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Measure of rotatability of modified five-level second-order rotatable design using supplementary difference sets

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ABSTRACT

Rotatability is a desirable quality of fitting response surface experimental designs. The property states that the variance of the estimated response made from the Taylor's series expansion are constant on circles, spheres and hyper-spheres about the centre of the design. In this article, a measure of rotatability of modified second-order rotatable design is presented. The variance function of a second-order response design and an infinite class of supplementary difference sets is used in coming up with the design.

1. Introduction

Response surface methodology is used in situation where the form of the relationship between the response and independent variables is unknown. Therefore, the first step in Response Surface Methodology is to find a suitable approximation for the true functional relationship between the response variable and the set of independent variables. The technique to be used is to fit a low order polynomial to the response and if it is inadequate then we graduate it to a higher order polynomial. We use a second-order model when the response of interest has curvature.

Response surface methodology is useful for the modelling and analysis of problems in which a response is influenced by several variables and the objective is to optimise this response. The property of rotatability is a desirable quality of an experimental design and requires that the variances of the estimated response made from the Taylor's series are constant on circles, spheres and hyper-spheres about the centre of the design. The article presents a measure of rotatability of modified second-order rotatable designs.

The construction method is based on the work of Das (1961) where he obtained rotatable design for both second and third orders up to eight factors as a fractional replicate of factorial design. The work was extended by Das and Narasimham (1962), where they constructed rotatable design through balanced incomplete block design. Most recently, Koukouvinos,

Mylona, Skountzou, and Goos (2013) gave a general construction method for five-level second-order rotatable designs. The construction of five-level rotatable design using supplementary difference sets was done by Mutiso, Kerich, and Ng'eno (2016a, 2016b). The article presents a measure of rotatability of modified five-level rotatable designs using an infinite class of supplementary difference sets.

2. Model and design criteria

Suppose we want to use second-order response surface design $D = ((x_{iu}))$ to fit the surface

$$
y_u = b_o + \sum_{i=1}^{V} b_i x_{iu} + \sum_{i=1}^{V} b_{ii} x_{iu}^2
$$
$$
+ \sum_{i=1}^{v} \sum_{j=1}^{v} b_{ij} x_{iu} x_{ju} + \mathcal{E}_u,
$$

where y_u is the response observed at the u-th design point, x_{iu} denotes the level of the ith factor $i = (1, 2, \ldots, v)$ in the u-th run and $u = (1, 2, \ldots, N)$ of the experiment and $\mathcal{E}_u$'s are uncorrelated random error with mean zero and variance δ^2.

Here b_o, b_i, b_{ii}, b_{ij} are parameters of the model and y_u is the response observed at the uth design point. The parameters in the response relation are estimated using least squares techniques.

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Box and Hunter (1957) developed the following symmetry conditions (Moments conditions) for a second-order response surface to form a rotatable arrangement.

(i) $\sum_{u=i}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0$ if any α_i is odd for $\sum \alpha_i \leq 4$,

(ii) $\sum_{u=1}^{N} x_{iu}^2 = $ Constant $= N\lambda_2$ for $i = 1, 2, \ldots, v$,

(iii) $\sum_{u=1}^{N} x_{iu}^4 = $ Constant $= cN\lambda_4$ for $i = 1, 2, \ldots, v$,

(iv) $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = $ Constant $= N\lambda_4$,

(v) $\frac{\sum_{u=1}^{N} x_{iu}^4}{\sum_{u=1}^{N} X_{iu}^2 x_{ju}^2} = c$ for $i \neq j$,

where c, λ_2 and λ_4 are constants.

Using these symmetry conditions the variance and the covariance of the estimates are obtained and are shown below

(i) $V(b_0) = \frac{(c+v-1)\lambda_4 \delta^2}{N[(c+v-1)\lambda_4 - v\lambda_2^2]}$,

(ii) $V(b_i) = \frac{\delta^2}{N\lambda_2}$,

(iii) $V(b_{ij}) = \frac{\delta^2}{N\lambda_4}$,

(iv) $V(b_{ii}) = \frac{\delta^2}{N(c-1)\lambda_4} \left[\frac{(c+v-2)\lambda_4 - \lambda_2^2(v-1)}{(c+v-1)\lambda_4 - v\lambda_2^2} \right]$,

(v) $\text{Cov}(b_0, b_{ii}) = \frac{-\lambda_2 \delta^2}{N[(c+v-1)\lambda_4 - v\lambda_2^2]}$,

(vi) $\text{Cov}(b_{ii}, b_{ij}) = \frac{\delta^2}{N\lambda_4(c-1)} \left[\frac{\lambda_2^2 - \lambda_4}{(c+v-1)\lambda_4 - v\lambda_2^2} \right]$,

and all other covariances are zero.

An inspection of the variances shows that a necessary condition for the existence of a non-singular second-order design is $(c+v-1)\lambda_4 - v\lambda_2^2 > 0$ which leads to the following non-singularity condition first developed by Box and Hunter (1957):

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1}.$$

Hader and Park (1978) studied estimates in response at two different points in the factor space. They introduced the analogous form of Box Hunter rotatability and termed it as slope rotatability. The technique utilises the concept of partial derivatives to estimate the response surface at two different points in a factor space. Hader and Park (1978) and Park (1987) stated that the necessary and sufficient condition for slope rotatability is $4V(b_{ii}) = V(b_{ij})$. The condition was simplified by Victorbabu and Narasimham (1991) where they developed the necessary and sufficient condition for a second-order design to be slope rotatable and expressed the relationship as

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0,$$

where

$$c = \frac{\sum_{u=1}^{N} x_{iu}^4}{\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2}.$$

3. Modified second-order rotatable design

A design is said to be a second-order rotatable if it satisfy moment's conditions and non-singularity conditions. The usual method of constructing a second-order rotatable design is by putting some restrictions indicating some relations among second-order moments. The paper explores modified condition of rotatability which is obtained by solving the equation below which is the necessary and sufficient condition for a second-order design to be slope rotatable.

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0.$$

The equation gives a solution of $c = 5$ if and only if $\lambda_2^2 = \lambda_4$. In particular, we investigate the restriction

$$\lambda_2^2 = \lambda_4, \text{ i.e., } \sum_{u=1}^{N} x_{iu}^2 = N \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2.$$

The restriction $\lambda_2^2 = \lambda_4$ and $c = 5$ are similar conditions and gives another series of rotatable designed called modified second-order rotatable designs. The modified symmetry condition therefore becomes

$$\sum_{u=1}^{N} x_{iu}^4 = 5 \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 5N\lambda_4.$$

4. Construction of the design

The construction method of second and third order rotatable design using Balanced Incomplete Block Design was presented by Das and Narasimham (1962). In constructing the design, they considered each point in the design as essentially a combination of levels of different factors. First, we have unknown level denoted by a, b, c etc. Secondly, we have another design in v factors of the form 2^v where the two levels of each factor are $+1$ and -1. Thirdly, we can get another combination when any combination of the first design is associated with the second design by multiplication.

Seberry (1973) defined supplementary difference sets (SDS) and stated that the parameters of e-$[v; k_1, k_2, \ldots, k_e, \lambda]$ SDS satisfies

$$\lambda(v-1) = \sum_{i=1}^{e} k_i(k_i - 1).$$

If $k_1 = k_2 = \ldots k_e = k$, we shall write e-$[v; k; \lambda]$ to denote the e SDS and the equation becomes

$$\lambda(v-1) = ek(k-1).$$

In this paper we use the supplementary difference sets developed by Koukouvinos, Manta, and Mylona (2008). Let $C_1, C_2, \ldots, C_e$ be two-subsets of Z_v (or any finite abelian group of order v) where $v = n-1 = 2e+1$, $C_i = [i, v-i] i = 1, 2, \ldots, (v-1)/2 = 1, 2, \ldots, e$.

Then the sets $C_1, C_2, \ldots, C_e$ will be an e-[v;2;1] SDS. Based on this SDS we construct rotatable second-order design in m-factors, constituted of a factorial part with level combinations $(-1,1,0)$ plus a set of $2\,m$ axial points at a distance b from the origin plus centre points denoted by n_0. Therefore, the total number of experimental points is given as

$$N = m.2^{t(m)} + 2m + n_0.$$

Koukouvinos et al. (2013) came up with the following useful relationships which are utilised in this study.

(i) $\sum_{u=1}^{N} x_{1u} = \sum_{u=1}^{N} x_{2u} = \sum_{u=1}^{N} x_{3u} = \sum_{u=1}^{N} x_{4u} = 0$,

(ii) $\sum_{u=1}^{N} x_{iu}^2 = 2^{t(m)}(e-1) + 2b^2 = N\lambda_2$,

(iii) $\sum_{u=1}^{N} x_{iu}^4 = 2^{t(m)}(e-1) + 2b^4 = cN\lambda_4$,

(iv) $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e-2) = N\lambda_4$.

From the above equation, we have

$$\sum x_{iu}^2 = N\lambda_2 \quad and \quad \sum x_{iu}^2 x_{ju}^2 = N\lambda_4.$$

Therefore $\lambda_2 = \sum x_{iu}^2/N$ and $\lambda_4 = \sum x_{iu}^2 x_{ju}^2/N$.

Mutiso et al. (2016a) stated in a theorem that supplementary difference sets with parameters e-[v:2:1] gives a five-level modified second-order rotatable design in

$$b^4 = \frac{2^{t(m)}(4e-9)}{2} \text{ and,}$$

$$N = \frac{[(e-1)2^{t(m)} + 2b^2]^2}{(e-2)2^{t(m)}},$$

and $n_0 = N - 2^{t(m)} - 2m$.

The modified restriction under investigation is $\lambda_2^2 = \lambda_4$ which is equivalent to

$$\frac{\left(\sum x_{iu}^2\right)^2}{N^2} = \frac{\sum x_{iu}^2 x_{ju}^2}{N}.$$

Hence

$$N = \frac{\left(\sum x_{iu}^2\right)^2}{\sum x_{iu}^2 x_{ju}^2},$$

but

$$\sum_{u=1}^{N} x_{iu}^2 = 2^{t(m)}(e-1) + 2b^2,$$

and

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e-2).$$

Therefore,

$$N = \frac{[2^{t(m)}(e-1) + 2b^2]^2}{2^{t(m)}(e-2)}.$$

The modified symmetry condition for modified second-order rotatability is expressed as

$$\sum_{u=1}^{N} x_{iu}^4 = 5\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2,$$

where

$$\sum_{u=1}^{N} x_{iu}^4 = 2^{t(m)}(e-1) + 2b^4 \text{ and}$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e-2).$$

On simplification we obtain

$$2^{t(m)}(e-1) + 2b^4 = 5[2^{t(m)}(e-2)],$$

$$2b^4 = 5[2^{t(m)}(e-2) - 2^{t(m)}(e-1),$$

$$2b^4 = 2^{t(m)}[5(e-2) - (e-1)],$$

$$2b^4 = 2^{t(m)}(4e-9),$$

$$b^4 = \frac{2^{t(m)}(4e-9)}{2}.$$

Example 1: Consider a 3-(7:2:1) SDS therefore $v = 9$ m $= (v-1)/2 = 3$, $e = (v-1)/2 = 3$

$$b^4 = \frac{2^{t(m)}(4e-9)}{2} = \frac{2^2[4(3)-9]}{2} = 6,$$

$$b^2 = \sqrt{6} = 2.4495,$$

$$\sum_{u=1}^{N} x_{iu}^4 = 2^{t(m)}(e-1) + 2b^4$$

$$= 2(2^2) + 2(\sqrt{6})^2 = 20,$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e-2) = (3-2)2^2 = 4,$$

$$\frac{\sum_{u=1}^{N} x_{iu}^4}{\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2} = \frac{20}{4} = 5,$$

hence the design forms a rotatable arrangement.

Also,

$$N = \frac{[2^{t(m)}(e-1) + 2b^2]^2}{2^{t(m)}(e-2)} = \frac{[(3-1)2^2 + 2\left(\sqrt{6}\right)]^2}{(3-2)2^2}$$

$$= \frac{166.3837}{4} = 41.5959 \approx 42,$$

and

$$n_0 = N - 2^{t(m)} - 2m$$

$$= 42 - 3(2^2) - 2(3) = 24 \text{ centre points.}$$

For rotatability, we need to check if non-singularity condition is satisfied.

$$\frac{\lambda_4}{\lambda_2^2} > \frac{m}{c+m-1},$$

where

$$\lambda_4 = \frac{2^{t(m)}(e-2)}{N} = \frac{4}{24} = 0.1666,$$

$$\lambda_2 = \frac{2^{t(m)}(e-1) + 2b^2}{N} = \frac{12.8989}{24} = 0.5375,$$

$$m = \frac{v-1}{2} = 3, e = \frac{v-1}{2} = 3, c = 5.$$

Thus

$$\frac{\lambda_4}{\lambda_2^2} = \frac{0.1666}{0.5375^2} = 0.5767,$$

also

$$\frac{m}{c+m-1} = \frac{3}{7} = 0.4286.$$

Since $0.5767 > 0.4286$ the non-singularity condition is satisfied hence the design constructed using supplementary difference sets is second-order rotatable. A list of modified second-order rotatable designs is tabulated in Table A1 in Appendix.

5. Proposed method: measure of rotatability of modified second-order rotatable design

A design whose moments do not conform to the moment conditions of rotatability is said to be non-rotatable. Quite often, a non-rotatable design may exhibit surface of constant prediction variances that are nearly spherical. In this case, a design is described as near rotatable. If circumstances are such that exact rotatability is unattainable, it is still a good idea to make the design nearly rotatable as stated by Park, Lim, and Baba (1993). Thus it is important to know if a particular design is rotatable or, if it is not, to know how rotatable it is.

The traditional way of measuring rotatability was to inspect contour plots of constant prediction variance to see how close they are to being circular. This practice, however, in addition to being subjective had a number of limitations when the number of input variables exceeds 3.

Khuri (1988) proposed a quantitative measure of response surface design which attracted further studies and other measures were developed by Draper and Guttman (1988) and Draper and Pukelsheim (1990). In this article, the author uses the measure that was developed by Park and Kim (1992). The measure is expressed as follows

$$Q_v(D) = \frac{1}{\delta^4}[4V(b_{ii}) - V(b_{ij})]^2.$$

Theorem 5.1: *The measure of rotatability of five-level modified second-order rotatable design is given as*

$$Q_m(D) = \left[\frac{(\lambda_2^2 - \lambda_4)}{N\lambda_4[\lambda_4(4+m) - m\lambda_2^2]}\right]^2.$$

Proof:

$$Q_v(D) = \frac{1}{\delta^4}[4V(b_{ii}) - V(b_{ij})]^2,$$

$$V(b_{ii}) = \frac{\delta^2}{N(c-1)\lambda_4}\left[\frac{(c+v-2)\lambda_4 - \lambda_2^2(v-1)}{(c+v-1)\lambda_4 - v\lambda_2^2}\right],$$

$$V(b_{ij}) = \frac{\delta^2}{N\lambda_4}.$$

We therefore substitute $v(b_{ii})$ and $v(b_{ij})$ in $Q_v(D)$ and proceed as follows:

$$Q_v(D)$$

$$= \frac{1}{\delta^4}[4V(b_{ii}) - V(b_{ij})]^2,$$

$$= \frac{1}{\delta^4}\left\{4\left[\frac{\delta^2}{N(c-1)\lambda_4}\right.\right.$$
$$\left.\left.\times\left(\frac{(c+v-2)\lambda_4 - \lambda_2^2(v-1)}{(c+v-1)\lambda_4 - v\lambda_2^2}\right)\right] - \frac{\delta^2}{N\lambda_4}\right\}^2,$$

$$= \left\{4\left[\frac{1}{N(c-1)\lambda_4}\right.\right.$$
$$\left.\left.\times\left(\frac{(c+v-2)\lambda_4 - \lambda_2^2(v-1)}{(c+v-1)\lambda_4 - v\lambda_2^2}\right)\right] - \frac{1}{N\lambda_4}\right\}^2,$$

$$= \left\{\left[\frac{4[(c+v-2)\lambda_4 - \lambda_2^2(v-1)]}{N(c-1)\lambda_4(c+v-1)\lambda_4 - v\lambda_2^2}\right] - \frac{1}{N\lambda_4}\right\}^2,$$

$$= \left\{\frac{\begin{array}{c}4[(c+v-2)\lambda_4 - \lambda_2^2(v-1)]\\ -(c-1)[(c+v-1)\lambda_4 - v\lambda_2^2]\end{array}}{N(c-1)\lambda_4(c+v-1)\lambda_4 - v\lambda_2^2}\right\}^2,$$

$$= \left\{\frac{\begin{array}{c}4(c+v-2)\lambda_4 - 4\lambda_2^2(v-1) - (c-1)\\ (c+v-1)\lambda_4 + (c-1)v\lambda_2^2\end{array}}{N\lambda_4(c-1)[(c+v-1)\lambda_4 - v\lambda_2^2]}\right\}^2,$$

$$= \left\{\frac{\begin{array}{c}4(c+v-2)\lambda_4 - (c-1)(c+v-1)\lambda_4\\ -4\lambda_2^2(v-1) + (c-1)v\lambda_2^2\end{array}}{N\lambda_4(c-1)[(c+v-1)\lambda_4 - v\lambda_2^2]}\right\}^2,$$

$$= \left\{\frac{\begin{array}{c}[4(c+v-2) - (c-1)(c+v-1)]\lambda_4\\ -[4(v-1) + (c-1)v]\lambda_2^2\end{array}}{N\lambda_4(c-1)[(c+v-1)\lambda_4 - v\lambda_2^2]}\right\}^2,$$

on simplification we have

$$= Q_v(D)$$

$$= \left[\frac{\lambda_4[v(5-c) - (c-3)^2] - \lambda_2^2[v(c-5) + 4]}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]}\right]^2.$$

Particularly for modified condition we fix $c = 5$ and therefore we have

$$Q_v(D) = \left[\frac{-4\lambda_4 + 4\lambda_2^2}{4N\lambda_4[\lambda_4(4 + V) - V\lambda_2^2]}\right]^2,$$

$$= \left[\frac{4(\lambda_2^2 - \lambda_4)}{4N\lambda_4[\lambda_4(4 + V) - V\lambda_2^2]}\right]^2,$$

$$= \left[\frac{(\lambda_2^2 - \lambda_4)}{N\lambda_4[\lambda_4(4 + V) - V\lambda_2^2]}\right]^2.$$

For our construction technique we have m factors. We therefore replace v with m to obtain

$$Q_m(D) = \left[\frac{(\lambda_2^2 - \lambda_4)}{N\lambda_4[\lambda_4(4 + m) - m\lambda_2^2]}\right]^2,$$

where

$$N = \frac{[(e - 1)2^{t(m)} + 2b^2]^2}{(e - 2)2^{t(m)}},$$

$$\lambda_4 = \frac{2^{t(m)}(e - 2)}{N} \text{ and } \lambda_2 = \frac{2^{t(m)}(e - 1) + 2b^2}{N}.$$

but for modified rotatability $\lambda_2^2 = \lambda_4$ meaning that the numerator of the measure is zero. Hence $Q_m(D) = 0$ meaning that $Q_m(D)$ takes the value of zero if the design is rotatable and otherwise if the design in not rotatable. ∎

Example 2: Suppose we want to construct an experimental design with four factors and nine treatments. In this case $e = m = (v - 1)/2$ and $v = 2e + 1 = 9$. $C_i = [i, v - i]$ $i = 1, 2, \ldots, (v - 1)/2 = 1, 2, \ldots, e$. The experimental set becomes $C_1 = [1, 8]$, $C_2 = [2, 7]$, $C_3 = [3, 6]$, $C_4 = [4, 5]$. The C_i's, $i = 1, \ldots, 4$ are 4-[9:2:1] SDS.

Step 1

We consider the sets $C_1 = [1, 8]$, $C_2 = [2, 7]$, $C_3 = [3, 6]$, $C_4 = [4, 5]$ which are 4-[9:2:1] supplementary difference sets.

The incidence matrix produced by the above SDS is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and its mirror image obtained by replacing 0 with 1 and 1 with 0 is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Step 2

We consider the first $(v - 1)/2$ column of the above matrix, i.e.,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and this produces an array with e rows and e columns where $e = (v - 1)/2$ whose every column has one zero element and e-1 elements equal to 1.

Step 3

We consider a fractional fraction 2^{4-1} with 8 runs and superimpose one column of the fraction onto each 1 of the array. We also superimpose $2^{4-1} \times 1 = 8 \times 1$ vector of 0's into each zeros. Step three is what was referred to by Das and Narasimham (1962) as association by multiplication. A three-level design is obtained which has the form

$$\begin{bmatrix} 0 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 \end{bmatrix}$$

where 0 represents a column vector of eight 0's and $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ represent a 2^{4-1} fractional factorial.

Step 4

In order to attain rotatability we add an axial point $\pm b$ in every column of the design leading to

$$\begin{bmatrix} 0 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 \\ \pm b & 0 & 0 & 0 \\ 0 & \pm b & 0 & 0 \\ 0 & 0 & \pm b & 0 \\ 0 & 0 & 0 & \pm b \end{bmatrix}$$

Let's take the first column to be x_{1u}, second column to be x_{2u}, third column to be x_{3u} and fourth column to be x_{4u}.

The measure of rotatability developed in this article is expressed as

$$Q_m(D) = \left[\frac{(\lambda_2^2 - \lambda_4)}{N\lambda_4[\lambda_4(4+m) - m\lambda_2^2]} \right]^2,$$

where

$$N = \frac{[(e-1)2^{t(m)} + 2b^2]^2}{(e-2)2^{t(m)}}, \lambda_4 = \frac{2^{t(m)}(e-2)}{N}, \lambda_2$$

$$= \frac{2^{t(m)}(e-1) + 2b^2}{N}, \text{ and}$$

$$b^2 = \sqrt{\left[\frac{2^{t(m)}(4e-9)}{2} \right]}.$$

Therefore,

$$b^2 = \sqrt{\left[\frac{2^{t(m)}(4e-9)}{2} \right]} = \sqrt{\left[\frac{2^{4-1}(4 \times 4 - 9)}{2} \right]}$$

$$= \sqrt{\left[\frac{8(16-9)}{2} \right]} = \sqrt{28} = 5.2915,$$

$$N = \frac{[(e-1)2^{t(m)} + 2b^2]^2}{(e-2)2^{t(m)}}$$

$$= \frac{[(4-1)2^3 + 2(5.2915)]^2}{(4-2)2^3} = 74.73 \approx 75,$$

$$\lambda_4 = \frac{2^{t(m)}(e-2)}{N} = \frac{2^3(4-2)}{75} = 0.2133,$$

$$\lambda_2 = \frac{2^{t(m)}(e-1) + 2b^2}{N}$$

$$= \frac{2^3(4-1) + 2(5.2915)}{75} = 0.4611,$$

$$Q_m(D) = \left[\frac{(\lambda_2^2 - \lambda_4)}{N\lambda_4[\lambda_4(4+m) - m\lambda_2^2]} \right]^2$$

$$= \left[\frac{(0.4611^2 - 0.2133)}{75(0.2133)[0.2133(4+4) - 4(0.4611^2)]} \right]^2,$$

$$= \left[\frac{-0.0007}{13.6922} \right]^2 = 2.6 \times 10^{-9} \approx 0.$$

The measure $Q_m(D)$ takes the value of zero if the design is rotatable and otherwise if the design in not rotatable. Thus a design consisting of four factors and nine treatment constructed using supplementary difference sets exits. The design will require 75 experimental units.

The computation of the measures for different class of supplementary difference sets is tabulated in Tables A2–A10 in Appendix.

Disclosure statement

No potential conflict of interest was reported by the author.

Notes on contributor

Haron Mutai Ng'eno is a PhD student at Moi University, Kenya.

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Appendix

Table A1. Five-level modified second-order rotatable designs constructed using SDS.

SDS	$m = \dfrac{v-1}{2}$	$e = \dfrac{v-1}{2}$	$2^{t(m)}$	$f = m.2^{t(m)}$	b^2	N	n_0
3-(7:2:1)	3	3	4	12	2.4495	42	24
4-(9:2:1)	4	4	8	32	5.2915	75	35
5-(11:2:1)	5	5	$\frac{1}{2}$ repl. of $2^{t(m)} = 8$	40	6.6322	86	36
			16	80	9.3808	143	53
6-(13:2:1)	6	6	$\frac{1}{4}$ repl. of $2^{t(m)} = 8$	48	7.7459	96	36
			32	192	15.4919	285	81
7-(15:2:1)	7	7	$\frac{1}{4}$ repl. of $2^{t(m)} = 16$	112	12.3288	182	56
			64	448	24.6576	522	60
8-(17:2:1)	8	8	$\frac{1}{8}$ repl. of $2^{t(m)} = 16$	128	13.5647	202	58
9-(19:2:1)	9	9	$\frac{1}{16}$ repl. of $2^{t(m)} = 16$	144	14.6969	222	60
10-(21:2:1)	10	10	$\frac{1}{32}$ repl. of $2^{t(m)} = 16$	160	15.7480	241	61
11-(23:2:1)	11	11	$\frac{1}{64}$ repl. of $2^{t(m)} = 16$	176	16.7332	260	62

Table A2. Measure of rotatability of five-level modified SORD using 3-[7:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	2.50	0.095	0.057	3.718×10^{-4}
1.20	3.04	0.095	0.067	2.287×10^{-4}
1.40	3.92	0.095	0.081	6.913×10^{-5}
1.60	4.52	0.095	0.089	1.427×10^{-5}
1.50	5.00	0.095	0.095	0.000
1.57	5.28	0.095	0.098	4.107×10^{-6}
1.80	7.25	0.095	0.119	3.814×10^{-4}
2.00	10.00	0.095	0.145	2.969×10^{-3}
2.20	13.71	0.095	0.177	2.3521896×10^{-2}
2.40	18.59	0.095	0.216	3.182
2.60	24.85	0.095	0.263	0.115
2.80	32.73	0.095	0.318	3.739×10^{-2}
3.00	42.50	0.095	0.383	2.224×10^{-2}
3.20	54.43	0.095	0.460	1.637×10^{-2}
3.40	68.81	0.095	0.549	1.343×10^{-2}
3.60	85.98	0.095	0.652	1.169×10^{-2}
3.80	106.26	0.095	0.771	1.057×10^{-2}
4.00	130.00	0.095	0.907	9.798×10^{-3}
4.20	157.58	0.095	1.062	9.242×10^{-3}
4.40	189.40	0.095	1.237	8.829×10^{-3}
Rotatable $b = 1.57$				

Note: $m = 3, v = 7, N = 42, b = 1.57, \lambda_4 = 0.095$.

Table A3. Measure of rotatability of five-level modified SORD using 4-[9:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.63	0.213	0.120	2.262×10^{-5}
1.20	1.76	0.213	0.128	1.993×10^{-5}
1.40	1.98	0.213	0.139	1.628×10^{-5}
1.60	2.32	0.213	0.151	1.245×10^{-5}
1.80	2.81	0.213	0.165	8.283×10^{-6}
2.00	3.50	0.213	0.182	3.953×10^{-6}
2.20	4.43	0.213	0.202	5.906×10^{-7}
2.30	5.00	0.213	0.213	0.000
2.40	5.65	0.213	0.224	7.262×10^{-7}
2.60	7.21	0.213	0.250	1.082×10^{-5}
2.80	9.18	0.213	0.280	1.757×10^{-5}
3.00	11.63	0.213	0.314	1.991×10^{-4}
3.20	14.61	0.213	0.352	8.641×10^{-4}
3.40	18.20	0.213	0.395	8.441×10^{-3}
3.60	22.50	0.213	0.443	4.483×10^{-2}
3.80	27.56	0.213	0.497	3.918×10^{-3}
4.00	33.50	0.213	0.558	1.673×10^{-3}
4.20	40.40	0.213	0.625	1.049×10^{-3}
4.40	48.35	0.213	0.699	7.762×10^{-4}
Rotatable $b = 2.30$				

Note: $m = 4, v = 9, N = 75, b = 2.30, \lambda_4 = 0.21$.

Table A4. Measure of rotatability of five-level modified SORD using 5-[11:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.42	0.279	0.516	2.049×10^{-2}
1.20	1.51	0.279	0.164	8.033×10^{-6}
1.40	1.65	0.279	0.174	7.111×10^{-6}
1.60	1.88	0.279	0.186	6.010×10^{-6}
1.80	2.21	0.279	0.200	4.748×10^{-6}
2.00	2.67	0.279	0.216	3.367×10^{-6}
2.20	3.29	0.279	0.235	1.884×10^{-6}
2.40	4.10	0.279	0.256	6.064×10^{-7}
2.50	4.59	0.279	0.267	1.809×10^{-7}
2.58	5.00	0.279	0.279	0.000
2.60	5.14	0.279	0.280	1.407×10^{-9}
2.80	6.46	0.279	0.301	8.307×10^{-7}
3.00	8.08	0.279	0.338	8.970×10^{-6}
3.20	10.07	0.279	0.372	3.545×10^{-5}
3.40	12.47	0.279	0.411	1.456×10^{-4}
3.60	15.33	0.279	0.454	9.159×10^{-4}
3.80	18.71	0.279	0.501	2.378
4.00	22.67	0.279	0.554	1.958×10^{-3}
4.20	27.26	0.279	0.612	6.391×10^{-4}
4.40	32.57	0.279	0.676	3.625×10^{-4}
Rotatable $b = 2.57$				

Note: $m = 5, v = 11, N = 86, b = 2.57, \lambda_4 = 0.279$.

Table A5. Measure of rotatability of five-level modified SORD using 6-[13:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.31	0.333	0.191	4.137×10^{-6}
1.20	1.38	0.333	0.199	3.851×10^{-6}
1.40	1.49	0.333	0.209	3.491×10^{-6}
1.60	1.66	0.333	0.221	3.057×10^{-6}
1.80	1.91	0.333	0.234	2.585×10^{-6}
2.00	2.25	0.333	0.250	2.013×10^{-6}
2.20	2.71	0.333	0.268	1.394×10^{-6}
2.40	3.32	0.333	0.288	7.721×10^{-7}
2.60	4.11	0.333	0.311	2.209×10^{-7}
2.70	4.57	0.333	0.323	5.049×10^{-7}
2.78	5.00	0.333	0.333	0.000
2.80	5.09	0.333	0.336	5.101×10^{-9}
3.00	6.31	0.333	0.365	7.710×10^{-7}
3.20	7.80	0.333	0.397	4.459×10^{-6}
3.40	9.60	0.333	0.432	1.761×10^{-5}
3.60	11.75	0.333	0.472	7.623×10^{-5}
3.80	14.28	0.333	0.515	5.627×10^{-3}
4.00	17.25	0.333	0.563	2.247×10^{-2}
4.20	20.70	0.333	0.615	6.004×10^{-4}
4.40	24.68	0.333	0.672	2.282×10^{-4}
Rotatable $b = 2.78$				

Note: $m = 6(\frac{1}{4}$ replicate$), v = 13, N = 96, b = 2.78, \lambda_4 = 0.333$.

Table A6. Measure of rotatability of five-level modified SORD using 7-[15:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.23	0.440	0.290	4.443×10^{-7}
1.20	1.25	0.440	0.295	4.258×10^{-7}
1.40	1.30	0.440	0.301	4.034×10^{-7}
1.60	1.36	0.440	0.309	3.734×10^{-7}
1.80	1.46	0.440	0.317	3.434×10^{-7}
2.00	1.60	0.440	0.327	3.059×10^{-7}
2.20	1.79	0.440	0.337	2.688×10^{-7}
2.40	2.03	0.440	0.349	2.247×10^{-7}
2.60	2.34	0.440	0.362	1.784×10^{-7}
2.80	2.74	0.440	0.377	1.278×10^{-7}
3.00	3.23	0.440	0.392	8.178×10^{-8}
3.20	3.82	0.440	0.410	3.616×10^{-8}
3.40	4.54	0.440	0.428	6.604×10^{-9}
3.50	4.95	0.440	0.438	1.982×10^{-10}
3.51	5.00	0.440	0.440	0.000
3.60	5.40	0.440	0.449	4.386×10^{-9}
3.80	6.41	0.440	0.471	6.294×10^{-8}
4.00	7.60	0.440	0.495	2.495×10^{-7}
4.20	8.98	0.440	0.520	6.931×10^{-7}
4.40	10.57	0.440	0.548	1.804×10^{-6}
Rotatable $b = 3.51$				

Note: $m = 7$ ($\frac{1}{4}$ replicate), $v = 15, N = 182, b = 3.51, \lambda_4 = 0.440$.

Table A7. Measure of rotatability of five-level modified SORD using 8-[17:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.19	0.475	0.318	2.688×10^{-7}
1.20	1.21	0.475	0.323	2.585×10^{-7}
1.40	1.25	0.475	0.329	2.459×10^{-7}
1.60	1.30	0.475	0.336	2.313×10^{-7}
1.80	1.39	0.475	0.344	2.145×10^{-7}
2.00	1.50	0.475	0.353	1.955×10^{-7}
2.20	1.65	0.475	0.363	1.743×10^{-7}
2.40	1.86	0.475	0.374	1.511×10^{-7}
2.60	2.12	0.475	0.386	1.261×10^{-7}
2.80	2.45	0.475	0.400	9.776×10^{-8}
3.00	2.85	0.475	0.414	7.088×10^{-8}
3.20	3.35	0.475	0.430	4.306×10^{-8}
3.40	3.95	0.475	0.447	1.888×10^{-8}
3.60	4.67	0.475	0.466	2.262×10^{-9}
3.68	5.00	0.475	0.475	0.000
3.80	5.51	0.475	0.486	4.003×10^{-9}
4.00	6.50	0.475	0.508	4.419×10^{-8}
4.20	7.65	0.475	0.532	1.692×10^{-7}
4.40	8.98	0.475	0.557	4.719×10^{-7}
Rotatable $b = 3.68$				

Note: $m = 8$ ($\frac{1}{8}$ replicate), $v = 17, N = 202, b = 3.68, \lambda_4 = 0.475$.

Table A8. Measure of rotatability of five-level modified SORD using 9-[19:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.16	0.504	0.343	1.725×10^{-7}
1.20	1.18	0.504	0.348	1.662×10^{-7}
1.40	1.21	0.504	0.353	1.598×10^{-7}
1.60	1.26	0.504	0.359	1.523×10^{-7}
1.80	1.33	0.504	0.367	1.420×10^{-7}
2.00	1.43	0.504	0.375	1.317×10^{-7}
2.20	1.56	0.504	0.385	1.187×10^{-7}
2.40	1.74	0.504	0.395	1.057×10^{-7}
2.60	1.96	0.504	0.406	9.135×10^{-8}
2.80	2.24	0.504	0.419	7.462×10^{-8}
3.00	2.59	0.504	0.433	5.712×10^{-8}
3.20	3.02	0.504	0.447	4.058×10^{-8}
3.40	3.53	0.504	0.463	2.361×10^{-8}
3.60	4.14	0.504	0.481	8.551×10^{-9}
3.80	4.87	0.504	0.499	4.701×10^{-10}
3.84	5.00	0.504	0.504	0.000
4.00	5.71	0.504	0.519	5.079×10^{-9}
4.20	6.70	0.504	0.541	3.861×10^{-8}
4.40	7.84	0.504	0.564	1.319×10^{-7}
Rotatable $b = 3.84$				

Note: $m = 9$ ($\frac{1}{16}$ replicate), $v = 19, N = 222, b = 3.84, \lambda_4 = 0.504$.

Table A9. Measure of rotatability of five-level modified SORD using 10-[21:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.14	0.531	0.367	1.159×10^{-7}
1.20	1.16	0.531	0.371	1.127×10^{-7}
1.40	1.19	0.531	0.377	1.079×10^{-7}
1.60	1.23	0.531	0.383	1.029×10^{-7}
1.80	1.29	0.531	0.389	9.803×10^{-8}
2.00	1.38	0.531	0.389	8.297×10^{-8}
2.20	1.49	0.531	0.407	7.534×10^{-8}
2.40	1.64	0.531	0.416	6.597×10^{-8}
2.60	1.84	0.531	0.427	5.578×10^{-8}
2.80	2.09	0.531	0.439	4.488×10^{-8}
3.00	2.39	0.531	0.452	3.353×10^{-8}
3.20	2.76	0.531	0.466	2.217×10^{-8}
3.40	3.21	0.531	0.481	1.163×10^{-8}
3.60	3.75	0.531	0.497	2.997×10^{-9}
3.80	4.38	0.531	0.515	6.216×10^{-10}
3.90	4.74	0.531	0.524	0.000
3.97	5.00	0.531	0.531	5.518×10^{-11}
4.00	5.12	0.531	0.533	5.518×10^{-11}
4.20	5.99	0.531	0.553	4.169×10^{-8}
4.40	6.98	0.531	0.575	1.159×10^{-7}
Rotatable $b = 3.97$				

Note: $m = 10$ ($\frac{1}{32}$ replicate), $v = 21, N = 241, b = 3.97, \lambda_4 = 0.531$.

Table A10. Measure of rotatability of five-level modified SORD using 11-[23:2:1] SDS.

b	c	λ_4	λ_2^2	$Q_m(D)$
1.00	1.13	0.554	0.388	8.129×10^{-8}
1.20	1.14	0.554	0.392	7.914×10^{-8}
1.40	1.16	0.554	0.397	7.641×10^{-8}
1.60	1.20	0.554	0.403	7.311×10^{-8}
1.80	1.26	0.554	0.409	6.977×10^{-8}
2.00	1.33	0.554	0.418	6.469×10^{-8}
2.20	1.44	0.554	0.426	6.012×10^{-8}
2.40	1.57	0.554	0.435	5.493×10^{-8}
2.60	1.75	0.554	0.445	4.910×10^{-8}
2.80	1.96	0.554	0.457	4.207×10^{-8}
3.00	2.24	0.554	0.469	3.507×10^{-8}
3.20	2.57	0.554	0.482	2.761×10^{-8}
3.40	2.97	0.554	0.496	1.991×10^{-8}
3.60	3.44	0.554	0.511	1.233×10^{-8}
3.80	4.01	0.554	0.528	5.205×10^{-9}
4.00	4.67	0.554	0.545	7.285×10^{-10}
4.09	5.00	0.554	0.554	0.000
4.20	5.43	0.554	0.564	1.086×10^{-9}
4.40	6.32	0.554	0.584	1.219×10^{-8}
Rotatable $b = 4.09$				

Note: $m = 11$ ($\frac{1}{64}$ replicate), $v = 23, N = 260, b = 4.09, \lambda_4 = 0.554$.