Statistical Theory and Related Fields

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tstf20

Multivariate extremes and max-stable processes: discussion of the paper by Zhengjun Zhang

R. L. Smith

To cite this article: R. L. Smith (2021) Multivariate extremes and max-stable processes: discussion of the paper by Zhengjun Zhang, Statistical Theory and Related Fields, 5:1, 41-44, DOI: 10.1080/24754269.2021.1871709

To link to this article: https://doi.org/10.1080/24754269.2021.1871709

Published online: 20 Jan 2021.

STATISTICAL THEORY AND RELATED FIELDS
2021, VOL. 5, NO. 1, 41–44
https://doi.org/10.1080/24754269.2021.1871709

Taylor & Francis
Taylor & Francis Group

Multivariate extremes and max-stable processes: discussion of the paper by Zhengjun Zhang

R. L. Smith

Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27514, USA

ABSTRACT
This discussion reviews the paper by Zhengjun Zhang in the context of broader research on multivariate extreme value theory and max-stable processes.

ARTICLE HISTORY
Received 1 January 2021
Accepted 1 January 2021

KEYWORDS
Brown-Resnick processes; extreme value theory; M4 processes; max-stable processes; maximum likelihood; multivariate extremes

It is a pleasure to congratulate Professor Zhang on his excellent survey of statistical developments in extreme value theory, with particular reference to extreme value methods for nonlinear time series and applications in finance and econometrics. I would like to comment on this paper from the perspective of how extreme value theory has developed as a subject over the last 30 years.

In the early 1990s, the future direction of extreme value theory seemed clear. The early work on the 'three types theorem' (Fisher & Tippett, 1928; Gnedenko, 1943) had led to a well-developed theory for univariate extremes based on the generalised extreme value distribution (Prescott & Walden, 1980, 1983; Smith, 1985), and a parallel theory based on the generalised Pareto distribution (Pickands, 1975) had led to a broad set of techniques based on exceedances over thresholds (Davison & Smith, 1990; Smith, 1987). Meanwhile, there were a number of conceptually different but mathematically equivalent representations of multivariate extreme value distributions (Deheuvels, 1978; de Haan & Resnick, 1977; Pickands, 1981, 1989) and the beginnings of a statistical theory for estimating such distributions both for block maxima (Tawn, 1988, 1990) and for threshold exceedances (Coles & Tawn, 1991; Joe et al., 1992). There were even representation theorems for max-stable (or min-stable) process (Giné et al., 1990; Haan, 1984; de Haan & Pickands, 1986; Penrose, 1992; Vatan, 1985) which suggested a framework for extreme value theory of temporal and/or spatial processes. It seemed just a matter of time before a well-developed set of statistical methodologies existed for all these models.

Things did not quite work out that way. Over the years, it has become apparent that many new constructions are needed to address statistical problems related to extremes in multivariate and spatially/temporally dependent data, and new models and new statistical techniques have emerged to deal with them.

One major development was the recognition that the original construction of multivariate extreme value distributions was not sufficient to model all the kinds of dependency that arise in practice, in particular, that whereas traditional multivariate extreme value theory is good for 'asymptotically dependent' classes of multivariate extremes, the alternative case of 'asymptotically independent' is at least as important, if not more so. The papers of Ledford and Tawn (1996, 1997) started this trend, and also led to a class of time series models incorporating the same concept (Ledford & Tawn, 2003). Recent research such as Wadsworth et al. (2017) has shown how such models may be extended to incorporate both asymptotic dependence and asymptotic independence within a common class of multivariate distributions.

Quite apart from that, the theory of max-stable (and more recently (Bopp et al. (2018); Huser et al. (2020)), max infinitely divisible) processes has developed far from its original motivation. Schlather (2002) proposed a class of max-stable processes related to Gaussian processes, then a more general construction by Kabluchko et al. (2009) led to a strong focus on 'Brown-Resnick processes', the original construction of which was due to Brown and Resnick (1977), but extended by Kabluchko and others to a construction that allows essentially any Gaussian process to act as the generator

CONTACT R. L. Smith rls@email.unc.edu

for a max-stable process. A few years later, this was further generalised to extremal to processes (Opitz, 2013). Statistical theory for these processes was initially lacking, primarily because they do not have a closed-form expression for the joint density function in higher than two dimensions, so direct application of maximum likelihood is not possible. A major breakthrough was the paper by Padoan et al. (2010), that proposed an estimation method based on the principle of composite (or pairwise) likelihood, in effect, a product of pairwise density functions instead of the full joint density of a sample. This method is applicable, in principle, to any of the classes of max-stable processes for which continuous densities exist, and is also extendable to threshold-exceedance processes (Huser & Davison, 2013, 2014). Meanwhile, researchers have continued to work on exact maximum-likelihood approaches, which while computationally intensive, have increasingly appeared to be a feasible option for moderate-dimension processes (Wadsworth, 2015; Wadsworth & Tawn, 2014) and even extendable to Bayesian inference (Thibaud et al., 2016), which is important because it raises the possibility of hierarchical models built on max-stable processes. Applications of these models have been, to my knowledge, nearly all in the environmental/climate context (Blanchet & Creutin, 2017; Blanchet & Davison, 2011; Oesting et al., 2017; Reich & Shaby, 2019; Reich et al., 2014) which raises the interesting question of whether any of these models would be useful in the financial context which has largely motivated the methods in Professor Zhang's review. More extended reviews of these methods have appeared elsewhere, for example Cooley et al. (2012); Davison et al. (2019, 2012).

In contrast to that rather rich line of development, Professor Zhang and his co-authors have pursued an alternative approach to max-stable processes, motivated more by financial than environmental applications, but which over the years has also resulted in a wide class of models and associated statistical methods. These are based on the M4 process proposed by Smith and Weissman (1996) (equation (4.1) of the present paper) but which Professor Zhang has made his own with his numerous contributions. The original idea was a multivariate generalisation of a representation for time-dependent extremal processes due to Deheuvels (1983) that could, in principle, approximate any temporally stationary multivariate max-stable process. However, there were a number of disadvantages that frustrated the use of these processes as practical models for extremes. One major difficulty was the occurrence of 'signature patterns', that is, recurrent patterns in data generated by these models that would be unlikely to occur in real data (Zhang & Smith, 2004). One consequence of that is that joint densities for such processes are not merely hard to compute (as in Brown–Resnick processes, for instance) but do not exist at all, so likelihood-based methods of estimation are impossible. Alternative estimators are possible based on empirical processes (Zhang & Smith, 2010) but that always seemed to me a rather artificial construction. Another problem with M4 processes is that the full representation involves infinitely many parameters, and while in practice these would always be truncated, that can still leave too many parameters for practical estimation, suggesting a need for sparse representations.

However, what we have seen over the past decade are some major extensions of these models that make them much more practical for applications. For me, a particularly impressive paper is by Tang et al. (2013), who proposed the SM4R model that extended the original M4 process in three ways: *sparsity* –a large number of the coefficients are predefined to be zero, thus reducing the dimension of the estimating problem; *randomness* – the fixed coefficients in the M4 process are replaced by random variables, which effectively eliminates the signature pattern problem; and *cross-sectional dependence* – the use of an extreme value copula function to define a more complex structure of the Z variables compared with the original M4 process. Parameter estimation was by Generalized Method of Moments (GMM), which is a common method of estimation in complex financial time series models, and they applied their model to the estimation of a conditional Value at Risk in 10-day returns from a portfolio composed of three well-known US stock indices. Compared with traditional models such as stochastic volatility or GARCH, their approach allows improved characterisation of probabilities of extreme co-movements by two or more indices, and for extreme movements by the same index on consecutive days.

A further development by Zhang and Zhu (2016) extended the SM4R model to the copula structured M4 model, or CSM4, which took an extension of the M4 process due to Heffernan et al. (2007) and combined it with ideas from Tang et al. (2013) to develop a model that allowed a combination of asymptotic independence and asymptotic dependence, and applied it to analyse joint movements of three indices based on exchange rates.

Of the many extensions of the M4 process discussed in Professor Zhang's paper, I think these two are some of the most important, that demonstrate that their real applicability. Nevertheless, I have some questions:

(1) Has any attempt been made at exact maximum-likelihood (or Bayesian) estimation for these processes? As noted earlier, the parallel methodology based on Brown–Resnick and similar processes has moved quite a bit in that direction, which potentially could lead to even more general methodologies such as hierarchical models.
(2) It seems to me that the models are still rather limited in dimensionality – the major examples in both papers (Tang et al., 2013; Zhang & Zhu, 2016)

involve just three dependent time series, whereas modern financial applications may involve many more than that, and other fields such as genetics and environmental science involve datasets in potentially thousands of dimensions. I wonder if Professor Zhang has any general thoughts about that problem. In passing, I might note that a different approach by Zhao and Zhang (2018) included an example of up to 28 co-dependent series extracted from the Dow Jones Industrial index, but the methodology there was rather different, combining semiparametric models with a new 'max-copula' construction.

In summary, much has been achieved, but as more complex models and applications are developed, it always seems that there are still more complicated problems just over the horizon. Nevertheless, I commend this review to anyone looking to start research in an area of statistical modelling that already has a rich history and much more potential for the future.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

R. L. Smith is Professor of Statistics at the University of North Carolina, Chapel Hill, USA.

References

Blanchet, J., & Creutin, J.-D. (2017). Co-occurrence of extreme daily rainfall in the French Mediterranean region. *Water Resources Research*, *53*(11), 9330–9349. https://doi.org/10.1002/2017WR020717

Blanchet, J., & Davison, A. C. (2011). Spatial modeling of extreme snow depth. *The Annals of Applied Statistics*, *5*(3), 1699–1725. https://doi.org/10.1214/11-AOAS464

Bopp, G., Shaby, B., & Huser, R. (2018). A hierarchical max-infinitely divisible process for extreme areal precipitation over watersheds, Arxiv:1804.04588.

Brown, B. M., & Resnick, S. (1977). Extreme values of independent stochastic processes. *Journal of Applied Probability*, *14*(4), 732–739. https://doi.org/10.2307/3213346

Coles, S. G., & Tawn, J. A. (1991). Modelling extreme multivariate events. *Journal of the Royal Statistical Society, Series B*, *53*(2), 377–392. https://doi.org/10.1111/j.2517-6161.1991.tb01830.x

Cooley, D., Cisewski, J., Erhardt, R. J., Jeon, S., Mannshardt, E., Omolo, B. O., & Sun, Y. (2012). A survey of spatial extremes: Measuring spatial dependence and modeling spatial effects. *REVSTAT*, *10*(1), 135–165.

Davison, A. C., Huser, R., & Thibaud, E. (2019). Spatial extremes. In A. Gelfand, M. Fuentes, J. Hoeting, & R. Smith (Eds.), *Handbook of environmental and ecological statistics* (pp. 711–744).

Davison, A. C., Padoan, S. A., & Ribatet, M. (2012). Statistical modeling of spatial extremes. *Statistical Science*, *27*(2), 161–186. https://doi.org/10.1214/11-STS376

Davison, A. C., & Smith, R. L. (1990). Models for exceedances over high thresholds (with discussion). *Journal of the Royal Statistical Society, Series B*, *52*(3), 393–442. https://doi.org/10.1111/j.2517-6161.1990.tb01796.x

de Haan, L., & Pickands, J. (1986). Stationary min-stable stochastic processes. *Probability Theory and Related Fields*, *72*(4), 477–492. https://doi.org/10.1007/BF00344716

Deheuvels, P. (1978). Caracterisation complète des lois extrêmes multivariées et de la convergence des types extrêmes. *Publications de l'Institut de statistique de l'Université de Paris*, *23*, 1–36.

Deheuvels, P. (1983). Point processes and multivariate extreme values. *Journal of Multivariate Analysis*, *13*(2), 257–272. https://doi.org/10.1016/0047-259X(83)90025-8

Fisher, R. A., & Tippett, L. H. C. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Mathematical Proceedings of the Cambridge Philosophical Society*, *24*(2), 180–190. https://doi.org/10.1017/S0305004100015681

Giné, E., Hahn, M. G., & Vatan, P. (1990). Max-infinitely divisible and max-stable sample continuous processes. *Probability Theory and Related Fields*, *87*(2), 139–165. https://doi.org/10.1007/BF01198427

Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *The Annals of Mathematics*, *44*(3), 423–453. https://doi.org/10.2307/1968974

Haan, L. D. (1984). A spectral representation for max-stable processes. *The Annals of Probability*, *12*(4), 1194–1204. https://doi.org/10.1214/aop/1176993148

Haan, L., & Resnick, S. I. (1977). Limit theory for multivariate sample extremes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, *40*(4), 317–337. https://doi.org/10.1007/BF00533086

Heffernan, J., Tawn, J. A., & Zhang, Z. (2007). Asymptotically (in)dependent multivariate maxima of moving maxima processes. *Extremes*, *10*(1–2), 57–82. https://doi.org/10.1007/s10687-007-0035-1

Huser, R., & Davison, A. C. (2013). Composite likelihood estimation for the Brown-Resnick process. *Biometrika*, *100*, 511–518. https://doi.org/10.1093/biomet/ass089

Huser, R., & Davison, A. C. (2014). Space-time modelling of extreme events. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *76*(2), 439–461. https://doi.org/10.1111/rssb.2014.76.issue-2

Huser, R., Opitz, T., & Thibaud, E. (2020). Max-infinitely divisible models and inference for spatial extremes. *Scandinavian Journal of Statistics*.

Joe, H., Smith, R. L., & Weissman, I. (1992). Bivariate threshold methods for extremes. *Journal of the Royal Statistical Society, Series B*, *54*(1), 171–183. https://doi.org/10.1111/j.2517-6161.1992.tb01871.x

Kabluchko, Z., Schlather, M., & De Haan, L. (2009). Stationary max-stable fields associated to negative definite functions. *The Annals of Probability*, *37*(5), 2042–2065. https://doi.org/10.1214/09-AOP455

Ledford, A. W., & Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, *83*(1), 169–187. https://doi.org/10.1093/biomet/83.1.169

Ledford, A. W., & Tawn, J. A. (1997). Modelling dependence within joint tail regions. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *59*(2), 475–499. https://doi.org/10.1111/rssb.1997.59.issue-2

Ledford, A. W., & Tawn, J. A. (2003). Diagnostics for dependence within time series extremes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *65*(2), 521–543. https://doi.org/10.1111/rssb.2003.65.issue-2

Oesting, M., Schlather, M., & Friedrichs, P. (2017). Statistical post-processing of forecasts for extremes using bivariate Brown-Resnick processes with an application to wind gusts. *Extremes*, *20*, 309–332. https://doi.org/10.1007/s10687-016-0277-x

Opitz, T. (2013). Extremal t processes: Elliptical domain of attraction and a spectral representation. *Journal of Multivariate Analysis*, *122*, 409–413. https://doi.org/10.1016/j.jmva.2013.08.008

Padoan, S. A., Ribatet, M., & Sisson, S. A. (2010). Likelihood-based inference for max-stable processes. *Journal of the American Statistical Association*, *105*(489), 263–277. https://doi.org/10.1198/jasa.2009.tm08577

Penrose, M. D. (1992). Semi-min-stable processes. *The Annals of Probability*, *20*(3), 1450–1463. https://doi.org/10.1214/aop/1176989700

Pickands, J. (1981). Multivariate extreme value distributions. *Bulletin of the International Statistical Institute*, *49*, 859–878.

Pickands, J. (1989). Multivariate negative exponential and extreme value distributions. In J. Hüsler and R. Reiss (Eds.), *Extreme value theory: Lecture notes in statistics* (p. 51). Springer.

Pickands, J. P., III. (1975). Statistical inference using extreme order statistics. *The Annals of Statistics*, *3*(1), 119–131. https://doi.org/10.1214/aos/1176343003

Prescott, P., & Walden, A. T. (1980). Maximum likelihood estimation of the parameters of the generalized extreme-value distribution. *Biometrika*, *67*(3), 723–724. https://doi.org/10.1093/biomet/67.3.723

Prescott, P., & Walden, A.T. (1983). Maximum likeiihood estimation of the parameters of the three-parameter generalized extreme-value distribution from censored samples. *Journal of Statistical Computation and Simulation*, *16*(3–4), 241–250. https://doi.org/10.1080/00949658308810625

Reich, B. J., & Shaby, B. A. (2019). A spatial Markov model for climate extremes. *Journal of Computational and Graphical Statistics*, *28*(1), 117–126. https://doi.org/10.1080/10618600.2018.1482764

Reich, B. J., Shaby, B. A., & Cooley, D. (2014). A hierarchical model for serially-dependent extremes: A study of heat waves in the western US. *Journal of Agricultural, Biological, and Environmental Statistics*, *19*(1), 119–135. https://doi.org/10.1007/s13253-013-0161-y

Schlather, M. (2002). Models for stationary max-stable random fields. *Extremes*, *5*(1), 33–44. https://doi.org/10.1023/A:1020977924878

Smith, R. L. (1985). Maximum likelihood estimation in a class of nonregular cases. *Biometrika*, *72*(1), 67–90. https://doi.org/10.1093/biomet/72.1.67

Smith, R. L. (1987). Estimating tails of probability distributions. *The Annals of Statistics*, *15*(3), 1174–1207. https://doi.org/10.1214/aos/1176350499

Smith, R. L., & Weissman, I. (1996). *Characterization and estimation of the multivariate extremal index* (Technical Report). University of North Carolina at Chapel Hill.

Tang, R., Shao, J., & Zhang, Z. (2013). Sparse moving maxima models for tail dependence in multivariate financial time series. *Journal of Statistical Planning and Inference*, *143*(5), 882–895. https://doi.org/10.1016/j.jspi.2012.11.008

Tawn, J. A. (1988). Bivariate extreme value theory: Models and estimation. *Biometrika*, *75*(3), 397–415. https://doi.org/10.1093/biomet/75.3.397

Tawn, J. A. (1990). Modelling multivariate extreme value distributions. *Biometrika*, *77*(2), 245–253. https://doi.org/10.1093/biomet/77.2.245

Thibaud, E., Aalto, J., Cooley, D. S., Davison, A. C., & Heikkinen, J. (2016). Bayesian inference for the Brown-Resnick process, with an application to extreme low temperatures. *The Annals of Applied Statistics*, *10*(4), 2303–2324. https://doi.org/10.1214/16-AOAS980

Vatan, P. (1985). Max-infinitely divisible and max-stablility in infinite dimensions. *Lecture Notes in Mathematics*, *1153*, 400–425. https://doi.org/10.1007/BFb0074941

Wadsworth, J. L. (2015). On the occurrence times of componentwise maxima and bias in likelihood inference for multivariate max-stable distributions. *Biometrika*, *102*(3), 705–711. https://doi.org/10.1093/biomet/asv029

Wadsworth, J. L., & Tawn, J. A. (2014). Efficient inference for spatial extreme value processes associated to log-gaussian random functions. *Biometrika*, *101*(1), 1–15. https://doi.org/10.1093/biomet/ast042

Wadsworth, J. L., Tawn, J. A., Davison, A. C., & Elton, D. M. (2017). Modelling across extremal dependence classes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *79*(1), 149–175. https://doi.org/10.1111/rssb.12157

Zhang, Z., & Smith, R. L. (2004). The behavior of multivariate maxima of moving maxima processes. *Journal of Applied Probability*, *41*(4), 1113–1123. https://doi.org/10.1239/jap/1101840556

Zhang, Z., & Smith, R. L. (2010). On the estimation and application of max-stable processes. *Journal of Statistical Planning and Inference*, *140*(5), 1135–1153. https://doi.org/10.1016/j.jspi.2009.10.014

Zhang, Z., & Zhu, B. (2016). Copula structured M4 processes with application to high-frequency financial data. *Journal of Econometrics*, *194*(2), 231–241. https://doi.org/10.1016/j.jeconom.2016.05.004

Zhao, Z., & Zhang, Z. (2018). Semiparametric dynamic max-copula model for multivariate time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *80*(2), 409–432. https://doi.org/10.1111/rssb.12256