Statistical Theory and Related Fields

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tstf20

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To cite this article: Sheng Chao Zhuang (2020) A hybrid model of optimal reinsurance: a discussion of 'Optimal reinsurance designs based on risk measures: a review' by Jun Cai and Yichun Chi, Statistical Theory and Related Fields, 4:1, 20-22, DOI: 10.1080/24754269.2020.1765127

To link to this article: https://doi.org/10.1080/24754269.2020.1765127

Published online: 18 May 2020.

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STATISTICAL THEORY AND RELATED FIELDS
2020, VOL. 4, NO. 1, 20–22
https://doi.org/10.1080/24754269.2020.1765127

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SHORT COMMUNICATION

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A hybrid model of optimal reinsurance: a discussion of 'Optimal reinsurance designs based on risk measures: a review' by Jun Cai and Yichun Chi

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ARTICLE HISTORY Received 28 April 2020; Revised 1 May 2020; Accepted 2 May 2020

I wholeheartedly congratulate Professors Cai and Chi (referred to as CC hereafter) for their extensive review paper that surveys the optimal reinsurance designs based on risk measures. This paper should not only facilitate a deeper understanding of the recent advances in optimal reinsurance designs, but also lead to many interesting and/or unsolved problems left for future studies.

My discussion, motivated by the need of developing a 'hybrid' model (in the sense of more than two agents), would provide a practical way to consider the optimal reinsurance problems. Notably, CC mentioned some related works in their section of concluding remarks and future studies. This discussion concentrates on one of my recent studies. To facilitate the discussion, I adopt the same notation as in CC.

In the reinsurance literature, the insurer's risk is often assumed to be exogenously given and the objective boils down to determining an optimal portion of its risk to a reinsurer. However, in practice, the risk of the insurer is directly related to the insured's claim. Motivated by this observation, Zhuang et al. (2017) assumes that the insurer's risk is endogenously determined from bargaining with an insured. In other words, in addition to determining optimal reinsurance, the insurer also needs to decide how much risk to underwrite from the insured. The problem is therefore phrased as a three-party problem involving insured, insurer and reinsurer, and with the objective of jointly analysing the optimal insurance–reinsurance designs from the insurer's perspective.

Since the model includes three agents, namely the insured, the insurer and the reinsurer, we use the notation P, I, R, to represent these agents, respectively. In a given time period, an insured with an initial wealth of W_P faces a non-negative and bounded risk X, which has a support on $[0, M]$ for some $M > 0$. We denote L^∞ as the class of bounded random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. As defined in CC, the general distortion premium principle is expressed as

$$\begin{aligned}\pi_k(Y) &= (1+\theta)\rho_{g_k}(Y)\\ &= \int_0^\infty (1+\theta) g_k(S_Y(z))\,\mathrm{d}z\\ &\qquad \text{for all non-negative } Y \in L^\infty,\end{aligned}$$

where $g_k(\cdot)$ denotes the distortion function of the agent k. We assume the insured wishes to insure a portion of its risk X with an insurer who has an initial wealth of W_I. Mathematically, the insurable risk X is decomposed into two parts: $f_I(X)$ and $X - f_I(X)$, where $f_I(X)$ represents the portion of loss that is ceded to the insurer, and $X - f_I(X)$ is the residual loss retained by the insured. We require that $f_I(\cdot)$ belongs to the following set $\mathcal{F}_{\hat{M}}$:

$$\begin{aligned}\mathcal{F}_{\hat{M}} := \{f(\cdot) : f(0) = 0,\ 0 \le f(x) - f(y) \le x - y,\\ \forall\, 0 \le y < x \le \hat{M}\}\end{aligned}$$

for $0 \le \hat{M} \le M$. Notably, this condition is consistent with the incentive compatibility condition discussed in CC as well as some recent studies on optimal insurance designs (see, e.g. Chi & Zhuang, 2020; Xu et al., 2019). Under such an insurance arrangement, the final wealth for the insured is $W_P - X + f_I(X) - \pi_I(f_I(X))$ and the final wealth for the insurer becomes $W_I - f_I(X) + \pi_I(f_I(X))$, where $\pi_I(f_I(X))$ represents the insurance premium for insuring the ceded risk $f_I(X)$. Moreover, we assume that at the time of insuring risk $f_I(X)$, the insurer wishes to reinsure its risk with a reinsurer. As a result, with the reinsurance, the risk $f_I(X)$ is further partitioned into $f_R(f_I(X))$ and $f_I(X) - f_R(f_I(X))$, where $f_R(f_I(X))$ captures the risk that is ceded to a reinsurer and $f_I(X) - f_R(f(X))$ is the net residual risk retained by the insurer. The final wealth for the insurer becomes $W_I - f_I(X) + \pi_I(f_I(X)) + f_R(f_I(X)) - \pi_R(f_R(f_I(X)))$, where $\pi_R(f_R(f_I(X)))$ corresponds to the reinsurance premium charged by the reinsurer for reinsuring the risk $f_R(f_I(X))$. In line with

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the incentive compatibility condition, we require that

$$(f_I, f_R) \in \left\{(\hat{f}_I, \hat{f}_R) : \hat{f}_I \in \mathcal{F}_M, \hat{f}_R \in \mathcal{F}_{\hat{f}_I(M)}\right\} := \mathcal{F}^2.$$

We further assume that the insured and the insurer are characterised by Yaari's dual utility. Notably, maximising the dual utility of gains is mathematically equivalent to minimising a distortion risk measure of losses. Therefore, the utilities of the insured and the insurer can be expressed as

$$U_k(W) := -\rho^{g_k}(-W) := E^{g_k}[W], \quad k \in \{P, I\}.$$

We propose baseline insurance–reinsurance model that was jointly integrating insured's decision, insurer's decision, as well as the corresponding optimal insurance premium:

$$\begin{aligned} \max_{f_I, f_R, \pi_I} \quad & E^{g_I}\Big[W_I - f_I(X) + \pi_I(f_I(X)) + f_R(f_I(X)) \\ & \quad - \pi_R(f_R(f_I(X)))\Big], \\ \text{s.t.} \quad & (f_I, f_R) \in \mathcal{F}^2, \pi_I(f_I(X)) \geq 0, \\ & E^{g_P}[W_P - X + f_I(X) - \pi_I(f_I(X))] \\ & \quad \geq E^{g_P}[W_P - X]. \end{aligned} \tag{1}$$

In the above model, the insurer's dual utility is maximised in the presence of both insurance and reinsurance. The left- and right-hand side of the second constraint measure the insured's dual utility with and without insurance. Therefore, the second constraint is the incentive condition for the insured to purchase the insurance contract. For this reason, this constraint can be viewed as the insurance participation constraint. Although problem (1) is a mathematical challenging problem, we solve it explicitly by using the technique of marginal indemnification function formulation approach (see, e.g. Assa, 2015; Cheung & Lo, 2017; Zhuang et al., 2016). This result extends the study of Cui et al. (2013) to the case that there are three agents involved in sharing the insurer's risk. In other words, we show that the classical risk-sharing models do apply to the setting with insurance and reinsurance jointly as well.

In practice, it is desirable to impose a reinsurance premium budget constraint as mentioned in CC. In what follows, we study two variants of our baseline model depending on how we incorporate the reinsurance premium budget constraint. First, we impose the premium budget by assuming that the insurer can only spend a given amount $C \geq 0$ on reinsurance:

$$\begin{aligned} \max_{f_I, f_R, \pi_I} \quad & E^{g_I}\Big[W_I - f_I(X) + \pi_I(f_I(X)) + f_R(f_I(X)) \\ & \quad - \pi_R(f_R(f_I(X)))\Big], \\ \text{s.t.} \quad & (f_I, f_R) \in \mathcal{F}^2, \pi_I(f_I(X)) \geq 0, \\ & E^{g_P}[W_P - X + f_I(X) - \pi_I(f_I(X))] \\ & \quad \geq E^{g_P}[W_P - X], \\ & \pi_R\big(f_R(f_I(X))\big) \leq C. \end{aligned} \tag{2}$$

We solve problem (2) explicitly by using the Lagrangian dual approach. The result clearly shows the effects of the reinsurance premium budget constraint. In particular, we demonstrate that the insurer provides less insurance coverage to the insured due to an upper limit on how much it could spend on reinsurance. Such a situation is aggravated by the larger risk retained by both insured and insurer.

Then, we similarly explore the baseline model by imposing the reinsurance premium budget in term of a certain percentage of the insurance premium collected. Specifically, we use the parameter $\alpha \in (0, 1]$ to capture the percentage of the insurance premium that an insurer could spend on reinsurance:

$$\begin{aligned} \max_{f_I, f_R, \pi_I} \quad & E^{g_I}\Big[W_I - f_I(X) + \pi_I(f_I(X)) + f_R(f_I(X)) \\ & \quad - \pi_R\big(f_R(f_I(X))\big)\Big], \\ \text{s.t.} \quad & (f_I, f_R) \in \mathcal{F}^2, \pi_I(f_I(X)) \geq 0, \\ & E^{g_P}[W_P - X + f_I(X) - \pi_I(f_I(X))] \\ & \quad \geq E^{g_P}[W_P - X], \\ & \pi_R\big(f_R(f_I(X))\big) \leq \alpha \pi_I(f_I(X)). \end{aligned} \tag{3}$$

Notably, the generalised baseline models (2) and (3) differ from each other merely on how we impose the reinsurance premium budget constraint. As a result, there are some similarities as well as differences on the effect of the reinsurance premium budget constraint. In particular, this budget constraint implies that the insurer's retained risk is higher for problem (3), as in problem (2). However, as opposed to problem (2), the risk that is ceded to a reinsurer decreases and that the insurance coverage $f_I(X)$ is not necessarily increasing or decreasing for problem (3).

The optimal insurance–reinsurance models that we have examined so far assume that reinsurer can only trade with the insurer. In the following, we assume that the insured can insure its risk exclusively with the insurer, or with the reinsurer, or with both. Therefore, we will need to choose a pricing function $\pi_I : L^\infty \to \mathbb{R}$ instead of the single price of the optimal contract. A plausible model that incorporates such a feature can be formulated as follows:

$$\max_{f_I, f_R, \pi_I} \quad E^{g_I}\Big[W_I - f_I(X) + \pi_I(f_I(X)) + f_R(f_I(X)) \\ - \pi_R\big(f_R(f_I(X))\big)\Big],$$

$$
\begin{aligned}
\text{s.t.}\quad & (f_I, f_R) \in \mathcal{F}^2, \pi_I(\hat{f}_I(X)) \geq 0 \quad \forall \hat{f}_I \in \mathcal{F}_M, \\
& E^{g_P}[W_P - X + \hat{f}_I(X) - \pi_I(\hat{f}_I(X))] \\
& \quad \geq E^{g_P}[W_P - X] \quad \forall \hat{f}_I \in \mathcal{F}_M, \\
& \pi_I(\hat{f}_I(X)) \leq \pi_I\big(\hat{f}_I(X) - \hat{f}_R(\hat{f}_I(X))\big) \\
& \qquad + \pi_R\big(\hat{f}_R(\hat{f}_I(X))\big) \quad \forall (\hat{f}_I, \hat{f}_R) \in \mathcal{F}^2.
\end{aligned} \tag{4}
$$

Notably, the individual rationality constraints are modified such that it holds for every insurance contract. The term $\pi_I(\hat{f}_I(X))$ on the left-hand side of the third constraint represents the insurance premium that the insurer needs to collect to provide the insurance coverage $\hat{f}_I(X)$. Suppose now the risk $\hat{f}_I(X)$ is partitioned into two portions (i.e. $\hat{f}_I(X) - \hat{f}_R(\hat{f}_I(X))$ and $\hat{f}_R(\hat{f}_I(X))$) and that these two parts of the risk are shared with the insurer and reinsurer, respectively. The right-hand side of the third constraint therefore indicates the total expense to the insured if it were to insure its risk from both insurer and reinsurer. The presence of the third constraint ensures that it is cheaper for the insured to simply trade exclusively with the insurer, even though the insured could also insure with the reinsurer. For this reason, we refer to this constraint as the competition constraint. After solving problem (4), we find that the competition constraint does not affect the optimal indemnity functions. But it lowers the insurance premium, and therefore affects the way the profits are shared between the insured and the insurer. As a consequence, the insured can pay a smaller price for the same coverage.

Such a hybrid model can be explored further in several directions. For example, it would be interesting to consider alternative objectives and premium principles (see, e.g. Balbás et al., 2009; Chi & Tan, 2013). However, the approach we used may not be readily applicable to such cases, as the marginal indemnification function formulation approach fails. In addition, we can consider a more advanced hybrid model by allowing multiple insureds (Bernard et al., 2020), multiple insurers (Asimit & Boonen, 2018) or multiple reinsurers (Boonen et al., 2016, 2018; Chi & Meng, 2014) to participate the insurance–reinsurance game.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes on contributor

Sheng Chao Zhuang is Assistant Professor of Actuarial Science in the Department of Finance at the University of Nebraska-Lincoln,US. His current main research interest is in the area of optimal (re)insurance, behavioral finance, portfolio selection and life insurance.

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