of the parameters $(\mu_0, \theta, \sigma, \sigma_\epsilon^2)$ are set as $(0.04, 3.9, 1, 0.4)$, and the temperature is taken as the stress level. Here, the Arrhenius model is assumed between the drift parameter and the temperature. The normal stress S_0 is specified as 50°C, and the accelerated levels are set as $S_1 = 83$°C, $S_2 = 133$°C and $S_3 = 173$°C. Moreover, each unit is observed four times, and the observation time are 100, 400, 1000 and 2000 h, respectively. Under the above setting, posterior samples based on the priors $\pi_{R_1}(\sigma, \theta, \eta, \mu_0)$ and $\pi_{R_2}(\sigma, \theta, \eta, \mu_0)$ are generated by the sampling method as mentioned in Section 4. The proposed Bayesian estimators are compared with the maximum likelihood estimator (MLE) in terms of the mean square error (MSE) and the frequentist coverage probability of 95% confidence interval under different sample sizes. By replicating the experiment 5000 times, we can obtain the estimated MSEs and coverage probabilities, which are shown in Table 1. It should be pointed out that the confidence intervals for MLE are approximated by the likelihood ratio based method as in Peng and Tseng (2009).

From Table 1, the following conclusions can be drawn:

- As is expected, the MSEs of all the estimators become smaller as the sample size increases. Furthermore, the MSEs of Bayesian estimators for all parameters are smaller than that of the MLEs in an obvious way.
- It can be observed that the coverage probabilities of the Bayesian estimators under π_{R_1} and π_{R_2} are much close to the nominal level 0.95. However, the coverage probabilities of the MLE are not satisfactory, since some of which do not reach 0.9 yet even for large samples. Of course, this may be normal in that the confidence intervals for the MLE are obtained by the asymptotic distribution.
- Generally speaking, the performance of the Bayesian estimators is superior to that of the MLEs in terms

Table 1. MSEs and coverage probabilities (within parentheses) of the Bayesian estimators and MLE.

(n_1, n_2, n_3)	MLE/priors	σ	σ_ϵ^2	θ	μ_0
(5,5,5)	MLE	0.751	0.401	0.344	9.734×10^{-3}
		(0.8032)	(0.8128)	(0.8308)	(0.8214)
	π_{R_1}	0.737	0.317	0.299	4.284×10^{-3}
		(0.9378)	(0.9382)	(0.9618)	(0.9348)
	π_{R_2}	0.728	0.308	0.298	4.273×10^{-3}
		(0.9380)	(0.9382)	(0.9612)	(0.9392)
(10,15,15)	MLE	0.1487	0.1256	9.474×10^{-2}	5.164×10^{-3}
		(0.8426)	(0.8662)	(0.8628)	(0.8618)
	π_{R_1}	0.1347	0.1082	8.276×10^{-2}	9.128×10^{-4}
		(0.9406)	(0.9588)	(0.9432)	(0.9412)
	π_{R_2}	9.312×10^{-2}	8.374×10^{-2}	8.147×10^{-2}	9.136×10^{-4}
		(0.9412)	(0.9532)	(0.9464)	(0.9426)
(20,20,20)	MLE	7.892×10^{-2}	8.637×10^{-2}	5.382×10^{-2}	1.358×10^{-3}
		(0.8672)	(0.8904)	(0.9018)	(0.8846)
	π_{R_1}	7.346×10^{-2}	6.382×10^{-2}	4.831×10^{-2}	6.573×10^{-4}
		(0.9442)	(0.9468)	(0.9548)	(0.9534)
	π_{R_2}	6.732×10^{-2}	4.317×10^{-2}	4.774×10^{-2}	6.427×10^{-4}
		(0.9524)	(0.9512)	(0.9498)	(0.9508)
(25,25,25)	MLE	4.297×10^{-2}	3.428×10^{-2}	2.116×10^{-2}	6.213×10^{-4}
		(0.8892)	(0.9124)	(0.9248)	(0.9024)
	π_{R_1}	4.025×10^{-2}	2.114×10^{-2}	1.0781×10^{-2}	2.754×10^{-4}
		(0.9468)	(0.9518)	(0.9512)	(0.9518)
	π_{R_2}	3.841×10^{-2}	2.115×10^{-2}	1.0778×10^{-2}	2.749×10^{-4}
		(0.9488)	(0.9504)	(0.9502)	(0.9508)

of the MSE and the coverage probability. And for Bayesian estimators, the performance of the prior π_{R_1} is similar to that of π_{R_2}, although π_{R_2} is somewhat better than π_{R_1} for small sample sizes.

6. Real data analysis

Now we apply the proposed Bayesian approach to analyse the real data of LEDs in Zhao and Elsayed (2004). In the original experiment, each unit was measured for five times (50 h, 100 h, 150 h, 200 h, 250 h, respectively), and the maximum test duration allowed was 250 h. The normal stress was $s_0 = 28$ mA, and two stress levels were $s_1 = 35$ mA and $s_2 = 40$ mA, respectively.

We use to the Wiener accelerated degradation model with measurement errors to fit the data. And the Power law model is adopted between the drift parameter and the stress level. Besides, the threshold value ω is specified as 0.5 as in Lee and Tang (2007).

The 95% credible intervals of $(\mu_0, \sigma, \theta, \sigma_\epsilon^2)$ and the posterior means are shown in Table 2. It can be seen that the Bayesian estimates based on the priors π_{R_1} and π_{R_2} are relatively close to each other. Finally, the estimated MTTF_0s under the priors π_{R_1} and π_{R_2} are 6124 h and 6133 h, respectively.

Table 2. Posterior means and 95% credible intervals (within parentheses) of parameters.

Prior	Parameter	Mean	95% Credible interval
π_{R_1}	θ	1.4802	(1.0824, 2.1147)
	μ_0	8.1648×10^{-5}	$(7.4571, 8.8102) \times 10^{-5}$
	σ	1.7012×10^{-3}	$(1.4198, 2.3112) \times 10^{-3}$
	σ_ϵ^2	1.4794×10^{-2}	$(1.0981, 1.7932) \times 10^{-2}$
π_{R_2}	θ	1.4765	(1.0176, 2.0595)
	μ_0	8.1518×10^{-5}	$(7.4257, 8.7328) \times 10^{-5}$
	σ	1.7132×10^{-3}	$(1.4341, 2.3475) \times 10^{-3}$
	σ_ϵ^2	1.5134×10^{-2}	$(1.1297, 1.8147) \times 10^{-2}$

7. Concluding remarks

In this paper, we propose an objective Bayesian approach to investigate the acceleration degradation model based on the Wiener process with measurement errors. The Jeffreys prior and reference priors under different group orderings are derived for the model. The propriety of the posterior distribution under the non-informative priors is validated. A simulation study is carried out to see the performance of the Bayesian approach, which indicates that the proposed method is superior to the MLE in terms of the MSE and the coverage probability. Finally, the method is applied to analyse a real data set, and the MTTF of the product is estimated.

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