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Optimal mean-variance reinsurance and investment strategy with constraints in a non-Markovian regime-switching model

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ABSTRACT
This paper is devoted to study the proportional reinsurance/new business and investment problem under the mean-variance criterion in a continuous-time setting. The strategies are constrained in the non-negative cone and all coefficients in the model except the interest rate are stochastic processes adapted the filtration generated by a Markov chain. With the help of a backward stochastic differential equation driven by the Markov chain, we obtain the optimal strategy and optimal cost explicitly under this non-Markovian regime-switching model. The cases with one risky asset and Markov regime-switching model are considered as special cases.

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Markov chain; mean-variance problem; non-negative constraints; BSDE; regime-switching

1. Introduction

Insurers usually control their risks via some business activities, such as investing in a financial market, purchasing reinsurance, and acquiring new business. The problem of finding the optimal reinsurance/new business and investment strategies has been investigated in a vast of literature under various criteria, among which minimising ruin probability, maximising expected utility of terminal wealth, and minimising the mean-variance cost are most popular.

Browne (1995) first studied the optimal investment problem for an insurer aiming to maximise the terminal utility function and minimise the ruin probability under a diffusion risk model. Under a similar diffusion model, Promislow and Young (2005) obtained the optimal investment and quota-share reinsurance strategies minimising the probability of ruin. Yang and Zhang (2005) derived the optimal reinsurance-investment strategy maximising the expected utility of terminal wealth under a jump-diffusion model. Chen, Li, and Li (2010) studied the reinsurance-investment problem with a dynamic value-at-risk (VaR) constraint and got the optimal strategy minimising the probability of ruin via the dynamic programming technique and Lagrange multiplier method. More recent work on minimising ruin probability and maximising utility includes Yi, Li, Viens, and Zeng (2013), Liang and Bayraktar (2014) and Xu, Zhang, and Yao (2017), among others.

Since the pioneer work of Markowitz (1952), the mean-variance optimisation problem has become a key topic in modern portfolio selection theory. By embedding the original mean-variance problem into a linear-quadratic (LQ, for short) control problem, Li and Ng (2000) and Zhou and Li (2000) derived the pre-commitment optimal solution to the dynamic mean-variance problem in a multi-period model and a continuous-time model, respectively. Bäuerle (2005) studied the optimal proportional reinsurance problem under the benchmark and the mean-variance criterion. Bai and Zhang (2008) considered the optimal reinsurance/new business and investment problem with no-shorting constraint under the mean-variance criterion. They considered both classical and diffusion risk models in the Markovian framework and solved the problem through the viscosity solution to the Hamilton–Jacobi–Bellman equation. Similarly, with the help of stochastic LQ control theory and viscosity solution, Bi (2013) studied the optimal investment and reinsurance problem for an insurer under the mean-variance criterion with non-negative constraint on the strategies in a jump-diffusion financial market. By using the martingale method, Bi, Meng, and Zhang (2014) investigated the mean-variance optimal investment and reinsurance problem with bankruptcy prohibition. For more recent work on mean-variance reinsurance/new business and investment problem, we refer the reader to Shen and Zeng (2015), Zeng, Li, and Gu (2016), Wang, Wang, and Wei (2019), etc.

In this paper, we study the mean-variance reinsurance/new business and investment problem under a non-Markovian regime-switching model. It is well-known that in the Markov regime-switching model the coefficients are deterministic functions of a Markov

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chain. Due to its flexibility, the Markov regime-switching model is usually used to capture the business cycle and changes in the environment, etc. Zhang and Siu (2012) studied the optimal proportional reinsurance and investment problem with no short-selling constraint in a Markovian regime-switching models. Chen and Yam (2013) considered the mean-variance reinsurance-investment problem under a regime-switching model by using the similar method in Zhou and Yin (2003). In contrast to the Markov regime-switching model, the coefficients are stochastic processes adapted to the filtration generated by a Markov chain (or jointly by a Markov chain and a Brownian motion, see Siu & Shen, 2017) in the non-Markovian regime-switching model. The advantage of this kind of model is that it may capture the path-dependence and memory effect in the financial market, since the parameters depend on not only the current state but also the historical information of the Markov chain. Under non-Markovian regime-switching models, Shen, Wei, and Zhao (2018) investigated the mean-variance asset-liability management problem; Wang and Wei (2019) studied the mean-variance portfolio selection problem via mean-field formulation.

We assume that the insurer can purchase proportional reinsurance and access a financial market consisting of a riskless asset and multiple risky assets. The claim process of the insurer and the price processes of the risky assets are correlated and modelled by drifted and geometric Brownian motions, respectively. The coefficients in the model depend on the historical information of a Markov chain. Similar to Bai and Zhang (2008), we impose the non-negative constraint on the retention level and the investment strategy. Since our model is non-Markovian, the viscosity method used in their paper fails. Instead, we follow the method proposed by Hu and Zhou (2005) which studied a stochastic LQ control problem with control constrained in a cone. By using a backward stochastic differential equation (BSDE, for short) driven by the Markov chain, we obtain the optimal strategy and efficient frontier in closed-form. We also study the Markov regime-switching model as a special case, where the BSDE degenerates to a system of ordinary differential equations (ODE, for short). It is worthy of noting that under the Markov regime-switching model, Chen and Yam (2013) gave a condition under which the optimal reinsurance and investment strategy without constraints are indeed non-negative. We show that our results are consistent with theirs.

The remainder of this paper is organised as follows. In Section 2, we introduce some notations and formulate the mean-variance problem. In Section 3, we show the main results of the paper. In Section 4, we consider two special cases with one risky asset and Markovian regime-switching model. Finally, Section 5 concludes the paper.

2. The model

2.1. Notation

Let $[0, T]$ be a fixed time duration, where $0 < T < +\infty$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space on which an $(n+1)$-dimensional standard Brownian motion $\boldsymbol{W}(\cdot) := (W_0(\cdot), W_1(\cdot), W_2(\cdot), \ldots, W_n(\cdot))^\top$ and an irreducible continuous-time Markov chain $\boldsymbol{\alpha}(\cdot)$ with finite states are defined. Here, the filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{0\le t\le T}$ is the augmentation under $\mathbb{P}$ of the natural filtration generated by $\boldsymbol{W}(\cdot)$ and $\boldsymbol{\alpha}(\cdot)$. We also assume that the Brownian motion and the Markov chain are independent of each other.

Without loss of generality, let $\mathcal{M} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ be the state pace of $\boldsymbol{\alpha}(\cdot)$, where $\boldsymbol{e}_i = (0, \ldots, 1, \ldots, 0)^\top$, $i = 1, \ldots, m$ are the ith unit column vectors in $\mathbb{R}^m$. Let $\boldsymbol{Q}(t) = (q_{ij}(t))_{m\times m}$ be the Q-matrix of the Markov chain $\boldsymbol{\alpha}(t)$ at time t. We assume that the entries in $\boldsymbol{Q}(\cdot)$ are uniformly bounded and continuous. Denote by $\mathbb{G} \equiv \{\mathcal{G}_t\}_{0\le t\le T}$ the augmentation under $\mathbb{P}$ of the natural filtration generated by the Markov chain $\boldsymbol{\alpha}(\cdot)$.

From Elliott, Aggoun, and Moore (2008, Appendix B), we have the following semi-martingale representation of the Markov chain $\boldsymbol{\alpha}(\cdot)$:

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(0) + \int_0^t \boldsymbol{Q}^\top(s)\boldsymbol{\alpha}(s)\,\mathrm{d}s + \boldsymbol{M}(t), \qquad (1)$$

where $\boldsymbol{M}(\cdot)$ is a martingale valued in $\mathbb{R}^m$.

Similar to Cohen and Elliott (2008), we define

$$\begin{aligned}\boldsymbol{\psi}(t) := \operatorname{diag}\left(\boldsymbol{Q}^\top(t)\boldsymbol{\alpha}(t-)\right) - \boldsymbol{Q}^\top(t)\operatorname{diag}(\boldsymbol{\alpha}(t-)) \\ - \operatorname{diag}(\boldsymbol{\alpha}(t-))\,\boldsymbol{Q}(t).\end{aligned}$$

For a matrix $\boldsymbol{A}$ with proper dimension, we denote $\|\boldsymbol{A}\|^2_{\boldsymbol{\alpha}(t-)} := \mathrm{Tr}(\boldsymbol{A}^\top\boldsymbol{\psi}(t)\boldsymbol{A})$.

Throughout this paper, we will use the same notations as in Hu and Zhou (2005). We list here for the reader's convenience. Denote the inverse of a non-singular square matrix $\boldsymbol{A}$ by $\boldsymbol{A}^{-1}$ and the n-dimensional identity matrix by $\boldsymbol{I}_n$. The transpose and the norm of any vector or matrix $\boldsymbol{A}$ are denoted by $\boldsymbol{A}^\top$ and $|\boldsymbol{A}| = \sqrt{\sum_{i,j} a_{ij}^2}$, respectively. If $\boldsymbol{A} \in \mathbb{R}_+^{k\times l}$, it means that $\boldsymbol{A}$ is a $k\times l$-dimensional real matrix in which all entries are *non-negative*. We denote the set of symmetric $n\times n$ (square) matrices by $\mathbb{S}^n$. We list the following spaces about random variables or stochastic processes involved in this paper. For $\mathbb{H} = \mathbb{F}, \mathbb{G}$, a positive integral number k, $\mathscr{R} = \mathbb{R}^k, \mathbb{R}_+^k, \mathbb{R}_+^k \setminus \{0\}, \mathbb{S}^n$, etc., let

$$\begin{aligned}L^2_{\mathbb{H}}(s,t;\mathscr{R}) := \Big\{\phi : \Omega\times[s,t] \to \mathscr{R} \,\Big|\, \phi(\cdot) \\ \text{is } \mathbb{H}\text{-adapted and } \mathbb{E}\left[\int_s^t |\phi(\tau)|^2\,\mathrm{d}\tau\right] < \infty\Big\},\end{aligned}$$

$$S^2_{\mathbb{H}}(s,t;\mathscr{R}) := \Big\{\phi : \Omega\times[s,t] \to \mathscr{R} \,\Big|\, \phi(\cdot)$$

is $\mathbb{H}$-adapted, RCLL and

$$\mathbb{E}\left[\sup_{\tau\in[s,t]}|\phi(\tau)|^2\right]<\infty\Bigg\},$$

$$L^2_{\mathbb{G},\boldsymbol{\alpha}}(s,t;\mathscr{R}):=\Bigg\{\phi:\Omega\times[s,t]\to\mathscr{R}\Big|\,\phi(\cdot)$$

$$\text{is } \mathbb{H}\text{-adapted and } \mathbb{E}\left[\int_s^t\|\phi(\tau)\|^2_{\boldsymbol{\alpha}(\tau-)}\,\mathrm{d}\tau\right]<\infty\Bigg\},$$

$$L^\infty_{\mathbb{H}}([s,t];\mathscr{R}):=\Bigg\{\phi:\Omega\times[s,t]\to\mathscr{R}\Big|\,\phi(\cdot)$$

$$\text{is } \mathbb{H}\text{-adapted and } \operatorname*{ess\,sup}_{(\tau,\omega)\in[s,t]\times\Omega}|\phi(\tau)|<\infty\Bigg\},$$

$$L^2_{\mathbb{H}}(\Omega;C([s,t];\mathscr{R})):=\Bigg\{\phi:\Omega\times[s,t]\to\mathscr{R}\Big|\,\phi(\cdot)$$

is $\mathbb{H}$-adapted, continuous and

$$\mathbb{E}\left[\sup_{\tau\in[s,t]}|\phi(\tau)|^2\right]<\infty\Bigg\}.$$

$\boldsymbol{A}\in L^2_{\mathbb{H}}(s,t;\mathbb{S}^k)$ is called uniformly positive definite, if there exists a deterministic constant $c>0$ such that $\boldsymbol{A}(\tau,\omega)>c\boldsymbol{I}_n$ for a.e. $\tau\in[s,t]$ and $\mathbb{P}$-a.s.. For any real number we define $x\vee y:=\max\{x,y\}$, $x\wedge y:=\min\{x,y\}$, and especially, $x^+:=x\vee 0$ and $x^-:=-\{x\wedge 0\}$.

2.2. Mean-variance reinsurance and investment problem

Assume that the insurer is allowed to invest the surplus into a financial market, consisting of a risk-free asset and n risky assets. The price of the risk-free asset $S_0(\cdot)$ satisfies

$$\begin{aligned}\mathrm{d}S_0(t)&=r(t)S_0(t)\,\mathrm{d}t,\quad t\in[0,T],\\ S_0(0)&=s_0>0,\end{aligned}\tag{2}$$

where the interest rate $r(\cdot)>0$ is a deterministic, uniformly bounded, scalar-valued function. For $k=1,2,\dots,n$, the price of the kth risky asset $S_k(\cdot)$ is given by

$$\begin{aligned}\mathrm{d}S_k(t)&=S_k(t)\left[\mu_k(t)\,\mathrm{d}t+\sum_{l=1}^n\sigma_{kl}(t)\,\mathrm{d}W_l(t)\right],\\ &\quad t\in[0,T],\\ S_k(0)&=s_k>0,\end{aligned}\tag{3}$$

where $\mu_k(\cdot)(>r(\cdot))$ is the expected return rate of the kth risky asset and $\boldsymbol{\sigma}_k(\cdot):=(\sigma_{k1}(\cdot),\dots,\sigma_{kn}(\cdot))^\top\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}$ is the volatility rate. $\boldsymbol{\beta}(t):=(\mu_1(t)-r(t),\dots,\mu_n(t)-r(t))^\top$ is usually called the risk premium vector at time t. Usually, $\boldsymbol{\beta}(\cdot)\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}$ are more popular and reasonable for every investor or insurer. For convenience, denote by $\boldsymbol{\sigma}(\cdot):=(\boldsymbol{\sigma}_1(\cdot),\dots,\boldsymbol{\sigma}_n(\cdot))^\top=(\sigma_{kl}(\cdot))_{n\times n}\in\mathbb{R}^{n\times n}$ in following.

Following Promislow and Young (2005), we model the claim process $C(t)$ according to a Brownian motion with drift as

$$\mathrm{d}C(t)=a(t)\,\mathrm{d}t-\overline{\boldsymbol{b}}^\top(t)\,\mathrm{d}\boldsymbol{W}(t),\tag{4}$$

where $a(\cdot)$, $\overline{\boldsymbol{b}}(\cdot):=(b_0(\cdot),b_1(\cdot),\dots,b_n(\cdot))^\top$ are bounded $\mathbb{G}$-adapted stochastic processes. We assume that $a(\cdot)>0$, $b_0(\cdot)>0$, $\boldsymbol{b}(\cdot):=(b_1(\cdot),\dots,b_n(\cdot))^\top\in\mathbb{R}^n_+$. In this paper, the claim process $C(\cdot)$ and the risky assets are correlated while they are independent to each other in Chen and Yam (2013).[1]

We assume that the premium is paid continuously at rate $c_0(\cdot)=(1+\theta(\cdot))a(\cdot)$ with safety loading $\theta(\cdot)>0$. Then before introducing reinsurance and investment, the surplus process $U(\cdot)$ is given by

$$\begin{aligned}\mathrm{d}U(t)&=c_0(t)\,\mathrm{d}t-\mathrm{d}C(t)\\ &=\theta(t)a(t)\,\mathrm{d}t+\overline{\boldsymbol{b}}^\top(t)\,\mathrm{d}\boldsymbol{W}(t).\end{aligned}\tag{5}$$

To control the original insurance risk, we assume the insurer can purchase proportional reinsurance or acquire new business as described in Bäuerle (2005) and Bai and Zhang (2008). Let the retention level for the original insurer/ceded company at time t be $q(t)\in[0,1)$. Then for the claim Y arriving at time t, the part of the claim the insurer pays is $q(t)Y$ and that paid by the reinsurance/ceded company is $(1-q(t))Y$. In this paper, we only consider the cheap reinsurance. Then, the insurer pays reinsurance premiums continuously at rate $c_1(t)=(1+\theta(t))a(t)(1-q(t))$. After the purchase of reinsurance, the surplus process $U(t)$ becomes

$$\begin{aligned}\mathrm{d}U(t)&=c_0(t)\,\mathrm{d}t-q(t)\,\mathrm{d}C(t)-c_1(t)\,\mathrm{d}t\\ &=a(t)\theta(t)q(t)\,\mathrm{d}t+q(t)\overline{\boldsymbol{b}}^\top(t)\,\mathrm{d}\boldsymbol{W}(t).\end{aligned}\tag{6}$$

Since $q(t)\in[1,+\infty)$ can be interpreted as acquiring new business, we restrict the reinsurance strategy $q(t)\in[0,+\infty)$ in this paper.

Let $\pi_k(t)$ be the amount of the insurer's wealth invested in the kth risky asset at time t. In this paper, short-selling of the risky assets is not allowed, i.e. it must be satisfied that $\pi_k(t)\geq 0, t\in[0,T]$, $k=1,\dots,n$. The process $\boldsymbol{\pi}(\cdot):=(\pi_1(\cdot),\dots,\pi_n(\cdot))^\top\in\mathbb{R}^n_+$ is called an investment portfolio of the insurer at the risk financial market. Then the joint strategy of the reinsurance and risk investment can be described by a $(n+1)$-dimensional stochastic process, denoted by $\boldsymbol{u}^\top(\cdot):=$

[1] For example, Credit Default Swap (CDS) is a popular credit derivative to enhance the credit ratings of the reference risky assets. Thus, the claim processes of insurers providing CDS protections are related to the financial risks.

$(q(\cdot),\boldsymbol{\pi}^\top(\cdot))$. Let $X(\cdot) := X^{\boldsymbol{u}}(\cdot) := X^{(q,\boldsymbol{\pi}^\top)^\top}(\cdot)$ be the wealth of the insurer, who adopts the reinsurance strategy $q(\cdot)$ and the investment portfolio $\boldsymbol{\pi}(\cdot)$. Given an investment portfolio $\boldsymbol{\pi}(\cdot)$, the amount of the insurer's wealth invested in the free-risk asset can be determined by $\pi_0(\cdot) := X(\cdot) - \sum_{k=1}^n \pi_k(\cdot)$. Therefore, for an initial wealth $x_0 > 0$, the dynamics of the wealth process $X(\cdot)$ is given by the following stochastic differential equation:

$$
\begin{aligned}
\mathrm{d}X(t) &= \Big(r(t)X(t) + \boldsymbol{u}^\top(t)\boldsymbol{B}(t)\Big)\,\mathrm{d}t \\
&\quad + \boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t), \quad t \in [0,T], \\
X(0) &= x_0 > 0,
\end{aligned}
\tag{7}
$$

where

$$
\boldsymbol{\Sigma}(t) := \begin{pmatrix} b_0(t) & \boldsymbol{b}^\top(t) \\ \boldsymbol{0} & \boldsymbol{\sigma}(t) \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}
$$

$$
\text{and} \quad \boldsymbol{B}(t) := \Big(a(t)\theta(t), \boldsymbol{\beta}^\top(t)\Big)^\top \in \mathbb{R}^{n+1}.
$$

Assumption 2.1: $\theta(\cdot), a(\cdot), b_0(\cdot), \mu_k(\cdot), \sigma_{kl}(\cdot) \in L^\infty_{\mathbb{G}}([0,T];\mathbb{R}_+ \setminus \{0\})$, $b_k(\cdot) \in L^\infty_{\mathbb{G}}([0,T];\mathbb{R}_+)$, $k,l = 1,\ldots,n$, *are* $\mathrm{d}t \times \mathrm{d}\mathbb{P}$*-a.s. predictable processes.* $\boldsymbol{\sigma}(\cdot)$ *is uniformly non-degenerate, i.e. there exists a constant* $\delta > 0$ *such that*

$$
\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t) \geq \delta \boldsymbol{I}_n, \quad \forall\, t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{8}
$$

Remark 2.1: By some elementary matrix operations, we can show the following results under Assumption 2.1:

(1) For all $t \in [0,T]$, $\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)$ and $\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)$ are positive definite;
(2) There are two positive constants $\hat{\rho}$ and $\check{\rho}$ such that for a.e. $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\hat{\rho}\,|\boldsymbol{v}|^2 \leq \boldsymbol{v}^\top \boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v} \leq \check{\rho}\,|\boldsymbol{v}|^2, \\
&\quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n+1}.
\end{aligned}
\tag{9}
$$

Definition 2.2: A strategy $\boldsymbol{u}(\cdot) = (q(\cdot),\boldsymbol{\pi}^\top(\cdot))^\top$ is *admissible* if

$$
\Big(q(\cdot), \boldsymbol{\pi}^\top(\cdot)\Big)^\top \in L^2_{\mathbb{F}}(0,T;\mathbb{R}_+) \times L^2_{\mathbb{F}}\big(0,T;\mathbb{R}^n_+\big).
$$

We denote by $\mathcal{A}$ the set of all admissible strategies.

Similar to Hu and Zhou (2005), we consider the following mean-variance optimisation problem.

Definition 2.3: The mean-variance reinsurance and investment problem is a constrained stochastic optimisation problem such that for any given $z \geq x_0 e^{\int_0^T r(t)\,\mathrm{d}t}$:

$$
\begin{aligned}
&\text{minmize} \quad J_{MV}(x_0, \boldsymbol{u}(\cdot)) := \mathbb{E}\left[(X(T)-z)^2\right] \\
&\qquad = \mathbb{E}\left[(X(T))^2\right] - z^2, \\
&\text{subject to} \quad \mathbb{E}X(T) = z, \quad \boldsymbol{u}(\cdot) \in \mathcal{A}.
\end{aligned}
\tag{10}
$$

Moreover, the problem is called *feasible* if there is at least one strategy satisfying all the constraints of (10). The problem is called *finite* if it is feasible and the infimum $J_{MV}(x_0,\boldsymbol{u}(\cdot))$ is finite.

Remark 2.2: The restriction of the targeted payoff $z \geq x_0 e^{\int_0^T r(t)\,\mathrm{d}t}$ is natural as the latter can always be achieved by putting all the money in the bank and conducting proportional reinsurance policies with the retention ratio $q(t) = 0$ for all $t \in [0,T]$ (i.e. the insurer transfers all of the claim to the reinsurer).

Similar to Hu and Zhou (2005, Theorem 6.1), if we choose $\boldsymbol{v}^\top(\cdot) = (a(\cdot)\theta(\cdot), \boldsymbol{0})$ and adopt a family of admissible strategies $\boldsymbol{u}^\delta(\cdot) := \delta\boldsymbol{v}(\cdot), \delta \geq 0$, by adjusting the coefficient $\delta \geq 0$, it is easy to get the following proposition about the feasibility of the problem.

Proposition 2.4: *The mean-variance reinsurance-investment problem* (10) *is always feasible for every* $z \in [x_0 e^{\int_0^T r(t)\,\mathrm{d}t}, +\infty)$.

3. Solution to the problem

3.1. Preliminaries

In this section, some mathematical preliminaries are presented. In the subsequent analysis, a vital technical tool is Tanaka's formula.

Lemma 3.1 (Tanaka's formula): *Let* $X(t)$ *be a continuous semi-martingale. Then*

$$
\begin{aligned}
\mathrm{d}X^+(t) &= 1_{(X(t)>0)}\,\mathrm{d}X(t) + \tfrac{1}{2}\,\mathrm{d}L(t), \\
\mathrm{d}X^-(t) &= -1_{(X(t)\leq 0)}\,\mathrm{d}X(t) + \tfrac{1}{2}\,\mathrm{d}L(t),
\end{aligned}
$$

where $L(\cdot)$ *is an increasing continuous process, called the local time of* $X(\cdot)$ *at 0, satisfying*

$$
\int_0^t |X(s)|\,\mathrm{d}L(s) = 0, \quad \mathbb{P}\text{-a.s.} \tag{11}
$$

For $(t, \boldsymbol{v}(t)) \in [0,T] \times \mathbb{R}^{n+1}$, define

$$
\begin{aligned}
H_+(t,\boldsymbol{v}(t)) &:= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) \\
&\quad + 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t), \\
H_-(t,\boldsymbol{v}(t)) &:= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) \\
&\quad - 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t),
\end{aligned}
\tag{12}
$$

Let

$$H_+^*(t) := \min_{\boldsymbol{v}\in\mathbb{R}_+^{n+1}} H_+(t,\boldsymbol{v}(t)),$$
$$H_-^*(t) := \min_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1}} H_-(t,\boldsymbol{v}(t)). \tag{13}$$

$$\boldsymbol{v}_+(t) := \left(\zeta_+(t), \boldsymbol{\xi}_+^\top(t)\right)^\top := \arg\min_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1}} H_+(t,\boldsymbol{v}(t)),$$
$$\boldsymbol{v}_-(t) := \left(\zeta_-(t), \boldsymbol{\xi}_-^\top(t)\right)^\top := \arg\min_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1}} H_-(t,\boldsymbol{v}(t)), \tag{14}$$

where $\zeta_+(\cdot),\zeta_-(\cdot)\in\mathbb{R}_+$ and $\boldsymbol{\xi}_+(\cdot),\boldsymbol{\xi}_-(\cdot)\in\mathbb{R}_+^n$. For the convenience of analysis, $\boldsymbol{v}(\cdot)$ is sometimes partitioned as $\boldsymbol{v}(\cdot)=(\kappa(\cdot),\boldsymbol{v}_1^\top(\cdot))^\top$ where $\kappa(\cdot)\in\mathbb{R}_+$ and $\boldsymbol{v}_1(\cdot)=(v_1(\cdot),\dots,v_n(\cdot))^\top\in\mathbb{R}_+^n$.

Lemma 3.2: *Under Assumption* 2.1. *we have for a.e.* $t\in[0,T]$ *and* $\mathbb{P}$*-a.s.*

$$H_+^*(t) \equiv H_+(t,\boldsymbol{v}_+(t)) = 0, \quad \boldsymbol{v}_+(t)=\boldsymbol{0} \tag{15}$$

and

$$-H_0(t) \le H_-^*(t) \equiv \inf_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1}\setminus\{\boldsymbol{0}\},|\boldsymbol{v}(t)|\le\frac{2C_1}{\hat\rho}} H_-(t,\boldsymbol{v}(t))$$
$$\le -\frac{|B(t)|^2}{\check\rho}, \tag{16}$$

where $H_0(t) := \boldsymbol{B}^\top(t)(\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t))^{-1}\boldsymbol{B}(t)$ *and*

$$C_1 := (n+1)^{1/2} \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} \{a(t)\theta(t), \mu_1(t)-r(t), \dots, \mu_n(t)-r(t)\}.$$

Proof: Since $\boldsymbol{B}(t)\in\mathbb{R}_+^{n+1}\setminus\{\boldsymbol{0}\}$ and $\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)$ is positive definite (see Remark 2.1), it is easy to find that $H_+^*(t)=\min_{\boldsymbol{v}\in\mathbb{R}_+^{n+1}} H_+(t,\boldsymbol{v})\ge 0$ and the equality holds if and only if $\boldsymbol{v}(t)=\boldsymbol{0}$. Therefore (15) is obtained.

For the second part of the lemma, firstly by noting $\boldsymbol{B}(\cdot)\in\mathbb{R}_+^{n+1}\setminus\{\boldsymbol{0}\}$, $\boldsymbol{v}(\cdot)\in\mathbb{R}_+^{n+1}$ and (9), we obtain for a.e. $t\in[0,T]$ and $\mathbb{P}$-a.s.,

$$\begin{aligned}\boldsymbol{v}^\top(t)\boldsymbol{B}(t) &= \left|\boldsymbol{v}^\top(t)\boldsymbol{B}(t)\right| \\ &= \left|a(t)\theta(t)\kappa(t) + \sum_{l=1}^n (\mu_l(t)-r(t))\,v_l(t)\right| \\ &\le \sqrt{\begin{array}{c}(n+1)\left[a^2(t)\theta^2(t)\kappa^2(t)\right.\\ \left.+\sum_{l=1}^n(\mu_l(t)-r(t))^2 v_l^2(t)\right]\end{array}} \\ &\le C_1|\boldsymbol{v}(t)|,\end{aligned}$$

where the last inequality is from that

$$C_1 = (n+1)^{1/2} \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} \{a(t)\theta(t), \mu_1(t)-r(t), \dots, \mu_n(t)-r(t)\} > 0.$$

If $|\boldsymbol{v}(t)|>2C_1/\hat\rho$, we have for a.e. $t\in[0,T]$ and $\mathbb{P}$-a.s.,

$$\begin{aligned}H_-(t,\boldsymbol{v}(t)) &= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) - 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t) \\ &= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) - 2\left|\boldsymbol{v}^\top(t)\boldsymbol{B}(t)\right| \\ &\ge \hat\rho\left(|\boldsymbol{v}(t)| - \frac{2C_1}{\hat\rho}\right)|\boldsymbol{v}(t)| > 0\end{aligned}$$

and

$$\inf_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1},|\boldsymbol{v}(t)|>2C_1/\hat\rho} H_-(t,\boldsymbol{v}(t)) \ge \inf_{|\boldsymbol{v}(t)|>2C_1/\hat\rho} H_-(t,\boldsymbol{v}(t)) > 0.$$

But $H_-^*(t)\le H_-(t,\boldsymbol{0})=0$. Thus the minimum value $H_-^*(t)$ will be obtained somewhere in the bounded range $\{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1}||\boldsymbol{v}(t)|\le 2C_1/\hat\rho\}$, that is to say

$$H_-^*(t) \equiv \inf_{\boldsymbol{v}(t)\in\mathbb{R}_+^{n+1},|\boldsymbol{v}(t)|\le 2C_1/\hat\rho} H_-(t,\boldsymbol{v}(t)).$$

Since

$$\begin{aligned}H_-(t,\boldsymbol{v}(t)) &= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) - 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t) \\ &= \left|\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) - \boldsymbol{\Sigma}^{-1}(t)\boldsymbol{B}^\top(t)\right|^2 - H_0(t) \\ &\ge -H_0(t), \quad \text{for } \boldsymbol{v}(t)\in\mathbb{R}^{n+1},\end{aligned} \tag{17}$$

it is not difficult to get that $H_-^*(t)\ge -H_0(t)$. Moreover, by (9), we have for a.e. $t\in[0,T]$ and $\mathbb{P}$-a.s.

$$\begin{aligned}H_-(t,\boldsymbol{v}(t)) &= \boldsymbol{v}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{v}(t) - 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t) \\ &\le \check\rho\boldsymbol{v}^\top(t)\boldsymbol{v}(t) - 2\boldsymbol{v}^\top(t)\boldsymbol{B}(t) \\ &= \check\rho\left|\boldsymbol{v}^\top(t) - \frac{\boldsymbol{B}^\top(t)}{\check\rho}\right|^2 - \frac{|\boldsymbol{B}(t)|^2}{\check\rho}\end{aligned}$$

Therefore

$$\begin{aligned}H_-^*(t) &= \min_{\boldsymbol{v}\in\mathbb{R}_+^{n+1}} H_-(t,\boldsymbol{v}(t)) \le H_-\left(t,\frac{\boldsymbol{B}(t)}{\check\rho}\right) \\ &= -\frac{|\boldsymbol{B}(t)|^2}{\check\rho} < 0.\end{aligned} \tag{18}$$

Moreover, considering $H_-(t,\boldsymbol{v}(t))|_{\boldsymbol{v}(t)=\boldsymbol{0}}=0$, we obtain (16). Consequently, by recalling Assumption 2.1, $H_-^*(t)$ is finite. ■

Remark 3.1: If the claim process is independent of the risky assets, i.e. $\boldsymbol{b}(t)=\mathbf{0}$, it is easy to see

$$\begin{aligned}H_{-}(t,\boldsymbol{v}(t)) &= q^2(t)\left(\kappa(t)-\frac{a(t)\theta(t)}{q(t)}\right)^2\\&\quad+\left|\left[\boldsymbol{v}^\top(t)-\boldsymbol{\beta}(t)\left(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\right)^{-1}\right]\boldsymbol{\sigma}(t)\right|^2\\&\quad-\frac{a^2(t)\theta^2(t)}{q^2(t)}-\boldsymbol{\beta}^\top(t)\left(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\beta}(t)\\&\geq-\left[\frac{a^2(t)\theta^2(t)}{q^2(t)}+\boldsymbol{\beta}^\top(t)\left(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\beta}(t)\right].\end{aligned}$$

Then

$$\begin{aligned}H_{-}^{*}(t) &= -\left[\frac{a^2(t)\theta^2(t)}{q^2(t)}\right.\\&\qquad\left.+\boldsymbol{\beta}^\top(t)\left(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\beta}(t)\right],\\ \boldsymbol{v}_{-}(t) &= \left(\zeta_{-}(t),\boldsymbol{\xi}_{-}(t)\right)\\&=\left(\frac{a(t)\theta(t)}{q(t)},\left(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\beta}(t)\right).\end{aligned}$$

Given $\boldsymbol{v}_{-}(\cdot)$, we consider the SDE

$$\begin{aligned}\mathrm{d}Y(t) &= \left\{r(t)Y(t)+Y^{-}(t)\boldsymbol{v}_{-}^\top(t)\boldsymbol{B}(t)\right\}\mathrm{d}t\\&\quad+Y^{-}(t)\boldsymbol{v}_{-}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t),\quad t\in[0,T],\\ Y(0)&=y_0.\end{aligned}\tag{19}$$

Lemma 3.3: *SDE* (19) *has a unique solution*

$$\begin{aligned}Y^*(t) &= y_0^+\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}-y_0^-\phi(t)\,\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}\\&\in L^2_{\mathbb{F}}\left(\Omega;C(0,T;\mathbb{R})\right),\end{aligned}\tag{20}$$

where

$$\begin{aligned}\phi(t) &:= \exp\left\{\int_0^t\left(\boldsymbol{v}_{-}^\top(s)\boldsymbol{B}(s)\right.\right.\\&\qquad\left.-\frac{1}{2}\boldsymbol{v}_{-}^\top(s)\boldsymbol{\Sigma}(s)\boldsymbol{\Sigma}^\top(s)\boldsymbol{v}_{-}(s)\right)\mathrm{d}s\\&\qquad\left.+\int_0^t\boldsymbol{v}_{-}^\top(s)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s)\right\}.\end{aligned}\tag{21}$$

Proof: Firstly, we introduce two linear SDEs as follows

$$\begin{aligned}\mathrm{d}Y_{+}(t)&=r(t)Y_{+}(t)\,\mathrm{d}t,\quad t\in[0,T],\\ Y_{+}(0)&=y_0^+,\end{aligned}\tag{22}$$

and

$$\begin{aligned}\mathrm{d}Y_{-}(t) &= \left(r(t)-\boldsymbol{v}_{-}^\top(t)\boldsymbol{B}(t)\right)Y_{-}(t)\,\mathrm{d}t\\&\quad-Y_{-}(t)\boldsymbol{v}_{-}^\top(t)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(t),\quad t\in[0,T],\\ Y_{-}(0)&=y_0^-.\end{aligned}\tag{23}$$

It is easy to get the unique solution to (22)

$$Y_{+}^{*}(t):=y_0^+\mathrm{e}^{\int_0^t r(s)\mathrm{d}s}.$$

Considering Assumption 2.1, it is easy to obtain that $r(\cdot)-\boldsymbol{v}_{-}^\top(\cdot)\boldsymbol{B}(\cdot)\in L^\infty_{\mathbb{G}}(0,T;\mathbb{R})$ and $\boldsymbol{v}_{-}^\top(\cdot)\boldsymbol{\Sigma}(\cdot)\in L^\infty_{\mathbb{G}}(0,T;\mathbb{R})$. It is well-known that the unique continuous $\mathbb{F}$-adapted solution to (23) is given by

$$\begin{aligned}Y_{-}^{*}(t) &:= y_0^-\exp\left\{\int_0^t\left[r(s)+\boldsymbol{v}_{-}^\top(s)\boldsymbol{B}(s)\right.\right.\\&\qquad\left.-\frac{1}{2}\boldsymbol{v}_{-}^\top(s)\boldsymbol{\Sigma}(s)\boldsymbol{\Sigma}^\top(s)\boldsymbol{v}_{-}(s)\right]\mathrm{d}s\\&\qquad\left.+\int_0^t\boldsymbol{v}_{-}^\top(s)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s)\right\}.\end{aligned}\tag{24}$$

Define

$$Y^*(t):=Y_{+}^{*}(t)-Y_{-}^{*}(t).$$

The fact that $Y_{+}^{*}(t)\geq 0$, $Y_{-}^{*}(t)\geq 0$ and $Y_{+}^{*}(t)Y_{-}^{*}(t)=0$ implies that

$$Y^{*+}(t)=Y_{+}^{*}(t),\quad Y^{*-}(t)=Y_{-}^{*}(t),\quad\forall\, t\in[0,T].$$

By applying Itô formula to $Y^*(t):=Y_{+}^{*}(t)-Y_{-}^{*}(t)$, the resulted SDE is exactly (19).

If we can prove the uniqueness of the solution, then the proof of the lemma will be accomplished. To this end, first suppose that $Y_1^*(\cdot)$ and $Y_2^*(\cdot)$ are two continuous adapted solution to SDE (19). Denote $\hat{Y}(\cdot):=Y_1^*(\cdot)-Y_2^*(\cdot)$. Now in order to generate a linear SDE for $\hat{Y}(\cdot)$, the following procedure was implemented. Set

$$\begin{aligned}\gamma_{+}(t)&:=\frac{Y_1^{*+}(t)-Y_2^{*+}(t)}{Y_1^*(t)-Y_2^*(t)}1_{\{Y_1^*(t)\neq Y_2^*(t)\}},\\ \gamma_{-}(t)&:=\frac{Y_1^{*-}(t)-Y_2^{*-}(t)}{Y_1^*(t)-Y_2^*(t)}1_{\{Y_1^*(t)\neq Y_2^*(t)\}}.\end{aligned}$$

Then $\hat{Y}(\cdot)$ is a continuous adapted solution to the following linear SDE:

$$\begin{aligned}\mathrm{d}\hat{Y}(t) &= \left(r(t)+\gamma_{-}(t)\boldsymbol{v}_{-}^\top(s)\boldsymbol{B}(s)\right)\hat{Y}(t)\,\mathrm{d}t\\&\quad+\gamma_{-}(t)\boldsymbol{v}_{-}^\top(s)\boldsymbol{\Sigma}(s)\hat{Y}(t)\,\mathrm{d}\boldsymbol{W}(t),\\&\quad t\in[0,T],\\ \hat{Y}(0)&=0.\end{aligned}$$

Therefore, $\hat{Y}(\cdot)=0$, which implies that $Y^*(\cdot)=Y_{+}^{*}(\cdot)-Y_{-}^{*}(\cdot)$ is the unique the solution to SDE (19). ■

Next, we introduce the following backward stochastic differential equation driven by the Markov chain $\boldsymbol{\alpha}(\cdot)$

$$\begin{aligned}\mathrm{d}P(t)&=f(t,P(t))\,\mathrm{d}t+\boldsymbol{\Psi}^\top(t)\,\mathrm{d}\boldsymbol{M}(t),\quad t\in[0,T]\\ P(T)&=1,\end{aligned}\tag{25}$$

where $f(t,P(t)):=-(2r(t)+H_{-}^{*}(t))P(t)$.

By Lemma 3.2 and Cohen and Elliott (2010, Theorem 3.11), we have the following result:

Lemma 3.4: *There exists a unique solution* $(P(\cdot),\boldsymbol{\Psi}(\cdot)) \in S^2_{\mathbb{G}}(0,T;\mathbb{R}) \times L^2_{\mathbb{G},\boldsymbol{\alpha}}(0,T;\mathbb{R}^m)$ *to the BSDE* (25) *such that*

$$P(t) = \mathrm{e}^{2\int_t^T r(s)\,\mathrm{d}s}\mathbb{E}_t\left[\mathrm{e}^{\int_t^T H^*_-(s)\,\mathrm{d}s}\right], \quad t \in [0,T], \tag{26}$$

and $K_1 \le P(\cdot) \le K_2$, *for some constants* $K_1, K_2 > 0$.

3.2. Main results

In this section, we give explicit solutions to the mean-variance reinsurance-investment problem (10) in terms of the solution to the BSDE (25).

It is well-known that the Lagrange multiplier method is the standard method to cope with the constraint equation $\mathbb{E}X(T) = z$ on the mean-variance problem (10). By inserting the Lagrange multiplier $\lambda \in \mathbb{R}$, we can consider the following cost functional:

$$\begin{aligned} J(x_0,\boldsymbol{u}(\cdot),\lambda) &:= \mathbb{E}\left\{(X(T))^2 - z^2 - 2\lambda\left[X(T) - z\right]\right\} \\ &= \mathbb{E}\left[|X(T)-\lambda|^2\right] - (\lambda - z)^2, \\ &\lambda \in \mathbb{R}. \end{aligned} \tag{27}$$

Similar to Hu and Zhou (2005), we may first solve the following unconstrained problem parameterized by the Lagrange multiplier $\lambda \in \mathbb{R}$:

$$\begin{aligned} &\text{minmize} \quad J(x_0,\boldsymbol{u}(\cdot),\lambda) = \mathbb{E}\left[|X(T)-\lambda|^2\right] \\ &\quad - (\lambda - z)^2 \\ &\text{subject to} \quad \mathbb{E}X(T) = z, \quad \boldsymbol{u}(\cdot) \in \mathcal{A}. \end{aligned} \tag{28}$$

Define $J^*(x_0,\lambda) := \inf_{\boldsymbol{u}(\cdot)\in\mathcal{A}} J(x_0,\boldsymbol{u}(\cdot),\lambda)$.

Theorem 3.5: *Let* $(P(\cdot),\boldsymbol{\Psi}(\cdot)) \in S^2_{\mathbb{G}}(0,T;\mathbb{R}) \times L^2_{\mathbb{G},\boldsymbol{\alpha}}(0,T;\mathbb{R}^m)$ *be the solution to BSDE* (25). *Then the strategy*

$$\boldsymbol{u}^*(t) = \boldsymbol{v}_-(t)\left(X(t) - \lambda\mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}\right)^- \tag{29}$$

is optimal for the problem (28). *Moreover, the corresponding cost is*

$$\begin{aligned} J^*(x_0,\lambda) &= \mathrm{e}^{2\int_0^T r(s)\,\mathrm{d}s}\left[\left(x_0 - \lambda\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}\right)^+\right]^2 \\ &\quad + P(0)\left[\left(x_0 - \lambda\,\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}\right)^-\right]^2 \\ &\quad - (\lambda - z)^2. \end{aligned} \tag{30}$$

Proof: First note that Lemma 3.4 ensures that (25) has an unique bounded, uniformly positive solution $(P(\cdot),\boldsymbol{\Psi}(\cdot)) \in S^2_{\mathbb{G}}(0,T;\mathbb{R}) \times L^2_{\mathbb{G},\alpha}(0,T;\mathbb{R}^m)$. Let $X(\cdot)$ be the solution to (7) under an arbitrary admissible strategy $\boldsymbol{u}(\cdot) = (q(\cdot),\boldsymbol{\pi}^\top(\cdot))^\top$. Noting that the admissible strategy satisfies $\boldsymbol{u}(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+1})$, by the theory of stochastic differential equation (SDE, for short), there is a unique solution $X(\cdot) \in L^2_{\mathbb{F}}(\Omega;C(0,T;\mathbb{R}))$ for the SDE (7).

Set $Y(t) := X(t) - \lambda\mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}$. By Itô's formula and (7), we have

$$\begin{aligned} \mathrm{d}Y(t) &= \left(r(t)Y(t) + \boldsymbol{u}^\top(t)\boldsymbol{B}(t)\right)\mathrm{d}t \\ &\quad + \boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t), \quad t \in [0,T], \\ Y(0) &= x_0 - \lambda\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}. \end{aligned} \tag{31}$$

Therefore, the cost function (27) can be written as

$$J(y_0,\boldsymbol{u}(\cdot)) = \mathbb{E}\left[Y(T)^2\right] - (\lambda - z)^2.$$

Applying Tanaka's formula (3.1) to $Y(\cdot)$, we have

$$\begin{aligned} \mathrm{d}Y^+(t) &= \left(r(t)Y^+(t) + 1_{(Y(t)>0)}\boldsymbol{u}^\top(t)\boldsymbol{B}(t)\right)\mathrm{d}t \\ &\quad + 1_{(Y(t)>0)}\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t) + \tfrac{1}{2}\mathrm{d}L(t), \end{aligned} \tag{32}$$

and

$$\begin{aligned} \mathrm{d}Y^-(t) &= \left(r(t)Y^-(t) - 1_{(Y(t)\le 0)}\boldsymbol{u}^\top(t)\boldsymbol{B}(t)\right)\mathrm{d}t \\ &\quad - 1_{(Y(t)<0)}\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t) + \tfrac{1}{2}\mathrm{d}L(t), \end{aligned} \tag{33}$$

where $L(\cdot)$ is the local time of $Y(\cdot)$ at 0. Applying Itô's formula, we get

$$\begin{aligned} \mathrm{d}\left(Y^+(t)\right)^2 &= \left(2r(t)\left(Y^+(t)\right)^2 + 2Y^+(t)\boldsymbol{u}^\top(t)\boldsymbol{B}(t)\right. \\ &\quad \left. +1_{(Y(t)>0)}\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{u}(t)\right)\mathrm{d}t \\ &\quad + 2Y^+(t)\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t), \end{aligned} \tag{34}$$

where we have used the fact that $Y^+(t)\,\mathrm{d}L(t) = 0$ by virtue of (11). Similarly, we have

$$\begin{aligned} \mathrm{d}\left(Y^-(t)\right)^2 &= \left(2r(t)\left(Y^-(t)\right)^2\right. \\ &\quad - 2Y^-(t)\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}(t) \\ &\quad \left. +1_{(Y(t)\le 0)}\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{u}(t)\right)\mathrm{d}t \\ &\quad - 2Y^-(t)\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t). \end{aligned} \tag{35}$$

Applying Itô's formula to $\mathrm{e}^{2\int_t^T r(s)\,\mathrm{d}s}(Y^+(t))^2$, we have (after some reorganisation)

$$\begin{aligned} &\mathrm{d}\left[\mathrm{e}^{2\int_t^T r(s)\mathrm{d}s}\left(Y^+(t)\right)^2\right] \\ &= \mathrm{e}^{2\int_t^T r(s)\,\mathrm{d}s}\left(2Y^+(t)\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\right. \\ &\quad \left. +1_{(Y(t)>0)}\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\boldsymbol{u}(t)\right)\mathrm{d}t \\ &\quad + 2\mathrm{e}^{2\int_t^T r(s)\mathrm{d}s}Y^+(t)\boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,\mathrm{d}W(t). \end{aligned} \tag{36}$$

Similarly, we can derive

$$\begin{aligned} \mathrm{d}\left[P(t)\left(Y^{-}(t)\right)^2\right] = \Big[& -H_{-}^{*}(t)P(t)\left(Y^{-}(t)\right)^2 \\ & - 2P(t)Y^{-}(t)\boldsymbol{u}^{\top}(t)\boldsymbol{B}(t) \\ & + 1_{(Y(t)\le 0)}P(t)\boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{u}(t)\Big]\mathrm{d}t \\ & - 2P(t)Y^{-}(t)\boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\mathrm{d}W(t) \\ & + Y^{-}(t)^2\boldsymbol{\Psi}^{\top}(t)\,\mathrm{d}\boldsymbol{M}(t). \end{aligned} \tag{37}$$

Next, we define, for any integer $\bar{n} \ge 1$, the following stopping time $\iota_{\bar{n}}$ as follows

$$\begin{aligned} \iota_{\bar{n}} := \inf\Bigg\{ t \ge 0 \,\Bigg|\, & \int_0^t \Big| 2\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s} \\ & \times Y^{+}(s)\boldsymbol{u}^{\top}(s)\boldsymbol{\Sigma}(t)\Big|^2 \mathrm{d}s \\ & + \int_0^t \left|2P(s)Y^{-}(s)\boldsymbol{u}^{\top}(s)\boldsymbol{\Sigma}(t)\right|^2 \mathrm{d}s \\ & + \int_0^{\mathrm{T}} \left(Y^{-}(s)\right)^4 \|\boldsymbol{\Psi}(s)\|^2_{\boldsymbol{\alpha}(s-)}\mathrm{d}s \ge \bar{n} \Bigg\} \wedge T, \end{aligned} \tag{38}$$

where $\inf\emptyset := T$. Obviously, $\iota_{\bar{n}}$, $\bar{n} \ge 1$, is an increasing sequence of stopping times converging to T almost surely.

Summing (36) and (37), we get

$$\begin{aligned} & \mathrm{d}\left[\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\,\mathrm{d}s}\left(Y^{+}(t)\right)^2 + P(t)\left(Y^{-}(t)\right)^2\right] \\ & = \Big\{\Big(1_{(Y(t)>0)}\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s} + 1_{(Y(t)\le 0)}P(t)\Big) \\ & \quad \times \boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{u}(t) \\ & \quad + 2\left(\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}Y^{+}(t) - P(t)Y^{-}(t)\right) \\ & \quad \times \boldsymbol{u}^{\top}(t)\boldsymbol{B}(t) - H_{-}^{*}(t)P(t)\left(Y^{-}(t)\right)^2\Big\}\mathrm{d}t \\ & \quad + 2\left(\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}Y^{+}(t) - P(t)Y^{-}(t)\right) \\ & \quad \times \boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\mathrm{d}\boldsymbol{W}(t) \\ & \quad + \left(Y^{-}(t)\right)^2\boldsymbol{\Psi}^{\top}(t)\,\mathrm{d}\boldsymbol{M}(t). \end{aligned} \tag{39}$$

Integrating the above formula from 0 to $\iota_{\bar{n}}$, and then taking expectation, after arranging we get

$$\begin{aligned} & \mathbb{E}\left[\mathrm{e}^{2\int_{\iota_{\bar{n}}}^{\mathrm{T}} r(s)\mathrm{d}s}\left(Y^{+}(\iota_{\bar{n}})\right)^2 + P(\iota_{\bar{n}})\left(Y^{-}(\iota_{\bar{n}})\right)^2\right] \\ & = \mathbb{E}\left[\mathrm{e}^{2\int_0^{\mathrm{T}} r(s)\mathrm{d}s}\left(Y^{+}(0)\right)^2 + P(0)\left(Y^{-}(0)\right)^2\right] \\ & \quad + \mathbb{E}\int_0^{\iota_{\bar{n}}} \varphi(Y(t),\boldsymbol{u}(t))\,\mathrm{d}t, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \varphi(Y(t),\boldsymbol{u}(t)) := & \Big(1_{(Y(t)>0)}\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s} + 1_{(Y(t)\le 0)}P(t)\Big) \\ & \times \boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{u}(t) \\ & + 2\left(\mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}Y^{+}(t) - P(t)Y^{-}(t)\right) \\ & \times \boldsymbol{u}^{\top}(t)B(t) - H_{-}^{*}(t)P(t)\left(Y^{-}(t)\right)^2. \end{aligned} \tag{41}$$

Now let us send $\bar{n} \to \infty$. In addition, noticing that $Y(\cdot) \in L^2_{\mathbb{F}}(\Omega; C(0,T;\mathbb{R}))$, $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C(0,T;\mathbb{R}))$, and $K_1 \le P(\cdot) \le K_2$, applying the dominated convergence theorem, from (27) we get

$$\begin{aligned} & J(x_0,\boldsymbol{u}(\cdot),\lambda) \\ & = \lim_{\bar{n}\to\infty}\mathbb{E}\left[\mathrm{e}^{2\int_{\iota_{\bar{n}}}^{\mathrm{T}} r(s)\mathrm{d}s}\left(Y^{+}(\iota_{\bar{n}})\right)^2 + P(\iota_{\bar{n}})\left(Y^{-}(\iota_{\bar{n}})\right)^2\right] \\ & \quad - (\lambda - z)^2 = \mathbb{E}\left[\mathrm{e}^{2\int_0^{\mathrm{T}} r(s)\mathrm{d}s}\left(y_0^{+}\right)^2 + P(0)\left(y_0^{-}\right)^2\right] \\ & \quad + \mathbb{E}\int_0^{\mathrm{T}} \varphi(Y(t),\boldsymbol{u}(t))\,\mathrm{d}t - (\lambda - z)^2. \end{aligned} \tag{42}$$

The next step is to show that $\varphi(Y(t),\boldsymbol{u}(t))) \ge 0$ for any $t \in [0,T]$. Specifically, the analysis is as follows.

If $Y(t) > 0$ for some t, then set $\boldsymbol{u}(t) = Y(t)\boldsymbol{v}(t)$. In this case, notice $\boldsymbol{u}(t) \in \mathbb{R}^{n+1}_{+}$ if only if $\boldsymbol{v}(t) \in \mathbb{R}^{n+1}_{+}$. Fixing t given before, substituting $Y(t)\boldsymbol{v}(t)$ for $\boldsymbol{u}(t)$, and noticing the definition of $H_{+}^{*}(t)$, then

$$\begin{aligned} \varphi(Y(t),\boldsymbol{u}(t)) & = \mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}\Big[\boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{u}(t) \\ & \quad + 2Y^{+}(t)\boldsymbol{u}^{\top}(t)\boldsymbol{B}(t)\Big] \\ & = \mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}\Big[\boldsymbol{v}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{v}(t) \\ & \quad + 2\boldsymbol{v}^{\top}(t)\boldsymbol{B}(t)\Big]Y^2(t) \\ & = \mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}H_{+}(t,\boldsymbol{v}(t))\,Y^2(t) \\ & \ge \mathrm{e}^{2\int_t^{\mathrm{T}} r(s)\mathrm{d}s}H_{+}^{*}(t)\,Y^2(t) = 0, \end{aligned}$$

where the equality holds at $\boldsymbol{v}(t) = \boldsymbol{v}_{+}(t) = (0,\boldsymbol{0})$, $t \in [0,T]$. Then, the equality in the above holds if and only if the strategy takes

$$\boldsymbol{u}^{*}(t) = \boldsymbol{0} \in \mathbb{R}^{n+1}_{+}.$$

If $Y(t) < 0$ for some t, then set $\boldsymbol{u}(t) = -Y(t)\boldsymbol{v}(t)$. In this case,

$$\begin{aligned} \varphi(Y(t),\boldsymbol{u}(t)) & = P(t)\boldsymbol{u}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{u}(t) \\ & \quad - 2P(t)Y^{-}(t)\boldsymbol{u}^{\top}(t)\boldsymbol{B}(t) \\ & \quad - H_{-}^{*}(t)P(t)\left(Y^{-}(t)\right)^2 \\ & = P(t)\boldsymbol{v}^{\top}(t)\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^{\top}(t)\boldsymbol{v}(t)Y^2(t) \end{aligned}$$

$$
\begin{aligned}
&- 2P(t)\,\boldsymbol{v}^\top(t)\,\boldsymbol{B}(t)\,Y^2(t)\\
&- H_-^*(t)P(t)Y^2(t)\\
&= P(t)\Big[\Big(\boldsymbol{v}^\top(t)\,\boldsymbol{\Sigma}(t)\,\boldsymbol{\Sigma}^\top(t)\,\boldsymbol{v}(t)\\
&\quad - 2\boldsymbol{v}^\top(t)\,\boldsymbol{B}(t)\Big) - H_-^*(t)\Big]Y^2(t)\\
&= P(t)\big(H_-(t,\boldsymbol{v}(t)) - H_-^*(t)\big)Y^2(t)\\
&\geq 0,
\end{aligned}
$$

where the equality holds at

$$\boldsymbol{u}^*(t) = Y^-(t)\,\boldsymbol{v}_-(t) \in \mathbb{R}_+^{n+1}.$$

Finally, when $Y(t) = 0$, then

$$\varphi(Y(t),\boldsymbol{u}(t)) = P(t)\,\boldsymbol{u}^\top(t)\,\boldsymbol{\Sigma}(t)\,\boldsymbol{\Sigma}^\top(t)\,\boldsymbol{u}(t) \geq 0$$

with the equality if and only if $\boldsymbol{u}^*(t) = \mathbf{0}$.

According to the above analysis, we have

$$\varphi(Y(t),\boldsymbol{u}(t)) \geq \varphi\big(Y(t),\boldsymbol{u}^*(t)\big) = 0.$$

Combining the above analysis and (42), we find that for all admissible strategies $\boldsymbol{u}(t) \in \mathcal{A}$,

$$
\begin{aligned}
J(x_0,\boldsymbol{u}(t),\lambda) \geq \mathbb{E}\left[\mathrm{e}^{2\int_0^{\mathrm{T}} r(s)\mathrm{d}s}\big(y_0^+\big)^2 + P(0)\big(y_0^-\big)^2\right]\\
- (\lambda - z)^2 \qquad (43)
\end{aligned}
$$

with the equality sign holding if and only if the strategy $\boldsymbol{u}(t)$ adopts $\boldsymbol{u}^*(t)$ as follows

$$
\begin{aligned}
\boldsymbol{u}^*(t) &= Y^+(t)\,\boldsymbol{v}_+(t) + Y^-(t)\,\boldsymbol{v}_-(t)\\
&= Y^-(t)\,\boldsymbol{v}_-(t), \qquad (44)
\end{aligned}
$$

i.e. the expression (29). Thus (30) follows from that

$$
\begin{aligned}
J^*(x_0,\lambda) &= J\big(x_0,\boldsymbol{u}^*(t),\lambda\big)\\
&= \mathrm{e}^{2\int_0^{\mathrm{T}} r(s)\mathrm{d}s}\big(y_0^+\big)^2 + P_-(0)\big(y_0^-\big)^2\\
&\quad - (\lambda - z)^2,
\end{aligned}
$$

where $y_0 = x_0 - \lambda\mathrm{e}^{-\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}$.

According to the definition of the admissible strategy, we need to show that the strategy $\boldsymbol{u}^*(\cdot)$ defined by (29) should belong to $L_{\mathbb{F}}^2(0,T;\mathbb{R}^{n+1})$ if it is admissible. Noting the definition of $\boldsymbol{v}_-(\cdot)$, and Assumption 2.1 and Remark 2.1, we have

$$\boldsymbol{v}_-(t) \in \left\{\boldsymbol{v}(t) \in \mathbb{R}_+^{m+1}\,\middle|\,|\boldsymbol{v}(t)| \leq \frac{2C_1}{\hat{\rho}}\right\}. \qquad (45)$$

Therefore $\boldsymbol{v}_-(\cdot)$ is uniformly bounded.

Now, under the state feedback strategy (44), the stochastic differential equation (31) is written as

$$
\begin{aligned}
\mathrm{d}Y(t) &= \left\{r(t)Y(t) + Y^-(t)\,\boldsymbol{v}_-^\top(t)\,\boldsymbol{B}(t)\right\}\mathrm{d}t\\
&\quad + Y^-(t)\,\boldsymbol{v}_-^\top(t)\,\boldsymbol{\Sigma}(t)\,\mathrm{d}\boldsymbol{W}(t), \quad t \in [0,T],
\end{aligned} \qquad (46)
$$

with $Y(0) = y_0$. By Lemma 3.3, (46) indeed admits a unique solution $Y^*(\cdot) \in L_{\mathbb{F}}^2(\Omega;C(0,T;\mathbb{R}))$. Recalling that $\boldsymbol{v}_+(\cdot) = 0$, it is easy to see that $\boldsymbol{u}^*(\cdot) = Y^{*-}(\cdot)\boldsymbol{v}_-(\cdot) \in L_{\mathbb{F}}^2(0,T;\mathbb{R}_+^{n+1})$. ■

Remark 3.2: It is easy to find that

$$\sup_{\lambda\in\mathbb{R}} J^*(x_0,\lambda) = \sup_{\lambda\in[x_0\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s},+\infty)} J^*(x_0,\lambda), \qquad (47)$$

from (30).

Now, we can show the optimal strategy and efficient frontier of the problem (10).

Theorem 3.6: *Let $(P(\cdot),\boldsymbol{\Psi}(\cdot)) \in S_{\mathbb{G}}^2(0,T;\mathbb{R}) \times L_{\mathbb{G}}^2(0,T;\mathbb{R}^m)$ be the unique solution to the BSDE (25). Then the efficient strategy corresponding to $z \geq x_0\,\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}$, as a feedback of the wealth process, for the problem (10) is*

$$\boldsymbol{u}^*(t) = \Big(\lambda^*\mathrm{e}^{-\int_t^{\mathrm{T}} r(s)\,\mathrm{d}s} - X^*(t)\Big)\,\boldsymbol{v}_-(t), \qquad (48)$$

where

$$
\begin{aligned}
\lambda^* &:= \frac{z - x_0P(0)\,\mathrm{e}^{-\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}}\\
&= \frac{z - x_0\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}\mathbb{E}\left(\mathrm{e}^{\int_0^{\mathrm{T}} H_-^*(s)\mathrm{d}s}\right)}{1 - \mathbb{E}\left(\mathrm{e}^{\int_0^{\mathrm{T}} H_-^*(s)\mathrm{d}s}\right)}.
\end{aligned} \qquad (49)
$$

Moreover, the efficient frontier is

$$
\begin{aligned}
\mathrm{Var}X^*(T) &= \frac{P(0)\,\mathrm{e}^{-2\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}}\\
&\quad \times \Big(\mathbb{E}X^*(T) - x_0\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\mathrm{d}s}\Big)^2,\\
\mathbb{E}X^*(T) &\geq x_0\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\mathrm{d}s}.
\end{aligned} \qquad (50)
$$

Proof: It follows from (16) that $1 - P(0)\mathrm{e}^{-2\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s} = 1 - \mathbb{E}(\mathrm{e}^{\int_0^{\mathrm{T}} H_-^*(s)\,\mathrm{d}s}) > 0$. Thus λ^* in (49) is well defined.

First, we intend to directly solve the problem (10) for $z = x_0\mathrm{e}^{\int_0^{\mathrm{T}} r(s)\,\mathrm{d}s}$. Let $X^*(\cdot)$ be the wealth process corresponding to the efficient strategy $\boldsymbol{u}^*(\cdot)$. By linearity, it is easy to see that $X^*(\cdot){=}X^0(\cdot) + X^1(\cdot)$ where $X^0(\cdot)$ and

$X^1(\cdot)$ are given by as follows, respectively

$$\mathrm{d}X^0(s) = X^0(s)r(s)\,\mathrm{d}s, \quad s \in [0,T],$$
$$X^0(0) = x_0 > 0, \tag{51}$$

and

$$\begin{aligned}\mathrm{d}X^1(s) &= \left(r(s)X^1(s) + \boldsymbol{u}^{*\top}(s)(s)\boldsymbol{B}(s)\right)\mathrm{d}s \\ &\quad + \boldsymbol{u}^{*\top}(s)(s)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s), \quad s \in [0,T], \\ X^1(0) &= 0.\end{aligned} \tag{52}$$

Applying Itô's formula, for $s \in [0,T]$, we obtain

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{-\int_0^s r(\tau)\,\mathrm{d}\tau}X^1(s)\right) &= \mathrm{e}^{-\int_0^s r(\tau)\,\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s)\boldsymbol{B}(s)\,\mathrm{d}s \\ &\quad + \mathrm{e}^{-\int_0^s r(\tau)\,\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s) \\ &\quad \times \boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s).\end{aligned}$$

Integrating from 0 to T and taking expectation, it yields that

$$\begin{aligned}X^1(t) &= \int_0^t \mathrm{e}^{\int_s^t r(\tau)\,\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s)\boldsymbol{B}(s)\,\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{\int_s^t r(\tau)\,\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s).\end{aligned} \tag{53}$$

Noting $X^0(t) := x_0\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}$ we have

$$\begin{aligned}X^*(t) &= X^0(t) + X^1(t) \\ &= x_0\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s} + \int_0^t \mathrm{e}^{\int_s^t r(\tau)\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s)\boldsymbol{B}(s)\,\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{\int_s^t r(\tau)\,\mathrm{d}\tau}\boldsymbol{u}^{*\top}(s)\boldsymbol{\Sigma}(s)\,\mathrm{d}\boldsymbol{W}(s)\end{aligned} \tag{54}$$

and

$$\begin{aligned}\mathbb{E}X^*(T) &= x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s} \\ &\quad + \mathbb{E}\int_0^T \mathrm{e}^{\int_t^T r(s)\,\mathrm{d}s}\boldsymbol{u}^{*\top}(t)\boldsymbol{B}(t)\,\mathrm{d}t.\end{aligned} \tag{55}$$

So if $z = x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$, under the constraint $\mathbb{E}X^*(T) = z$ in (10), by (55), we immediately obtain that the corresponding efficient strategy $\boldsymbol{u}^*(t) \equiv \boldsymbol{0}$ which means that all the wealth to be put in the bank (i.e. the risk-free asset) and all the risk of the insurance business to be passed to the reinsurer. By (54) and $\boldsymbol{u}^*(t) = \boldsymbol{0}$, $X^*(t) = x_0\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}$. Obviously it is easy to get the corresponding variance $\mathrm{Var}X^*(T) = 0$.

Putting $z = x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$, $X^*(t) = x_0\mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}$ into (49) and (48), we can easily obtain that $\lambda^* = x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$ and $\boldsymbol{u}^*(t) = \boldsymbol{0}$. Moreover, by substituting $x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$ for $\mathbb{E}X^*(T)$ in (50), we have $\mathrm{Var}X^*(T) = 0$. That is to say that, according to Definition 2.3, (50) and (48) are indeed the efficient frontier and the efficient strategy corresponding to $z = x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$. In the following, we only consider for any fixed $z > x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$.

As described in Hu and Zhou (2005)), by applying the duality theorem, we have for $z > x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$,

$$\begin{aligned}J^*_{MV}(x_0) &= \inf_{(q(\cdot),\boldsymbol{\pi}(\cdot))\in L^2_{\mathbb{F}}\left(0,T;\mathbb{R}^{n+1}_+\right)} J\left(x_0, q(\cdot), \pi(\cdot)\right) \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{(q(\cdot),\boldsymbol{\pi}(\cdot))\in L^2_{\mathbb{F}}\left(0,T;\mathbb{R}^{n+1}_+\right)} J\left(x_0, q(\cdot), \pi(\cdot), \lambda\right) \\ &= \sup_{\lambda\in\mathbb{R}} J^*(x_0,\lambda) = \sup_{\lambda\in\left[x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}, +\infty\right)} J^*(x_0,\lambda),\end{aligned}$$

where the last equality is from (47). Recalling (30), by making use of method of completing the square, we have, for any $\lambda \in [x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}, +\infty)$,

$$\begin{aligned}J^*(x_0,\lambda) &= \left(P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s} - 1\right)\lambda^2 \\ &\quad - 2\left(x_0P(0)\,\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s} - z\right)\lambda \\ &\quad + P(0)\,x_0^2 - z^2 \\ &= \left(P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s} - 1\right) \\ &\quad \times \left(\lambda - \frac{z - x_0P(0)\,\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}\right)^2 \\ &\quad + \frac{P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}} \\ &\quad \times \left(z - x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}\right)^2.\end{aligned}$$

Considering (16), under the constrained condition $z = \mathbb{E}X^*(T)$, it is apparent that for $z \geq x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$,

$$\begin{aligned}\mathrm{Var}X^*(T) &= J^*_{MV}(x_0) \\ &= \sup_{\lambda\in\left[x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}, +\infty\right)} J^*(x_0,\lambda) = J^*\left(x_0,\lambda^*\right) \\ &= \frac{P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}} \\ &\quad \times \left(\mathbb{E}X^*(T) - x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}\right)^2,\end{aligned}$$

where $\lambda^* := (z - x_0P(0)\,\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s})/(1 - P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}) \in (x_0\,\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}, +\infty)$. As $H^*_-(s) < 0$, we also can get for $z > x_0\,\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$,

$$\begin{aligned}y_0 &= x_0 - \lambda^*\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s} \\ &= x_0 - \frac{z - x_0P(0)\,\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}}{1 - P(0)\,\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s} \\ &= -\frac{z\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s} - x_0}{1 - \mathbb{E}\left(\mathrm{e}^{\int_0^T H^*_-(s)\mathrm{d}s}\right)} \leq 0.\end{aligned}$$

Then for $z > x_0 e^{\int_0^T r(s)\,ds}$, considering (20) in Lemma 3.3, we have

$$\begin{aligned} X^*(t) - \lambda^* e^{-\int_t^T r(s)\,ds} &= Y^*(t) \\ &= y_0^+ e^{\int_0^t r(s)ds} - y_0^- \phi(t) \\ &= -y_0^- \phi(t) = y_0 \phi(t) \le 0. \end{aligned} \tag{56}$$

By Theorem 3.5 and substituting λ^* for λ in (29), we have

$$\begin{aligned} \boldsymbol{u}^*(t) &= \boldsymbol{v}_-(t)\left(X(t) - \lambda e^{-\int_t^T r(s)\,ds}\right)^- \\ &= \boldsymbol{v}_-(t)\left(\lambda^* e^{-\int_t^T r(s)\,ds} - X^*(t)\right). \end{aligned}$$

The proof is completed. ■

4. Two special cases

4.1. One risky asset

In this section, we consider $n = 1$, i.e. there is only one risky asset. Then (3), (4) and (7) are rewritten as follows

$$\begin{aligned} dS_1(t) &= S_1(t)(\mu_1(t)\,dt + \sigma_{11}(t)\,dW_1(t)), \\ &\quad t \in [0,T], \\ S_1(0) &= s_1 > 0, \end{aligned}$$

$$dC(t) = a(t)\,dt - (b_0(t), b_1(t))\,d\boldsymbol{W}(t),$$

and

$$\begin{aligned} dX(t) &= \left(r(t)X(t) + \boldsymbol{u}^\top(t)\boldsymbol{B}(t)\right)dt \\ &\quad + \boldsymbol{u}^\top(t)\boldsymbol{\Sigma}(t)\,d\boldsymbol{W}(t), \quad t \in [0,T], \\ X(0) &= x_0 > 0, \end{aligned}$$

where $\boldsymbol{W}(t) = (W_0(t), W_1(t))^\top$. In this case, we have,

$$\begin{aligned} H_0(t) &= \boldsymbol{B}^\top(t)\left(\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}^\top(t)\right)^{-1}\boldsymbol{B}(t) \\ &= \left[\frac{a(t)\theta(t)}{b_0(t)} - \frac{b_1(t)(\mu_1(t) - r(t))}{b_0(t)\sigma_{11}(t)}\right]^2 \\ &\quad + \frac{(\mu_1(t) - r(t))^2}{\sigma_{11}^2(t)} > 0. \end{aligned}$$

From Proposition 3.2, we have $H_+^*(t) = H_+(t, \boldsymbol{v}_+(t)) = 0$, where $\boldsymbol{v}_+(t) = (0,0)$. In the next Lemma 4.1, we get the minimum value $H_-^*(t)$ and $\boldsymbol{v}_-(t) = (\zeta_-(t), \xi_-(t))$ for the case with $n = 1$.

Lemma 4.1: (i) *If $b_1(t) > 0,\ t \in [0,T]$, then*

$$\begin{aligned} H_-^* &= -\left(\frac{\mu_1 - r}{\sigma_{11}}\right)^2, \quad \boldsymbol{v}_- = \left(0, \frac{\mu_1 - r}{\sigma_{11}^2}\right)^\top, \\ &\quad \text{if } 0 < \theta < \frac{b_1(\mu_1 - r)}{a\sigma_{11}}, \\ H_-^* &= -H_0, \\ \boldsymbol{v}_- &= \left(\frac{a\theta\sigma_{11} - b_1(\mu_1 - r)}{b_0^2\sigma_{11}}, \right. \\ &\quad \left. \frac{(b_0^2 + b_1^2)(\mu_1 - r) - b_1 a\theta\sigma_{11}}{b_0^2\sigma_{11}^2}\right)^\top, \\ &\quad \text{if } \frac{b_1(\mu_1 - r)}{a\sigma_{11}} \le \theta < \frac{(b_1^2 + b_0^2)(\mu_1 - r)}{b_1 a\sigma_{11}}, \\ H_-^* &= -\frac{a^2\theta^2}{b_1^2 + b_0^2}, \quad \boldsymbol{v}_- = \left(\frac{a\theta}{b_1^2 + b_0^2}, 0\right)^\top, \\ &\quad \text{if } \theta \ge \frac{(b_1^2 + b_0^2)(\mu_1 - r)}{b_1 a\sigma_{11}}, \end{aligned} \tag{57}$$

where we have suppressed the variable t;

(ii) *If $b_1(t) = 0,\ t \in [0,T]$, we have for*

$$\begin{aligned} H_-^*(t) &= -H_0(t), \\ \boldsymbol{v}_-(t) &= \left(\frac{a(t)\theta(t)}{b_0^2(t)}, \frac{\mu_1(t) - r(t)}{\sigma_{11}^2(t)}\right)^\top. \end{aligned} \tag{58}$$

Proof: To get the the infimum of $H_+(t, \boldsymbol{v})$ for $\boldsymbol{v} := (\kappa, \upsilon_1)^\top \in \mathbb{R}_+^2$, we consider the the following optimisation problem

$$\begin{aligned} \min \quad & H_-(t, \boldsymbol{v}) = \left[b_0\kappa - \frac{a\theta}{b_0} + \frac{b_1(\mu_1 - r)}{b_0\sigma_{11}}\right]^2 \\ & \quad + \left(b_1\kappa + \sigma_{11}\upsilon_1 - \frac{\mu_1 - r}{\sigma_{11}}\right)^2, \\ \text{s.t.} \quad & -\kappa \le 0, \quad -\upsilon_1 \le 0. \end{aligned}$$

By Karush–Kuhn–Tucker conditions, it holds that

$$\begin{aligned} & \boldsymbol{v} \in \mathbb{R}_+^2, \quad \boldsymbol{\lambda} := (\lambda_1, \lambda_2)^\top \in \mathbb{R}_+^2, \\ & \frac{\partial H_-(t, \boldsymbol{v})}{\partial \kappa} + \lambda_1 = 0, \quad \frac{\partial H_-(t, \boldsymbol{v})}{\partial \upsilon_1} + \lambda_2 = 0, \\ & -\lambda_1\kappa = 0, \quad -\lambda_2\upsilon_1 = 0, \end{aligned}$$

i.e.

$$\begin{aligned} & \kappa \ge 0, \quad \upsilon_1 \ge 0, \quad \lambda_1 \ge 0, \quad \lambda_2 \ge 0, \\ & 2\left[\left(b_0^2 + b_1^2\right)\kappa + b_1\sigma_{11}\upsilon_1 - a\theta\right] - \lambda_1 = 0, \\ & 2\left[b_1\sigma_{11}\kappa + \sigma_{11}^2\upsilon_1 - (\mu_1 - r)\right] - \lambda_2 = 0, \\ & \lambda_1\kappa = 0, \quad \lambda_2\upsilon_1 = 0. \end{aligned}$$

It is easy to get

$$\begin{aligned}
\boldsymbol{v}^\top &= \left(0, \frac{\mu_1 - r}{\sigma_{11}^2}\right), \\
\boldsymbol{\lambda}^\top &= \left(2\left[\frac{b_1(\mu_1 - r)}{\sigma_{11}} - a\right]\theta, 0\right), \\
&\quad \text{if } a\theta\sigma_{11} \le b_1(\mu_1 - r), \\
\boldsymbol{v}^\top &= \left(\frac{a\theta\sigma_{11} - b_1(\mu_1 - r)}{b_0^2\sigma_{11}},\right. \\
&\quad \left.\frac{(b_0^2 + b_1^2)(\mu_1 - r) - b_1 a\theta\sigma_{11}}{b_0^2\sigma_{11}^2}\right), \\
&\quad \boldsymbol{\lambda}^\top = (0, 0), \\
&\quad \text{if } a\theta\sigma_{11} > b_1(\mu_1 - r) \quad \text{and} \\
&\quad b_1 a\theta\sigma_{11} < (b_0^2 + b_1^2)(\mu_1 - r), \\
\boldsymbol{v}^\top &= \left(\frac{a\theta}{b_0^2 + b_1^2}, 0\right), \\
\boldsymbol{\lambda}^\top &= \left(0, 2\left[\frac{a\theta b_1\sigma_{11}}{b_0^2 + b_1^2} - (\mu_1 - r)\right]\right), \\
&\quad \text{if } a\theta b_1\sigma_{11} \ge (\mu_1 - r)(b_0^2 + b_1^2).
\end{aligned}$$

The proof is completed. ■

Through applying Lemma 4.1 to Theorem 3.6 and 4.2, we can obtain the explicit expression of the optimal strategy in the case of one risk-free asset.

4.2. Markovian regime-switching model

In this section, we consider the Markovian regime-switching model, that is to say we suppose that all of random coefficients mentioned before in the paper are functions of the state of the Markov chain. To be precise, let

$$\begin{aligned}
&\theta(t) = \tilde{\theta}(t, \boldsymbol{\alpha}(t)), \nu(t) = \tilde{\nu}(t, \boldsymbol{\alpha}(t)), \\
&a(t) = \tilde{a}(t, \boldsymbol{\alpha}(\cdot)), \\
&\bar{\boldsymbol{b}}(t) = \tilde{\bar{\boldsymbol{b}}}(t, \boldsymbol{\alpha}(t)), \boldsymbol{B}(t) = \tilde{\boldsymbol{B}}(t, \boldsymbol{\alpha}(t)), \\
&\boldsymbol{\sigma}(t) = \tilde{\boldsymbol{\sigma}}(t, \boldsymbol{\alpha}(t)),
\end{aligned}$$

where $\tilde{\theta}(\cdot,\cdot)$, $\tilde{\nu}(\cdot,\cdot), \tilde{a}(\cdot,\cdot), \tilde{\bar{\boldsymbol{b}}}(\cdot,\cdot), \tilde{\boldsymbol{B}}(\cdot,\cdot), \tilde{\boldsymbol{\sigma}}(\cdot,\cdot)$, are deterministic and bounded. Then other notations are changed correspondingly, for instance, $\boldsymbol{\Sigma}(t) = \tilde{\boldsymbol{\Sigma}}(t, \boldsymbol{\alpha}(t))$ and

$$\begin{aligned}
H_-(t, \boldsymbol{v}(t)) &= \tilde{H}_-(t, \boldsymbol{\alpha}(t), \boldsymbol{v}(t)) \\
&:= \boldsymbol{v}^\top(t)\tilde{\boldsymbol{\Sigma}}(t, \boldsymbol{\alpha}(t))\tilde{\boldsymbol{\Sigma}}^\top(t, \boldsymbol{\alpha}(t))\boldsymbol{v}(t) \\
&\quad - 2\boldsymbol{v}^\top(t)\tilde{\boldsymbol{B}}(t, \boldsymbol{\alpha}(t)), (t, \boldsymbol{\alpha}(t), \boldsymbol{v}(t)) \\
&\in [0, T] \times \mathcal{M} \times \mathbb{R}^{n+1}.
\end{aligned}$$

For the BSDE (25), then we have

$$\begin{aligned}
f(t, P(t)) &= \tilde{f}(t, \boldsymbol{\alpha}(t), P(t)) \\
&:= -\left[2r(t) + \tilde{H}_-^*(t, \boldsymbol{\alpha}(t))\right]P(t),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{H}_-^*(t, \boldsymbol{\alpha}(t)) &:= \min_{\boldsymbol{v}(t)\in\mathcal{A}} \tilde{H}_-(t, \boldsymbol{\alpha}(t), \boldsymbol{v}(t)), \\
&\quad (t, \boldsymbol{\alpha}(t)) \in [0, T] \times \mathcal{M}.
\end{aligned}$$

In the following, to ease the presentation, sometimes we simplify the notations slightly by omitting the tilde character above those deterministic bounded functions.

By Cohen and Szpruch (2012), there exists a unique function $\boldsymbol{F}(t) := (F_1(t), \ldots, F_m(t))^\top$ such that

$$(P(t), \boldsymbol{\Psi}(t)) = \left(\boldsymbol{\alpha}^\top(t)\boldsymbol{F}(t), \boldsymbol{F}(t)\right), \tag{59}$$

and

$$\begin{aligned}
&\boldsymbol{F}_t(t) = -\boldsymbol{g}(t, \boldsymbol{F}(t)) - \boldsymbol{Q}(t)\boldsymbol{F}(t), \quad t \in [0, T], \\
&\boldsymbol{F}(T) = \mathbf{1},
\end{aligned}\tag{60}$$

where $\boldsymbol{F}_t(t) = (F_{1,t}(t), \ldots, F_{m,t}(t))^\top = (\mathrm{d}F_1(t)/\mathrm{d}t, \ldots, \mathrm{d}F_m(t)/\mathrm{d}t)^\top$ and

$$\begin{aligned}
\boldsymbol{g}(t, \boldsymbol{F}(t)) &= \left(f(t, \boldsymbol{e}_1, F_1(t)), \ldots, f(t, \boldsymbol{e}_m, F_m(t))\right)^\top \\
&= -\mathrm{diag}\left(2r(t) + H_-^*(t, \boldsymbol{e}_1), \ldots, 2r(t)\right. \\
&\quad \left. + H_-^*(t, \boldsymbol{e}_m)\right)\boldsymbol{F}(t).
\end{aligned}$$

System (60) is a homogeneous linear system of variable-coefficient first-order ordinary differential equations with continuous coefficients. Then, by the classical theory in the ODEs (see, e.g. Walter, 1998, P. 162), indeed there is exactly one solution to the system (60). If $r(\cdot)$, $H_-^*(\cdot, \boldsymbol{e}_i)$, $\boldsymbol{Q}(\cdot)$, $i = 1, \ldots, m$ are constants. Let $\boldsymbol{\Theta} := (\Theta_{ij})_{m\times m}$, where

$$\Theta_{ij} := \begin{cases} -q_{ij}, & j \ne i, \\ -2r - H_-^*(\cdot, \boldsymbol{e}_i), & j = i. \end{cases}$$

Then system (60) has the following unique solution

$$\boldsymbol{F}(t) = \sum_k c_k \mathbf{v}_k \mathrm{e}^{\lambda_k t},$$

where λ_k, $\mathbf{v}_k$ are the eigenvalues and corresponding eigenvectors of $\boldsymbol{\Theta}$ and the constants c_k is determined by $\sum_k c_k \mathbf{v}_k \mathrm{e}^{\lambda_k T} = \mathbf{1}$.

Applying (59) to (48) and (50) in Theorem 3.6, it easily to get the following result.

Theorem 4.2: *The efficient strategy corresponding to $z \ge x_0\,\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$, as a feedback of the wealth process, for* *the problem* (10) *is*

$$\boldsymbol{u}^*\left(t, X^*(t)\right) = \left(\frac{z - x_0 \boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}}{1 - \boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}} \times \mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s} - X^*(t)\right)\boldsymbol{v}_-(t), \quad (61)$$

where

$$\boldsymbol{v}_-(t, e_i) := \operatorname*{argmin}_{\boldsymbol{v}(t)\in\mathcal{A}} H_-(t, e_i, \boldsymbol{v}(t)), \quad (t,i) \in [0,T] \times \mathcal{M}$$

and the efficient frontier

$$\mathrm{Var}X^*(T) = \frac{\boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}{1 - \boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}} \times \left(\mathbb{E}X^*(T) - x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}\right)^2. \quad (62)$$

Remark 4.1: For the case with the one risky asset (i.e. $n = 1$), by Theorem 4.2, the efficient strategy corresponding to $z = \mathbb{E}X^*(T) \geq x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}$ is (61) with

$$\boldsymbol{v}_- = \begin{cases} \left(0, \dfrac{\mu_1 - r}{\sigma_{11}^2}\right)^\top, & \text{if } a\theta\sigma_{11} \leq b_1(\mu_1 - r), \\ \left(\dfrac{a\theta\sigma_{11} - b_1(\mu_1 - r)}{b_0^2\sigma_{11}}, \dfrac{(b_0^2 + b_1^2)(\mu_1 - r) - b_1 a\theta\sigma_{11}}{b_0^2\sigma_{11}^2}\right)^\top, & \text{if } a\theta\sigma_{11} > b_1(\mu_1 - r) \text{ and } b_1 a\theta\sigma_{11} < (b_0^2 + b_1^2)(\mu_1 - r), \\ \left(\dfrac{a\theta}{b_1^2 + b_0^2}, 0\right)^\top, & \text{if } a\theta b_1\sigma_{11} \geq (\mu_1 - r) b_0^2 + b_1^2, \end{cases}$$

and the efficient frontier is

$$\mathrm{Var}X^*(T) = \frac{\boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}}{1 - \boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r(s)\,\mathrm{d}s}} \times \left(z - x_0\mathrm{e}^{\int_0^T r(s)\,\mathrm{d}s}\right)^2. \quad (63)$$

Remark 4.2: If the interest rate in Chen and Yam (2013) is deterministic and $\boldsymbol{b}(\cdot) = \boldsymbol{0}$ in our paper, then the models in our paper are the same. In this case,

$$H_-^*(\cdot, \boldsymbol{e}_i) = H_0(\cdot, \boldsymbol{e}_i) = \frac{(\mu_1(\cdot, \boldsymbol{e}_i) - r(\cdot))^2}{\sigma_{11}^2(\cdot, \boldsymbol{e}_i)} + \left[\frac{a(\cdot, \boldsymbol{e}_i)\,\theta(\cdot, \boldsymbol{e}_i)}{b_0(\cdot, \boldsymbol{e}_i)} - \frac{b_1(\cdot, \boldsymbol{e}_i)(\mu_1(\cdot, \boldsymbol{e}_i) - r(\cdot))}{b_0(\cdot, \boldsymbol{e}_i)\,\sigma_{11}(\cdot, \boldsymbol{e}_i)}\right]^2$$

and the optimal strategy obtained in Chen and Yam (2013) is given by (61) with

$$\boldsymbol{v}_-(t) = \left(\frac{a(t)\,\theta(t)}{b_0^2(t)}, \frac{\mu_1(t) - r(t)}{\sigma_{11}^2(t)}\right)$$

and

$$\boldsymbol{F}_t(t) = \left\{\mathrm{diag}\left[2r(t) - H_0(t, \boldsymbol{e}_1), \ldots, 2r(t) - H_0(t, \boldsymbol{e}_m)\right] - \boldsymbol{Q}(t)\right\}\boldsymbol{F}(t), \quad t \in [0,T],$$
$$\boldsymbol{F}(T) = \boldsymbol{1}.$$

Under the deterministic interest rate, the assumption in Chen and Yam (2013, Theorem 5.1), which becomes

$$\frac{\mathrm{e}^{-\int_0^T r_0(s)\mathrm{d}s}\left(z - x_0\mathrm{e}^{\int_0^T r_0(s)\mathrm{d}s}\right)}{\boldsymbol{\alpha}^\top(0)\boldsymbol{F}(0)\mathrm{e}^{-2\int_0^T r_0(s)\mathrm{d}s} - 1} < 0,$$

always holds. It follows from Remark 4.1 that our results are the same.

5. Conclusion

We have investigated an optimal proportional reinsurance and investment problem for an insurer under the mean-variance criterion. We assumed that the claim process of the insurer and the prices of risky assets are correlated and the coefficients (except the interest rate) in the model are stochastic processes adapted to the filtration generated by a Markov chain. Such a non-Markovian model can capture the path-dependence and memory effects in the financial market. Furthermore, we considered the non-negative constraint on the reinsurance and investment strategies. By solving a unconstrained optimisation problem parameterized by the Lagrange multiplier, we obtained the optimal strategy in terms of the unique solution to a BSDE driven by the Markov chain.

Some relevant problems are worthy of being discussed in further. First, we can assume that the interest rate is also a stochastic process. Second, we may consider the non-cheap reinsurance. Third, we only studied the pre-commitment strategy in this paper and hope to investigate the time-consistent equilibrium strategy in our future works.

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