Construction of strong orthogonal arrays of strength three and three minus via Addelman–Kempthorne orthogonal arrays

Qiang Gao, Bochuan Jiang, Linyue Shang & Yaping Wang

Published online: 22 Jan 2026.

Submit your article to this journal ↗

Article views: 39

View related articles ↗

View Crossmark data ↗

Taylor & Francis
Taylor & Francis Group

Construction of strong orthogonal arrays of strength three and three minus via Addelman–Kempthorne orthogonal arrays

Qiang Gao[a], Bochuan Jiang [a], Linyue Shang[a] and Yaping Wang[b]

[a]School of Mathematics and Statistics, Beijing Jiaotong University, Beijing, People's Republic of China;
[b]KLATASDS-MOE, School of Statistics, East China Normal University, Shanghai, People's Republic of China

ABSTRACT

Space-filling designs with superior low-dimensional properties are highly required in computer experiments. Strong orthogonal arrays (SOAs) represent a class of such designs that outperform ordinary orthogonal arrays in their stratification properties within low dimensions. Nevertheless, current methods for constructing high-strength SOAs are rare, and they typically rely on regular designs, thereby limiting the number of runs in the final arrays to prime powers. This study presents new construction methods for three types of SOAs: SOAs of strength three, column-orthogonal SOAs (OSOAs) of strength three and three minus. The resulting designs have run sizes of twice an odd prime power without replications, filling the gaps in run sizes left by existing constructions. The projection properties of Addelman–Kempthorne orthogonal arrays are instrumental in the development of these construction methods.

1. Introduction

Deterministic computer simulations of physical phenomena are becoming increasingly prevalent in engineering and science. Space-filling designs are highly advantageous for these simulations (Fang et al., 2006; Santner et al., 2003). Inspired by (t, m, s)-nets from quasi-Monte Carlo methods, He and Tang (2013) introduced strong orthogonal arrays (SOAs) as a new class of space-filling designs. Compared to an orthogonal array of strength t, an SOA of strength t exhibits better space-filling properties in all dimensions lower than t, while maintaining the same space-filling properties in the t dimensions.

To attain the superior properties of an SOA, its strength should be greater than 2. He and Tang (2014) characterized SOAs of strength 3, with a focus on the space-filling properties in two and three dimensions. He et al. (2018) introduced SOAs of strength 2+, emphasizing space-filling properties in two dimensions and a relatively larger number of factors. Zhou and Tang (2019) further introduced SOAs of strength 3−, which have the same properties as SOAs of strength 3 but with a smaller number of levels. For a given run size, the higher the strength of the SOAs, the more superior their space-filling properties become. Meanwhile, for the same run size, a higher strength leads to a smaller number of factors. The construction of SOAs has garnered significant interest from researchers; see e.g., Liu and Liu (2015),

Shi and Tang (2019), Zhou and Tang (2019), Cheng et al. (2021), Jiang et al. (2021), Wang et al. (2021), Li et al. (2022), Chen and Tang (2022), Jiang et al. (2025) and so on.

Column orthogonality is important for designs of computer experiments, since column-orthogonal designs allow the main effects of the factors to be estimated independently. Moreover, Bingham et al. (2009) argued that column orthogonality can be viewed as a useful stepping stone to space-filling designs when Gaussian-process models are considered. Liu and Liu (2015) constructed column-orthogonal strong orthogonal arrays (OSOAs) of strength t through ordinary orthogonal arrays of strength t. Zhou and Tang (2019) employed Hadamard matrices and saturated orthogonal arrays to construct OSOAs of strength 3− and 2+. Bao et al. (2023) utilized difference matrices and generator matrices to construct OSOAs of strength 2+, 3− and 3.

Existing methods for constructing SOAs typically rely on orthogonal arrays, which can be classified as either regular or nonregular. Regular orthogonal arrays, characterized by their defining relation, are widely used to construct SOAs with s^n runs due to their clear structure, where s is a prime power and n is a positive integer. Nonregular orthogonal arrays offer greater flexibility in run sizes but possess more complex structures, which can present challenges in their application for constructing SOAs. Consequently, high-strength ($t \geq 3$) SOAs with λs^n runs, where λ is a positive integer, are usually derived by juxtaposing λ existing SOAs with s^n runs, which may result in a relatively small number of factors and potential repeated runs.

There is, however, a class of nonregular orthogonal arrays with $2s^n$ runs that have a clear structure, namely, Addelman–Kempthorne orthogonal arrays (Addelman & Kempthorne, 1961; Hedayat et al., 1999). Jiang et al. (2021) used these arrays to construct SOAs of strength 2+. This paper explores the application of three-level Addelman–Kempthorne orthogonal arrays to construct SOAs and OSOAs of strength 3 or 3−. We systematically derive all possible three-dimensional projection properties of Addelman–Kempthorne orthogonal arrays. Using these properties, we present new construction methods for three types of SOAs: SOAs of strength 3, column-orthogonal SOAs (OSOAs) of strength 3− and 3. Our methods avoid run repetition, address gaps in existing construction theory, and produce designs with twice the powers of three run sizes and typically more factors compared to existing designs.

The remainder of this article is organized as follows. Section 2 provides the necessary preliminaries and definitions. Section 3 introduces the Addelman–Kempthorne orthogonal arrays and explores their three-dimensional projection properties. Section 4 details the constructions of SOAs of strength 3, OSOAs of strength 3 and 3− based on the Addelman–Kempthorne orthogonal arrays. Section 5 concludes the paper. All proofs are deferred to appendices.

2. Preliminaries

Let $\mathbb{Z}_s = \{0, \ldots, s-1\}$ be the ring of integers modulo s. An $N \times m$ matrix with entries from $\mathbb{Z}_{s_j}$ in the jth column is an orthogonal array (OA) of N runs, m factors and strength t, denoted by $OA(N, m, s_1 \times \cdots \times s_m, t)$, if all possible combinations appear with the same frequency in any of its $N \times t$ submatrices. If $s_1 = \cdots = s_m = s$, the array is called symmetrical and a simpler notation $OA(N, m, s, t)$ or $OA(t)$ is used; otherwise, it is called asymmetrical.

An $OA(N, m, s^t, 1)$ with entries from $\mathbb{Z}_{s^t}$ is called a strong orthogonal array of N runs, m factors, s^t levels, and strength t, denoted by $SOA(N, m, s^t, t)$, if any subarray consisting of g columns can be collapsed into an $OA(N, g, s^{u_1} \times \cdots \times s^{u_g}, g)$ for any positive integers

$u_1, \ldots, u_g$ such that $u_1 + \cdots + u_g = t$. Here, the collapsing of s^t levels into s^{u_j} levels is according to $\lfloor a/s^{t-u_j} \rfloor$ for $a \in \mathbb{Z}_{s^t}$, where $\lfloor w \rfloor$ represents the largest integer not exceeding w (He & Tang, 2013).

A result from He and Tang (2013) is needed for the development of our construction methods.

Lemma 2.1: *An* SOA$(N, m, s^3, 3)$ *D exists if and only if there exist three arrays* $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_m)$ *and* $C = (c_1, \ldots, c_m)$ *such that* (a_i, a_j, a_k), (a_i, a_j, b_j) *and* (a_i, b_i, c_i) *are* OA$(N, 3, s, 3)$s *for all* $i \neq j$, $i \neq k$ *and* $j \neq k$. *These arrays are related through* $D = s^2A + sB + C$.

When the run size N is a prime power, He and Tang (2014) showed that an SOA$(N, m, s^3, 3)$ can be constructed from a regular OA$(N, m, s, 3)$. We summarize this result in Lemma 2.2. This approach will be employed for comparison with our methods.

Lemma 2.2: *For any prime power s and any integer* $n \geq 3$, *if there exists a regular* OA$(s^n, m, s, 3)$ *with* $m \leq (s^{n-1} - 1)/(s - 1)$, *an* SOA$(s^n, m, s^3, 3)$ *can be constructed.*

Inspired by SOAs of strength 3, Zhou and Tang (2019) propose the concept of SOAs of strength 3−. An OA$(N, m, s^2, 1)$ with entries from $\mathbb{Z}_{s^2}$ is called an SOA of strength 3−, denoted by SOA$(N, m, s^2, 3-)$, if any two-column subarray can be collapsed into an OA$(N, 2, s^2 \times s, 2)$ and an OA$(N, 2, s \times s^2, 2)$, and any three-column subarray can be collapsed into an OA$(N, 3, s, 3)$. The following result is due to Zhou and Tang (2019).

Lemma 2.3: *An* SOA$(N, m, s^2, 3-)$ *D exists if and only if there exist two arrays A and B, where* $A = (a_1, \ldots, a_m)$ *is an* OA$(N, m, s, 3)$ *and* $B = (b_1, \ldots, b_m)$ *is an* OA$(N, m, s, 1)$ *such that* (a_i, a_j, b_i) *is an* OA(3) *for any* $i \neq j$. *The three arrays are related through* $D = sA + B$.

Remark 2.1: By comparing Lemmas 2.1 and 2.3, we can infer that an SOA$(N, m, s^2, 3-)$ exists if and only if an SOA$(N, m, s^3, 3)$ exists. Specifically, if $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ satisfy the conditions in Lemma 2.3, we can simply define $C = (a_2, \ldots, a_m, a_1)$. It can be verified that the arrays A, B and C fulfil the conditions in Lemma 2.1. This equivalence was also noted by Bao et al. (2023).

An array D is called column-orthogonal if the inner product of any two columns of the centred array is zero. Throughout this article, centring a balanced s-level array means that each entry $a \in \mathbb{Z}_s$ is transformed to $a - (s-1)/2$. We use OSOA$(N, m, s^2, 3-)$ and OSOA$(N, m, s^3, 3)$ to denote column-orthogonal SOA$(N, m, s^2, 3-)$ and SOA$(N, m, s^3, 3)$, respectively.

By Theorem 2 of Zhou and Tang (2019), Remark 1 and Section 3 of Li et al. (2022), we obtain the following result, which will serve as the basis of our construction methods.

Lemma 2.4: *Assume there exists an* OA$(N, m, s, 3)$ $A = (a_1, \ldots, a_m)$ *and an* OA$(N, m, s, 2)$ $B = (b_1, \ldots, b_m)$ *such that* (a_i, a_j, b_i) *is an* OA(3) *for any* $i \neq j$. *Then three types of designs (i)* OSOA$(N, m, s^2, 3-)$, *(ii)* SOA$(N, m, s^3, 3)$ *and (iii)* OSOA$(N, 2\lfloor m/2 \rfloor, s^3, 3)$ *can be constructed.*

Remark 2.2: In Lemma 2.4, the second and third types of resulting designs enhance the number of levels, which is beneficial for improving the one-dimensional uniformity of the designs. Both the first and the third types of designs have column orthogonality, while the latter has almost the same number of factors as the others. Comprehensively, the third type of design is a preferable choice. For more in-depth insights, one can refer to Li et al. (2022).

Specifically, $D = sA + B$ is the desired OSOA$(N, m, s^2, 3-)$. The procedure for deriving the SOA$(N, m, s^3, 3)$ is detailed in Remark 2.1. We verify Lemma 2.4 in Appendix 2, which illustrates the construction of the OSOA$(N, 2\lfloor m/2 \rfloor, s^3, 3)$.

A space-filling design with no repeated runs is desirable for computer experiments. If the matrix A or B in Lemma 2.4 has no repeated rows, the resulting three types of designs immediately possess the following additional property.

Proposition 2.5: *(i) If the matrix A or B in Lemma 2.4 has no repeated rows, the constructed designs OSOA$(N, m, s^2, 3-)$ and SOA$(N, m, s^3, 3)$ will have no repeated runs. Furthermore, if m is even, the constructed design OSOA$(N, m, s^3, 3)$ will also have no repeated runs. (ii) If m is odd and there exists a subarray with $(m-1)$ columns in A or B that contains no repeated rows, the constructed OSOA$(N, m-1, s^3, 3)$ using the corresponding $(m-1)$ subarrays of A and B will have no repeated runs.*

In the following sections, we will focus on constructing arrays A and B from Addelman–Kempthorne OAs (Hedayat et al., 1999, Chap. 3.3), ensuring they meet the conditions specified in Lemma 2.4 and Proposition 2.5. To accomplish this, we will first enumerate all cases of three-column subarrays with strength 3 in an Addelman–Kempthorne OA in Section 3. Then, Section 4 will outline the specific methods for constructing A and B, as well as the final SOAs.

3. Enumeration of three-dimensional projections of Addelman–Kempthorne OAs

Let s be an odd prime power and GF(s) be a Galois field of order s with α being a primitive element. Write the elements of GF(s) as $\alpha_0 = 0$ and $\alpha_j = \alpha^j, j \geq 1$. Obviously, $\alpha_{s-1} = \alpha^{s-1} = 1$. Let $r = (s^{n-1} - 1)/(s - 1)$ and $l_i = (l_{1,i}, \ldots, l_{n-1,i})^\top, i = 1, \ldots, r$, be all possible nonzero $(n-1)$-vectors in $[\mathrm{GF}(s)]^{n-1}$ such that the first nonzero entry is 1. It is seen that $l_1, l_2, \ldots, l_r$ are pairwise linearly independent. The matrix formed by collecting $l_1, \ldots, l_r$, denoted by

$$L^{(s,n-1)} = (l_1, \ldots, l_r),$$

is termed as a generator. Further, let $X_1, \ldots, X_n$ be n indeterminate elements and $X = (X_2, \ldots, X_n)$. Define $f^{(0)} = g^{(0)} = X_1$. For each i, $1 \leq i \leq r$, define $4s$ functions of X_1 and X as

$$f_j^{(i)}(X_1, X) = \begin{cases} Xl_i, & \text{if } j = 1, \\ X_1 + \alpha_{j-1}Xl_i, & \text{if } 2 \leq j \leq s, \\ X_1^2 + \alpha_{j-s-1}X_1 + Xl_i, & \text{if } s+1 \leq j \leq 2s, \end{cases}$$

and

$$g_j^{(i)}(X_1, X) = \begin{cases} Xl_i, & \text{if } j = 1, \\ X_1 + \alpha_{j-1}Xl_i + \eta_{j-1}, & \text{if } 2 \le j \le s, \\ \alpha\left(X_1^2 + \alpha_{j-s-1}X_1\right) + Xl_i + \gamma_{j-s-1}, & \text{if } s+1 \le j \le 2s, \end{cases}$$

where $\eta_j = (\alpha - 1)/(4\alpha_{j+1})$ and $\gamma_j = \alpha_{2j}(\alpha - 1)/4, j = 0, \dots, s-1$.

Construct two $s^n \times (2rs + 1)$ arrays D_f and D_g as follows. Label the columns of D_f by $f^{(0)}, f_1^{(1)}, \dots, f_{2s}^{(1)}, \dots, f_1^{(r)}, \dots, f_{2s}^{(r)}$ and its rows by all possible n-tuples with entries from GF(s). The entries are obtained by evaluating the polynomial specified by the column label at the n-tuple specified by the row label. The array D_g is constructed in a similar way, but with column labels being $g^{(0)}, g_1^{(1)}, \dots, g_{2s}^{(1)}, \dots, g_1^{(r)}, \dots, g_{2s}^{(r)}$. By Theorem 3.16 of Hedayat et al. (1999), the juxtaposition of the two arrays

$$D_0 = \begin{pmatrix} D_f \\ D_g \end{pmatrix} \tag{1}$$

is an OA($2s^n, 2(s^n - 1)/(s - 1) - 1, s, 2$), and termed as an Addelman–Kempthorne OA. Denote $h^{(0)}$ as the column of D_0 obtained by $f^{(0)}$ and $g^{(0)}$, and $h_j^{(i)}$ as the column obtained by $f_j^{(i)}$ and $g_j^{(i)}$ for $i = 1, \dots, r$ and $j = 1, \dots, 2s$.

The rest of this section is devoted to investigating the strength of all possible three-dimensional projections of D_0. The results are summarized in Propositions 3.1–3.5. The proofs of these propositions are conceptually straightforward but quite tedious and are provided in Appendix 2.

Proposition 3.1: *The following results hold:*

(i) $(h^{(0)}, h_j^{(i)}, h_{j'}^{(i')})$ *forms an OA(3) if and only if $i \ne i'$;*

(ii) $(h_{j_1}^{(i)}, h_{j_2}^{(i')}, h_{j_3}^{(i')}), j_2 \ne j_3$, *forms an OA(3) if and only if $i \ne i'$;*

(iii) $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ *forms an OA(3) if $l_{i_1}, l_{i_2}, l_{i_3}$ are linearly independent.*

For linearly dependent distinct $l_{i_1}, l_{i_2}, l_{i_3}$, there always exist elements $\beta_1, \beta_2 \in$ GF(s) \ {0} satisfying $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$. For $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ with such $l_{i_1}, l_{i_2}, l_{i_3}$, we have the following Propositions 3.2–3.5.

Proposition 3.2: *Suppose i_1, i_2, i_3 are distinct, $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$ and $1 \le j_1 \le j_2 \le j_3 \le s$. Then $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ forms an OA(3) if and only if one of the following conditions holds:*

(i) $j_1 = j_2 = 1$ *and $j_3 > 1$;*

(ii) $j_1 = 1, j_2 > 1$ *and $\alpha_{j_2} \ne \beta_2 \alpha_{j_3}$;*

(iii) $j_1 > 1$ *and $\alpha_{j_1+j_2} \ne \beta_1 \alpha_{j_2+j_3} + \beta_2 \alpha_{j_1+j_3}$.*

Proposition 3.3: *Suppose i_1, i_2, i_3 are distinct, $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}, 1 \le j_1 \le j_2 \le s$ and $s + 1 \le j_3 \le 2s$. Then $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ forms an OA(3) if and only if one of the following conditions holds:*

(i) $j_1 = j_2 = 1$;

(ii) $j_1 = 1, j_2 > 1$ *and* $\beta_2 = 1$;

(iii) $j_1 > 1$ *and* $\beta_1 \alpha_{j_2}^2 + \beta_2 \alpha_{j_1}^2 = (\beta_1 \alpha_{j_2} + \beta_2 \alpha_{j_1})^2$.

Proposition 3.4: *Suppose* i_1, i_2, i_3 *are distinct,* $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}, 1 \le j_1 \le s$ *and* $s + 1 \le j_2, j_3 \le 2s$. *Then* $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ *forms an* OA(3) *if and only if one of the following conditions holds:*

(i) $j_1 = 1, \beta_2 = 1$ *and* $j_2 \ne j_3$;

(ii) $j_1 = 1, \beta_2 \ne 1$ *and* $j_2 = j_3$;

(iii) $j_1 > 1, \beta_2 = 1$ *and* $\beta_1 \notin \{\alpha_{j_1 - 1}(\alpha_{j_3 - s - 1} - \alpha_{j_2 - s - 1}), \alpha_{j_1}(\alpha_{j_3 - s - 1} - \alpha_{j_2 - s - 1})\}$;

(iv) $j_1 > 1, \beta_2 \ne 1$ *and* $\beta_1 \beta_2^{-1}(\beta_1 + \beta_2 - 1) = \alpha_{2j_1 - 1}(\alpha_{j_3 - s - 1} - \alpha_{j_2 - s - 1})^2$.

Proposition 3.5: *Suppose* i_1, i_2, i_3 *are distinct,* $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$ *and* $s + 1 \le j_1, j_2, j_3 \le 2s$. *Then* $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$ *forms an* OA(3) *if and only if one of the following conditions holds:*

(i) $\beta_1 + \beta_2 = 1$ *and* $\alpha_{j_3 - s - 1} \ne \beta_1 \alpha_{j_1 - s - 1} + \beta_2 \alpha_{j_2 - s - 1}$;

(ii) $\beta_1 + \beta_2 \ne 1$ *and* $(\alpha_{j_3 - s - 1} - \beta_1 \alpha_{j_1 - s - 1} - \beta_2 \alpha_{j_2 - s - 1})^2 = (1 - \beta_1 - \beta_2)(\alpha_{j_3 - s - 1}^2 - \beta_1 \alpha_{j_1 - s - 1}^2 - \beta_2 \alpha_{j_2 - s - 1}^2)$.

Note that Proposition 1 of Jiang et al. (2021) identifies only specific three-column subarrays with strength 3, rather than all cases. In contrast, Propositions 3.1–3.5 completely determine the strength of any three-column subarray in the Addelman–Kempthorne OA D_0, and thus cover the results of Jiang et al. (2021) as special cases.

To enable readers to quickly refer to the applicable scenarios of Propositions 3.1–3.5 during the design construction, we include the key conditions of these propositions in Table A1 in Appendix 1, while the process of how to apply these propositions to determine whether a three-column Addelman–Kempthorne subarray is an OA(3) is summarized in Algorithm 1. The primary computation of Algorithm 1 is concentrated on calculating β_1 and β_2, with an overall computational complexity of $O(sn)$. Since both s and n are relatively small, Algorithm 1 is highly efficient. In this article, we consider Algorithm 1 to impose a constant-time overhead.

4. Construction methods

This section presents two types of methods for constructing SOA($2s^n, m, s^3, 3$)s, OSOA($2s^n, m, s^2, 3-$)s and OSOA($2s^n, 2\lfloor m/2 \rfloor, s^3, 3$)s. According to Lemma 2.4, it suffices to find an OA($2s^n, m, s, 3$) $A = (a_1, \ldots, a_m)$ and an OA($2s^n, m, s, 2$) $B = (b_1, \ldots, b_m)$ such that (a_i, a_j, b_i) is an OA(3) for any $i \ne j$. We will first detail the algorithmic construction for small values of n and subsequently provide a recursive construction for larger values of n.

4.1. Construction using algorithms

This section presents the algorithmic construction for generating an OA($2s^n, m, s, 3$) $A = (a_1, \ldots, a_m)$ and an OA($2s^n, m, s, 2$) $B = (b_1, \ldots, b_m)$, such that (a_i, a_j, b_i) is an OA(3) for any $i \ne j$. Here, s is an odd prime power and $n \ge 3$ is an integer. The core of the construction involves appropriately selecting columns for A and B from the base orthogonal array, i.e.,

Algorithm 1 The process of determining whether a three-column subarray D' in the Addelman–Kempthorne OA is an OA(3)

1: Suppose $D' = (h^{(0)}, h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)})$ or $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$, where $1 \leq j_1 \leq j_2 \leq j_3 \leq 2s$.
2: **if** $h^{(0)} \in D'$ **then**
3:     use Proposition 3.1(i).
4: **else if** i_1, i_2, i_3 are not pairwise distinct **then**
5:     use Proposition 3.1(ii).
6: **else if** $l_{i_1}, l_{i_2}, l_{i_3}$ are linearly independent **then**
7:     use Proposition 3.1(iii).
8: **else if** $j_3 \leq s$ **then**
9:     use Proposition 3.2.
10: **else if** $j_2 \leq s < j_3$ **then**
11:     use Proposition 3.3.
12: **else if** $j_1 \leq s < j_2$ **then**
13:     use Proposition 3.4.
14: **else**
15:     use Proposition 3.5.
16: **end if**

the Addelman–Kempthorne $OA(2s^n, 2(s^n - 1)/(s - 1) - 1, s, 2)$ denoted by D_0. For any m-column subarray A of D_0, define

$$\rho(A) = k \left/ \binom{m}{3} \right., \tag{2}$$

where k is the number of strength-three three-column subarrays in A. Obviously, $0 \leq \rho(A) \leq 1$, and $\rho(A) = 1$ if and only if A is an OA(3). This means that we can obtain an OA(3) subarray A by maximizing $\rho(A)$. In this section, $\rho(A)$ can be efficiently calculated by using Propositions 3.1–3.5 as in Algorithm 1.

The construction of array A in Lemma 2.4 is crucial. It is initialized by randomly selecting m columns from D_0. Subsequently, A is iteratively refined into $\widetilde{A}$ through random column swaps with the set $D \setminus A$. This iterative process continues until the metric $\rho(A)$ attains its maximum value of 1. The heuristic local search algorithm is detailed in Algorithm 2. The algorithm parameters include $\tau = \mu\nu$ (the total number of iterations) and a threshold sequence $T_1, \ldots, T_\tau$. Here μ and ν are the numbers of inner and outer iterations, respectively, and the sequence of thresholds is determined as $T_1 = \cdots = T_\mu$, $T_{\mu+1} = \cdots = T_{2\mu} = \lambda T_1$, $T_{2\mu+1} = \cdots = T_{3\mu} = \lambda^2 T_1$, and so on, where $\lambda = (T_\tau/T_1)^{\frac{1}{\nu-1}}$.

The update formula for Δ in Step 5 of Algorithm 2 can be obtained. For array $A = (a_1, \ldots, a_m)$, the formula of $\rho(A)$ can be rewritten as

$$\rho(A) = (k_1 + \cdots + k_m) \left/ \binom{m}{3} \right., \tag{3}$$

where for $i = 1, \ldots, m$, k_i is the number of strength-three three-column combinations (a_i, a_p, a_q)s, where $1 \leq p < q \leq m$, $p \neq i$ and $q \neq i$. In Step 5 of Algorithm 2, suppose that $\widetilde{A} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m)$ is obtained by swapping the jth column a_j of A and the column d in

Algorithm 2 A heuristic local search algorithm for constructing an m-column subarray A with the largest $\rho(A)$-value

1: Initialize τ and the sequence of thresholds $T_1, \ldots, T_\tau$.
2: Let D_0 be the base Addelman–Kempthorne OA$(2s^n, 2(s^n-1)/(s-1)-1, s, 2)$. Let A be an OA$(2s^n, m, s, 2)$ whose columns come randomly from D_0.
3: Compute $\rho = \rho(A)$ in (2) by Algorithm 1. Let $A_{\text{opt}} := A$ and $\rho_{\text{opt}} := \rho$.
4: **for** $i = 1 : \tau$ **do**
5:     Randomly select columns $a_i \in A$ and $d \in D_0 \setminus A$. Swap columns a_i and d to obtain $\widetilde{A}$. Compute $\Delta = \tilde\rho - \rho$, where $\tilde\rho = \rho(\widetilde{A})$.
6:        **if** $\Delta > -\rho \cdot T_i$ **then**
7:            set $A = \widetilde{A}$ and $\rho = \tilde\rho$
8:            **if** $\rho > \rho_{\text{opt}}$ **then**
9:                update $A_{\text{opt}} := A$, $\rho_{\text{opt}} := \rho$
10:            **end if**
11:        **end if**
12:        **if** $\rho_{\text{opt}}=1$ **then**
13:            break
14:        **end if**
15: **end for**
16: return $A_{\text{opt}}, \rho_{\text{opt}}$;

$D_0 \setminus A$, where $1 \leq j \leq m$. Let $\tilde{k}_j$ denote the number of strength-three three-column combinations $(\tilde{a}_j, \tilde{a}_p, \tilde{a}_q)$, where $1 \leq p < q \leq m$, $p \neq j$ and $q \neq j$. It is clear that (a_i, a_p, a_q) and $(\tilde{a}_i, \tilde{a}_p, \tilde{a}_q)$ have the same strength, if $i \neq j$, $p \neq j$ and $q \neq j$. So $k_i = \tilde{k}_i$ for all $i \neq j$, and only $\tilde{k}_j$ needs to be calculated via Algorithm 1. Therefore, the updating formula for Δ simplifies to

$$\Delta = \tilde\rho - \rho = (\tilde{k}_j - k_j) \Big/ \binom{m}{3}, \tag{4}$$

with the computing complexity $O(m^2)$.

To obtain the largest strength-three Addelman–Kempthorne subarray A, we employ the bisection method to repeatedly call Algorithm 2. The specific process is outlined in Algorithm 3.

Once we obtain a strength-three Addelman–Kempthorne subarray $A = (a_1, \ldots, a_m)$ through Algorithms 2 and 3, a greedy algorithm, outlined as Algorithm 4, is performed to search the optimal array B. Initially, we set $B = A$. For each column a_i of A, we iterate b_i over all columns in $D_0 \setminus (A \cup B)$, evaluating the strength of all three-column combinations (a_i, b_i, a_j), $1 \leq j \neq i \leq m$. The column b_i yielding the highest count of strength-three combinations is selected. To determine whether (a_i, b_i, a_j) is an OA(3), Algorithm 1 is employed. The total complexity of Algorithm 4 is $O((m_0 - m)m^2)$, where $m_0 = 2(s^n - 1)/(s-1) - 1$.

As an illustration, we consider the case of $s = n = 3$. He and Tang (2014) constructed an SOA$(54, 5, 27, 3)$ whose number of factors reaches the maximum. By Theorem 4.3 of Bao et al. (2023), it is possible to obtain an OSOA$(27, 4, 9, 3-)$ and an OSOA$(27, 4, 27, 3)$. Therefore, a straightforward method to construct an OSOA$(54, 4, 9, 3-)$ and an OSOA$(54, 4, 27, 3)$ is to juxtapose two OSOA$(27, 4, 9, 3-)$ arrays and two OSOA$(27, 4, 27, 3)$ arrays, respectively.

Algorithm 3 Bisection method for maximal subarray A such that $\rho(A) = 1$

1: Initialize low = 3, high = $(s^n - 1)/(s - 1)$. Set $A_{opt} := \emptyset$.
2: **while** low $\leq$ high **do**
3:      Set mid = $\lfloor (\text{low} + \text{high})/2 \rfloor$.
4:      Call Algorithm 2 with parameters $m = \text{mid}$ to obtain $\widetilde{A}$ and $\tilde{\rho}$.
5:      **if** $\tilde{\rho} = 1$ **then**
6:          Update $A_{opt} := \widetilde{A}$ and low = mid + 1.
7:      **else**
8:          Set high = mid − 1.
9:      **end if**
10: **end while**
11: return A_{opt}.

Algorithm 4 Searching for the optimal array B based on a given array A

1: Let $A = (a_1, \ldots, a_m)$ be an m-column subarray of the Addelman–Kempthorne OA D_0. Initialize $B = (b_1, \ldots, b_m)$ with $b_i = a_i$ for $i = 1, \ldots, m$.
2: For $i = 1, \ldots, m$, let q_i record the number of strength-three (a_i, b_i, a_j)s, where $j = 1, \ldots, i - 1, i + 1, \ldots, m$. Initialize $q_i = 0$.
3: **for** $i = 1 : m$ **do**
4:      **for** $b'_i \in D_0 \setminus (A \cup B)$ **do**
5:          Let $q = 0$.
6:          **for** $j = 1 : i - 1, i + 1 : m$ **do**
7:              **if** (a_i, b'_i, a_j) is an OA(3) (by using Algorithm 1) **then**
8:                  $q := q + 1$
9:              **end if**
10:          **end for**
11:          **if** $q > q_i$ **then**
12:              update $q_i := q$ and $b_i := b'_i$
13:          **end if**
14:      **end for**
15: **end for**
16: return B and $(q_1, \ldots, q_m)$.

However, such a method may lead to repeated runs and the number of factors of the final designs may not reach the maximum possible number of factors, leaving potential for further improvements. We will use the Addelman–Kempthorne OA$(54, 25, 3, 2)$ as the base array to search the maximal SOA$(54, m, 27, 3)$, OSOA$(54, m, 9, 3-)$, and OSOA$(54, m, 27, 3)$.

Let $D_0 = (h^{(0)}, h_1^{(1)}, \ldots, h_6^{(1)}, \ldots, h_1^{(4)}, \ldots, h_6^{(4)})$ be an Addelman–Kempthorne OA$(54, 25, 3, 2)$ with its generator given by

$$L^{(3,2)} = (l_1, l_2, l_3, l_4) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Table 1. The constructed SOA(54, 5, 27, 3) (transposed).

0	6	3	13	10	16	26	23	20	11	17	14	21	18	24	7	4	1	18	24	21	4	1	7	17	14	11
0	7	5	12	10	17	24	22	20	21	19	26	6	4	2	9	16	14	6	4	2	9	16	14	21	19	26
0	16	23	2	15	22	1	17	21	4	11	24	3	10	26	5	9	25	8	12	19	7	14	18	6	13	20
0	11	19	22	3	14	17	25	6	9	20	1	4	12	23	26	7	15	20	1	9	12	23	4	7	15	26
0	18	9	16	7	25	23	14	5	4	22	13	11	2	20	24	15	6	26	17	8	3	21	12	10	1	19
19	25	22	5	2	8	15	12	9	2	8	5	12	9	15	25	22	19	10	16	13	23	20	26	6	3	0
15	13	11	18	25	23	3	1	8	18	25	23	3	1	8	15	13	11	12	10	17	24	22	20	0	7	5
4	11	24	3	10	26	5	9	25	8	12	19	7	14	18	6	13	20	0	16	23	2	15	22	1	17	21
10	18	2	5	13	21	24	8	16	19	0	11	14	22	3	6	17	25	2	10	18	21	5	13	16	24	8
14	5	23	18	9	0	7	25	16	15	6	24	22	13	4	2	20	11	19	10	1	8	26	17	12	3	21

Table 2. The constructed OSOA(54, 5, 9, 3−) (transposed).

0	2	1	4	3	5	8	7	6	3	5	4	7	6	8	2	1	0	6	8	7	1	0	2	5	4	3
0	2	1	4	3	5	8	7	6	7	6	8	2	1	0	3	5	4	2	1	0	3	5	4	7	6	8
0	5	7	0	5	7	0	5	7	1	3	8	1	3	8	1	3	8	2	4	6	2	4	6	2	4	6
0	3	6	7	1	4	5	8	2	3	6	0	1	4	7	8	2	5	6	0	3	4	7	1	2	5	8
0	6	3	5	2	8	7	4	1	1	7	4	3	0	6	8	5	2	8	5	2	1	7	4	3	0	6
6	8	7	1	0	2	5	4	3	0	2	1	4	3	5	8	7	6	3	5	4	7	6	8	2	1	0
5	4	3	6	8	7	1	0	2	6	8	7	1	0	2	5	4	3	4	3	5	8	7	6	0	2	1
1	3	8	1	3	8	1	3	8	2	4	6	2	4	6	2	4	6	0	5	7	0	5	7	0	5	7
3	6	0	1	4	7	8	2	5	6	0	3	4	7	1	2	5	8	0	3	6	7	1	4	5	8	2
4	1	7	6	3	0	2	8	5	5	2	8	7	4	1	0	6	3	6	3	0	2	8	5	4	1	7

Table 3. The constructed OSOA(54, 4, 27, 3) (transposed).

0	6	3	13	10	16	26	23	20	11	17	14	21	18	24	7	4	1	18	24	21	4	1	7	17	14	11
2	8	5	13	10	16	24	21	18	22	19	25	6	3	0	11	17	14	6	3	0	11	17	14	22	19	25
0	16	23	2	15	22	1	17	21	4	11	24	3	10	26	5	9	25	8	12	19	7	14	18	6	13	20
2	10	18	23	4	12	17	25	6	11	19	0	5	13	21	26	7	15	20	1	9	14	22	3	8	16	24
19	25	22	5	2	8	15	12	9	2	8	5	12	9	15	25	22	19	10	16	13	23	20	26	6	3	0
15	12	9	20	26	23	4	1	7	20	26	23	4	1	7	15	12	9	13	10	16	24	21	18	2	8	5
4	11	24	3	10	26	5	9	25	8	12	19	7	14	18	6	13	20	0	16	23	2	15	22	1	17	21
11	19	0	5	13	21	26	7	15	20	1	9	14	22	3	8	16	24	2	10	18	23	4	12	17	25	6

Construct the matrices A and B from D_0 by using Algorithms 1–4. We obtain

$$
\begin{cases}
A = (a_1, \ldots, a_5) = \left(h_3^{(1)}, h_6^{(1)}, h_1^{(2)}, h_2^{(4)}, h_5^{(4)} \right), \\
B = (b_1, \ldots, b_5) = \left(h_1^{(4)}, h_3^{(4)}, h_2^{(2)}, h_1^{(1)}, h_2^{(1)} \right),
\end{cases}
\tag{5}
$$

where A is an OA(54, 5, 3, 3) and B is an OA(54, 5, 3, 2), and they satisfy the condition in Lemma 2.4 and Proposition 2.5. Based on A and B, one can construct an SOA(54, 5, 27, 3), an OSOA(54, 5, 9, 3−) and an OSOA(54, 4, 27, 3), all with no repeated runs. The resulting SOA(54, 5, 27, 3), OSOA(54, 5, 9, 3−) and OSOA(54, 4, 27, 3) are presented in Tables 1, 2 and 3, respectively, with the top half of each table displaying runs 1–27 while the bottom half runs 28–54.

Given that the maximum number m of factors in an OA(54, m, 3, 3) is five (Hedayat et al., 1997), and considering that an m-column SOA of strength 3 or 3− can produce an m-column OA(3), the maximum number of factors for SOA(54, m, 27, 3) or OSOA(54, m, 9, 3−) is also restricted to five. Therefore, the obtained arrays SOA(54, 5, 27, 3) and OSOA(54, 5, 9, 3−) both attain the maximum number of factors.

He and Tang (2014) also constructed an SOA(54, 5, 27, 3). But to the best of our knowledge, the resulting arrays OSOA(54, 5, 9, 3−) and OSOA(54, 4, 27, 3) have not been documented in any literature. Using existing OSOA(27, 4, 9, 3−) and OSOA(27, 4, 27, 3) from

Table 4. Comparisons between the constructed $SOA(N, m, s^3, 3)$, $OSOA(N, m, s^2, 3-)$ or $OSOA(N, m, s^3, 3)$ and the existing designs.

Our results		Existing designs	
Design	Source	Design	Source
$SOA(54, 5, 27, 3)$	Algorithms 1–4	$SOA(54, 5, 27, 3)$	He and Tang (2014)
$SOA(162, 12, 27, 3)$	Algorithms 1–4	$SOA(162, 10, 27, 3)$	He and Tang (2014)[J]
$SOA(486, 25, 27, 3)$	Algorithms 1–4	$SOA(486, 20, 27, 3)$	He and Tang (2014)[J]
$SOA(1458, 56, 27, 3)$	Algorithms 1–4	$SOA(1458, 56, 27, 3)$	He and Tang (2014)[J]
$SOA(4374, 112, 27, 3)$	Theorem 4.1	$SOA(4374, 112, 27, 3)$	He and Tang (2014)[J]
$OSOA(54, 5, 9, 3-)$	Algorithms 1–4	$OSOA(54, 4, 9, 3-)$	Bao et al. (2023)[J]
$OSOA(162, 12, 9, 3-)$	Algorithms 1–4	$OSOA(162, 10, 9, 3-)$	Bao et al. (2023)[J]
$OSOA(486, 25, 9, 3-)$	Algorithms 1–4	–	–
$OSOA(1458, 56, 9, 3-)$	Algorithms 1–4	–	–
$OSOA(4374, 112, 9, 3-)$	Theorem 4.1	–	–
$OSOA(54, 4, 27, 3)$	Algorithms 1–4	$OSOA(54, 4, 27, 3)$	Bao et al. (2023)[J]
$OSOA(162, 12, 27, 3)$	Algorithms 1–4	$OSOA(162, 10, 27, 3)$	Bao et al. (2023)[J]
$OSOA(486, 24, 27, 3)$	Algorithms 1–4	–	–
$OSOA(1458, 56, 27, 3)$	Algorithms 1–4	–	–
$OSOA(4374, 112, 27, 3)$	Theorem 4.1	–	–

The superscript J indicates that the N-run design is obtained by juxtaposing two $(N/2)$-run designs obtained in the corresponding literature. The symbol '–' signifies the absence of relevant existing designs or methods.

Bao et al. (2023), it is possible to derive an $OSOA(54, 4, 9, 3-)$ and an $OSOA(54, 4, 27, 3)$ by the juxtaposition method. However, these arrays tend to have repeated runs, and the $OSOA(54, 4, 9, 3-)$ has one fewer column than the new $OSOA(54, 5, 9, 3-)$. Please refer to Section 4.2 and Table 4 for more comparisons.

4.2. Recursive constructions and comparisons

This subsection will show that if there exist matrices A and B satisfying the conditions specified in Lemma 2.4 and Proposition 2.5, we can recursively construct larger matrices A' and B' that also meet these conditions. Consequently, this result enables the recursive construction of $SOA(2s^n, m, s^3, 3)$s, $OSOA(2s^n, m, s^2, 3-)$s and $OSOA(2s^n, 2\lfloor m/2 \rfloor, s^3, 3)$s using Lemma 2.4.

Suppose $A = (a_1, \ldots, a_m)$ is an $OA(N, m, s, 3)$ and $B = (b_1, \ldots, b_m)$ is an $OA(N, m, s, 2)$, satisfying that (a_i, b_i, a_j) is an $OA(3)$ for any $i \neq j$. Let

$$A' = \begin{pmatrix} A & A \\ A & A+1 \\ \vdots & \vdots \\ A & A+(s-1) \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} B & B \\ B & B+1 \\ \vdots & \vdots \\ B & B+(s-1) \end{pmatrix}, \tag{6}$$

and denote their columns as $A' = (a'_1, \ldots, a'_{2m})$ and $B' = (b'_1, \ldots, b'_{2m})$. Then we have the following result, which is proved in Appendix 2.

Theorem 4.1: *The matrices A' and B' are an $OA(sN, 2m, s, 3)$ and an $OA(sN, 2m, s, 2)$, respectively, and they satisfy that (a'_i, a'_j, b'_i) is an $OA(3)$ for any $i \neq j$. Moreover, if A or B has no repeated rows, A' or B' in (6) has no repeated rows.*

Table 4 presents a comparison of the number of factors in our designs obtained in Section 4 versus existing ones (including those obtained via the juxtaposition method) for run sizes

$N = 2 \times 3^n$ where $n \geq 3$ and $N < 5000$. The superscript J indicates that the corresponding N-run design is obtained by juxtaposing two existing $(N/2)$-run designs which are constructed using methods from the literature. The symbol '–' signifies the absence of relevant existing designs or methods. Interested readers can access all designs obtained through our method in Table 4 at https://github.com/bcjiang0326/SOA3.

For SOA$(N, m, s^3, 3)$, when $n \in \{4, 5\}$ (i.e., $N \in \{162, 486\}$), the number of columns in our designs exceeds those in existing designs. When $n \in \{2, 6, 7\}$, the number of columns in our designs matches those of existing designs. Existing SOA$(N, m', s^3, 3)$ with run sizes that are twice prime powers are typically generally obtained by juxtaposing two SOA$(N/2, m', s^3, 3)$ with run sizes equal to prime powers. These SOA$(N/2, m', s^3, 3)$ with prime power run sizes are mainly constructed using strength-three regular designs (see He & Tang, 2014, Prop. 1). However, this juxtaposition method tends to produce repeated runs. In contrast, our designs are constructed from nonregular OAs (Addelman–Kempthorne OAs) directly and are guaranteed to contain no repeated runs.

When $n = 3$, our OSOA$(54, 5, 9, 3-)$ outperforms the existing design in terms of the number of columns and achieves the theoretical maximum. When $n = 4$, the number of columns in our OSOA$(162, 12, 9, 3-)$ or OSOA$(162, 12, 27, 3)$ also surpasses those of existing designs. For $n > 4$, there is no relevant construction theory on OSOA$(2 \cdot 3^n, m, 9, 3-)$s or OSOA$(2 \cdot 3^n, m, 27, 3)$s in the existing literature, and our results fill this gap.

For cases where $s > 3$, our exhaustive search revealed that, under small-scale parameters, the maximum number of columns for a strength-three Addelman–Kempthorne OA of $2s^n$ runs is the same as that for a strength-three regular OA of s^n runs. For instance, both OA$(125, 6, 5, 3)$ and OA$(250, 6, 5, 3)$ achieve the maximum number of columns among the strength-three regular and Addelman–Kempthorne OAs, respectively. Consequently, our constructions based on the Addelman–Kempthorne OA yield an SOA of strength 3 or 3− with the same number of columns as the juxtaposition method. Despite this, our approach retains a distinct advantage, as it can eliminate the occurrence of repeated rows compared to the juxtaposition method.

5. Concluding remarks

In this paper, we introduce new methods for constructing SOAs using Addelman–Kempthorne OAs. Our methods generate three types of SOAs with strength three or three minus, two of which are column-orthogonal. The resulting designs have run sizes of twice an odd prime power without replications, thereby filling the gaps in the run size limitations of existing constructions. The superior three-dimensional projection properties of the Addelman–Kempthorne OAs play a key role in developing our new construction methods. These projection properties offer a possibility for obtaining improved stratification and space-filling properties in the SOAs compared to using traditional regular orthogonal arrays.

There are several promising directions for future research. First, extending these construction methods to create higher-strength SOAs could further improve the flexibility and applicability of space-filling designs in computer experiments. Additionally, investigating the integration of these SOAs with other design frameworks, such as maximum distance designs or uniform designs, could lead to hybrid approaches that combine the strengths of different design methodologies. Finally, developing efficient algorithms for constructing and validating these new SOAs will be essential for their practical implementation and for advancing the field of experimental design.

Acknowledgments

Disclosure statement

Funding

ORCID

Bochuan Jiang 🄳 http://orcid.org/0009-0007-4177-1839

References

Addelman, S., & Kempthorne, O. (1961). Some main-effect plans and orthogonal arrays of strength two. *The Annals of Mathematical Statistics*, *32*(4), 1167–1176.

Bao, J., Ji, L., & Pan, Y. (2023). Constructions of column-orthogonal strong orthogonal arrays (in Chinese). *SCIENTIA SINICA Mathematica*, *53*(2), 233–248. https://doi.org/10.1360/SSM-2022-0062

Bingham, D., Sitter, R. R., & Tang, B. (2009). Orthogonal and nearly orthogonal designs for computer experiments. *Biometrika*, *96*(1), 51–65. https://doi.org/10.1093/biomet/asn057

Chen, G., & Tang, B. (2022). Using nonregular designs to generate space-filling designs. *Journal of Statistical Planning and Inference*, *221*, 201–211. https://doi.org/10.1016/j.jspi.2022.04.007

Cheng, C. S., He, Y., & Tang, B. (2021). Minimal second order saturated designs and their applications to space-filling designs. *Statistica Sinica*, *31*(2), 867–890.

Fang, K. T., Li, R., & Sudjianto, A. (2006). *Design and Modeling for Computer Experiments*. Chapman & Hall/CRC.

He, Y., Cheng, C. S., & Tang, B. (2018). Strong orthogonal arrays of strength two plus. *The Annals of Statistics*, *46*(2), 457–468. https://doi.org/10.1214/17-AOS1555

He, Y., Lin, C. D., & Sun, F. (2022). A new and flexible design construction for orthogonal arrays for modern applications. *The Annals of Statistics*, *50*(3), 1473–1489. https://doi.org/10.1214/21-AOS2159

He, Y., & Tang, B. (2013). Strong orthogonal arrays and associated Latin hypercubes for computer experiments. *Biometrika*, *100*(1), 254–260. https://doi.org/10.1093/biomet/ass065

He, Y., & Tang, B. (2014). A characterization of strong orthogonal arrays of strength three. *The Annals of Statistics*, *42*(4), 1347–1360. https://doi.org/10.1214/14-AOS1225

Hedayat, A. S., Seiden, E., & Stufken, J. (1997). On the maximal number of factors and the enumeration of 3-symbol orthogonal arrays of strength 3 and index 2. *Journal of Statistical Planning and Inference*, *58*(1), 43–63.

Hedayat, A. S., Sloane, N. J. A., & Stufken, J. (1999). *Orthogonal Arrays: Theory and Applications*. Springer.

Jiang, B., Wang, Y., & Sun, F. (2025). On construction of strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength two plus. *Science China Mathematics*, *68*(5), 1219–1242. https://doi.org/10.1007/s11425-023-2300-6

Jiang, B., Wang, Z., & Wang, Y. (2021). Strong orthogonal arrays of strength two-plus based on the Addelman-Kempthorne method. *Statistics & Probability Letters*, *175*, 109114. https://doi.org/10.1016/j.spl.2021.109114

Li, W., Liu, M.-Q., & Yang, J.-F. (2022). Construction of column-orthogonal strong orthogonal arrays. *Statistical Papers*, *63*(2), 515–530. https://doi.org/10.1007/s00362-021-01249-w

Liu, H., & Liu, M. (2015). Column-orthogonal strong orthogonal arrays and sliced strong orthogonal arrays. *Statistica Sinica*, *25*(4), 1713–1734.

Santner, T. J., Williams, B. J., & Notz, W. I. (2003). *The Design and Analysis of Computer Experiments*. Springer.

Shi, C., & Tang, B. (2019). Design selection for strong orthogonal arrays. *Canadian Journal of Statistics*, *47*(2), 302–314. https://doi.org/10.1002/cjs.v47.2

Wang, C., Yang, J., & Liu, M. (2021). Construction of strong group-orthogonal arrays. *Statistica Sinica*, *32*(3), 1225–1243.

Zhou, Y., & Tang, B. (2019). Column-orthogonal strong orthogonal arrays of strength two plus and three minus. *Biometrika*, *106*(4), 997–1004. https://doi.org/10.1093/biomet/asz043

Appendices

Appendix 1. A summary table that lists the key conditions for Propositions 3.1–3.5

Appendix 2. Proofs of Lemma 2.4, Propositions 3.1–3.5, and Theorem 4.1

Proof of Lemma 2.4: It follows directly from Theorem 2 of Zhou and Tang (2019) that $D = sA + B$ is an OSOA$(N, m, s^2, 3-)$. As noted in Remark 2.1, if an OSOA$(N, m, s^2, 3-)$ exists, there also exists an SOA$(N, m, s^3, 3)$, which shows (ii). Finally, denote $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_m)$ and $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_m)$ as the centred arrays of A and B, respectively. Let $p = \lfloor m/2 \rfloor$. Section 3 of Li et al. (2022) showed that the array defined as

$$D = \left(s^2\tilde{a}_1 + s\tilde{b}_1 + \tilde{a}_2, -\tilde{a}_1 + s^2\tilde{a}_2 + s\tilde{b}_2, s^2\tilde{a}_3 + s\tilde{b}_3 + \tilde{a}_4, -\tilde{a}_3 + s^2\tilde{a}_4 + s\tilde{b}_4, \right.$$

$$\left. \ldots, s^2\tilde{a}_{2p-1} + s\tilde{b}_{2p-1} + \tilde{a}_{2p}, -\tilde{a}_{2p-1} + s^2\tilde{a}_{2p} + s\tilde{b}_{2p} \right) + (s^3 - 1)/2 \tag{A1}$$

is an OSOA$(N, 2\lfloor m/2 \rfloor, s^3, 3)$, which confirms (iii). ∎

Proof of Proposition 3.1:

(i) Consider the case where $2 \le j \le s$ and $s + 1 \le j' \le 2s$. The number of times that (z_1, z_2, z_3) appears as a row in the columns $(f^{(0)}, f_j^{(i)}, f_{j'}^{(i')})$ is equal to the number of pairs (X_1, X) such that

$$X_1 = z_1, \quad X_1 + \alpha_{j-1}Xl_i = z_2, \quad X_1^2 + \alpha_{j'-s-1}X_1 + Xl_{i'} = z_3, \tag{A2}$$

plus the number of pairs (X_1, X) such that

$$X_1 = z_1, \quad X_1 + \alpha_{j-1}Xl_i + \eta_{j-1} = z_2, \quad \alpha\left(X_1^2 + \alpha_{j'-s-1}X_1\right) + Xl_{i'} + \gamma_{j'-s-1} = z_3. \tag{A3}$$

If $i \ne i'$, the two equations in X from (A2) are independent, yielding s^{n-3} solutions. A similar argument applies to (A3), so the total number of solutions is $2s^{n-3}$, as required. If $i = i'$, the two equations in X from (A2) are dependent, leading to 0 or s^{n-2} solutions. Similarly, there are also 0 or s^{n-2} solutions to (A3). Hence, the total number of solutions will be 0, s^{n-2} or $2s^{n-2}$, which is not desired. This verifies the case $2 \le j \le s$ and $s + 1 \le j' \le 2s$. Other cases follow similarly and are omitted for brevity.

(ii) Consider the case where $j_1 = 1, 2 \le j_2 \le s$ and $s + 1 \le j_3 \le 2s$. The number of times that (z_1, z_2, z_3) appears as a row in the columns $(f_1^{(i)}, f_{j_2}^{(i')}, f_{j_3}^{(i')})$ is equal to the number of pairs (X_1, X) such that

$$Xl_i = z_1, \quad X_1 + \alpha_{j_2-1}Xl_{i'} = z_2, \quad X_1^2 + \alpha_{j_3-s-1}X_1 + Xl_{i'} = z_3, \tag{A4}$$

plus the number of pairs (X_1, X) such that

$$Xl_i = z_1, \quad X_1 + \alpha_{j_2-1}Xl_{i'} + \eta_{j_2-1} = z_2,$$
$$\alpha\left(X_1^2 + \alpha_{j_3-s-1}X_1\right) + Xl_{i'} + \gamma_{j_3-s-1} = z_3. \tag{A5}$$

Suppose first that $i = i'$. Then we find $z_3 = (z_2 - \alpha_{j_2-1}z_1)^2 + \alpha_{j_3-s-1}(z_2 - \alpha_{j_2-1}z_1) + z_1$ from (A4) and $z_3 = \alpha(z_2 - \alpha_{j_2-1}z_1 - \eta_{j_2-1})^2 + \alpha_{j_3-s-1}(z_2 - \alpha_{j_2-1}z_1 - \eta_{j_2-1}) + z_1 + \gamma_{j_3-s-1}$

Table A1. Conditions for a three-column subarray of the Addelman–Kempthorne OA to be an OA(3).

Method	Three columns	Parameter conditions
Proposition 3.1	(i) $(h^{(0)}, h_j^{(i)}, h_{j'}^{(i')})$	(i) $i \neq i'$
	(ii) $(h_{j_1}^{(i)}, h_{j_2}^{(i')}, h_{j_3}^{(i')})$	(ii) $i \neq i', j_2 \neq j_3$
	(iii) $(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$	(iii) $l_{i_1}, l_{i_2}, l_{i_3}$ are linearly independent
Proposition 3.2	$(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$,	(i) $j_1 = j_2 = 1, j_3 > 1$; or
	i_1, i_2, i_3 are distinct,	(ii) $j_1 = 1, j_2 > 1, \alpha_{j_2} \neq \beta_2 \alpha_{j_3}$; or
	$l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$,	(iii) $j_1 > 1, \alpha_{j_1+j_2} \neq \beta_1 \alpha_{j_2+j_3} + \beta_2 \alpha_{j_1+j_3}$
	$1 \leq j_1 \leq j_2 \leq j_3 \leq s$	
Proposition 3.3	$(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$,	(i) $j_1 = j_2 = 1$; or
	i_1, i_2, i_3 are distinct,	(ii) $j_1 = 1, j_2 > 1, \beta_2 = 1$; or
	$l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$,	(iii) $j_1 > 1, \beta_1 \alpha_{j_2}^2 + \beta_2 \alpha_{j_1}^2 = (\beta_1 \alpha_{j_2} + \beta_2 \alpha_{j_1})^2$
	$1 \leq j_1 \leq j_2 \leq s < j_3 \leq 2s$	
Proposition 3.4	$(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$,	(i) $j_1 = 1, \beta_2 = 1, j_2 \neq j_3$; or
	i_1, i_2, i_3 are distinct,	(ii) $j_1 = 1, \beta_2 \neq 1, j_2 = j_3$; or
	$l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$,	(iii) $j_1 > 1, \beta_2 = 1$,
		$\quad \beta_1 \notin \{\alpha_{j_1-1}(\alpha_{j_3-s-1} - \alpha_{j_2-s-1}), \alpha_{j_1}(\alpha_{j_3-s-1} - \alpha_{j_2-s-1})\}$; or
	$1 \leq j_1 \leq s < j_2, j_3 \leq 2s$	(iv) $j_1 > 1, \beta_2 \neq 1$ and
		$\quad \beta_1 \beta_2^{-1}(\beta_1 + \beta_2 - 1) = \alpha_{2j_1-1}(\alpha_{j_3-s-1} - \alpha_{j_2-s-1})^2$
Proposition 3.5	$(h_{j_1}^{(i_1)}, h_{j_2}^{(i_2)}, h_{j_3}^{(i_3)})$,	(i) $\beta_1 + \beta_2 = 1, \alpha_{j_3-s-1} \neq \beta_1 \alpha_{j_1-s-1} + \beta_2 \alpha_{j_2-s-1}$; or
	i_1, i_2, i_3 are distinct,	(ii) $\beta_1 + \beta_2 \neq 1, (\alpha_{j_3-s-1} - \beta_1 \alpha_{j_1-s-1} - \beta_2 \alpha_{j_2-s-1})^2$
	$l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$,	$\quad = (1 - \beta_1 - \beta_2)(\alpha_{j_3-s-1}^2 - \beta_1 \alpha_{j_1-s-1}^2 - \beta_2 \alpha_{j_2-s-1}^2)$
	$s + 1 \leq j_1, j_2, j_3 \leq 2s$	

from (A5). It is easily verified that there must be 3-tuples (z_1, z_2, z_3) not satisfying any of the above two equations. Then for such (z_1, z_2, z_3), neither (A4) nor (A5) has a solution, so the total number of solutions is zero.

Next consider that $i \neq i'$. It follows from (A4) that X_1 must satisfy

$$X_1^2 + \left(\alpha_{j_3-s-1} - \alpha_{j_2-1}^{-1}\right) X_1 = z_3 - \alpha_{j_2-1}^{-1} z_2. \tag{A6}$$

Since the first two equations in X in (A4) are independent, for every X_1 satisfying (A6), there are s^{n-3} solutions for X in (A4). Similarly, we find from (A5) that for every X_1 satisfying

$$\alpha X_1^2 + \left(\alpha \alpha_{j_3-s-1} - \alpha_{j_2-1}^{-1}\right) X_1 = z_3 - \alpha_{j_2-1}^{-1} z_2 + \left(\alpha_{2j_2-1}^{-1} - \alpha_{2(j_3-s-1)}\right)(\alpha - 1)/4, \tag{A7}$$

there are s^{n-3} solutions for X in (A5). Thus it suffices to show that there are precisely two values of X_1 that satisfy (A6) or (A7). It is easily seen that (A6) has 0, 1 or 2 solutions depending on whether the discriminant $\Delta_1 = (\alpha_{j_3-s-1} - \alpha_{j_2-1}^{-1})^2 + 4(z_3 - \alpha_{j_2-1}^{-1} z_2)$ is, respectively, a nonresidue, zero or a quadratic residue. Similarly, (A7) has 0, 1 or 2 solutions depending on whether the discriminant $\Delta_2 = \alpha(\alpha_{j_3-s-1} - \alpha_{j_2-1}^{-1})^2 + 4\alpha(z_3 - \alpha_{j_2-1}^{-1} z_2) = \alpha \Delta_1$ is respectively a nonresidue, zero, or a quadratic residue. Since α is a nonresidue, the total number of solutions to (A6) and (A7) is always 2, as required. This completes the verification for the case where $j_1 = 1, 2 \leq j_2 \leq s$ and $s + 1 \leq j_3 \leq 2s$. For other cases, the verification is similar and thus omitted.

(iii) Consider the case where $j_1 = 1, 2 \leq j_2 \leq s$ and $s + 1 \leq j_3 \leq 2s$. Then the number of times that (z_1, z_2, z_3) appears as a row in the three columns $(f_{j_1}^{(i_1)}, f_{j_2}^{(i_2)}, f_{j_3}^{(i_3)})$ is equal to the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2-1} Xl_{i_2} = z_2, \quad X_1^2 + \alpha_{j_3-s-1} X_1 + Xl_{i_3} = z_3, \tag{A8}$$

plus the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2-1} Xl_{i_2} + \eta_{j_2-1} = z_2,$$
$$\alpha \left(X_1^2 + \alpha_{j_3-s-1} X_1\right) + Xl_{i_3} + \gamma_{j_3-s-1} = z_3. \tag{A9}$$

Note that l_{i_1}, l_{i_2} and l_{i_3} are linearly independent over GF(s). Then for each value of X_1, we have three independent equations in X in (A8), which leads to s^{n-4} solutions. Since there are s choices for X_1, there are s^{n-3} solutions to (A8). A similar argument applies to (A9), so the total number of solutions is $2s^{n-3}$, as required. This completes the verification for the case where $j_1 = 1, 2 \leq j_2 \leq s$ and $s + 1 \leq j_3 \leq 2s$. For other cases, the proof is similar and thus omitted.

■

Proof of Proposition 3.2:

(i) Assume first that $j_1 = j_2 = 1$. If $j_3 = 1$, then the number of times that (z_1, z_2, z_3) appears as a row in the three columns of D_0 is twice the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad Xl_{i_2} = z_2, \quad Xl_{i_3} = z_3. \tag{A10}$$

Since $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$, we find from (A10) that $z_3 = \beta_1 z_1 + \beta_2 z_2$. It is easily verified that there must be 3-tuples (z_1, z_2, z_3) not satisfying this equation. Then for such (z_1, z_2, z_3), there are no solutions to (A10).

If $2 \leq j_3 \leq s$, then the number of times that (z_1, z_2, z_3) appears is the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad Xl_{i_2} = z_2, \quad X_1 + \alpha_{j_3-1}Xl_{i_3} = z_3, \tag{A11}$$

plus the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad Xl_{i_2} = z_2, \quad X_1 + \alpha_{j_3-1}Xl_{i_3} + \eta_{j_3-1} = z_3. \tag{A12}$$

It follows from (A11) that $X_1 = z_3 - \alpha_{j_3-1}(\beta_1 z_1 + \beta_2 z_2)$. Since the first two equations in X in (A11) are independent, there are s^{n-3} solutions to (A11). A similar argument applies to (A12), so the total number of solutions is $2s^{n-3}$, as required. This completes the verification for Proposition 3.2(i).

(ii) Next, consider the case where $j_1 = 1$ and $2 \leq j_2 \leq j_3 \leq s$. Then the number of times that (z_1, z_2, z_3) appears is the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2-1}Xl_{i_2} = z_2, \quad X_1 + \alpha_{j_3-1}Xl_{i_3} = z_3, \tag{A13}$$

plus the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2-1}Xl_{i_2} + \eta_{j_2-1} = z_2, \quad X_1 + \alpha_{j_3-1}Xl_{i_3} + \eta_{j_3-1} = z_3. \tag{A14}$$

By noting that $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$, (A13) will not have a solution unless

$$\left(1 - \beta_2 \alpha_{j_3-1} \alpha_{j_2-1}^{-1}\right) X_1 = z_3 - \alpha_{j_3-1}\left(\beta_1 z_1 + \beta_2 \alpha_{j_2-1}^{-1} z_2\right). \tag{A15}$$

Similarly, (A14) will not have a solution unless

$$\left(1 - \beta_2 \alpha_{j_3-1} \alpha_{j_2-1}^{-1}\right) X_1$$
$$= z_3 - \alpha_{j_3-1}\left(\beta_1 z_1 + \beta_2 \alpha_{j_2-1}^{-1} z_2\right) - \left(\eta_{j_3-1} - \beta_2 \alpha_{j_3-1} \alpha_{j_2-1}^{-1} \eta_{j_2-1}\right). \tag{A16}$$

If $\alpha_{j_2} = \beta_2 \alpha_{j_3}$, which implies the left-hand sides of (A15) and (A16) are zero, then there must be 3-tuples (z_1, z_2, z_3) not satisfying any of (A15) and (A16), and at this time, neither (A13) nor (A14) will have a solution, which is not desired. Otherwise, since the first two equations in X in (A13) are independent, for the unique X_1 satisfying (A15), there are s^{n-3} solutions for X in (A13). A similar argument applies to (A14), so the total number of solutions is $2s^{n-3}$, as required. This completes the verification for Proposition 3.2(ii).

(iii) At last, we turn to the case where $2 \leq j_1 \leq j_2 \leq j_3 \leq s$. The corresponding verification can be obtained by utilizing the same technique as in the case of $j_1 = 1$ and $2 \leq j_2 \leq j_3 \leq s$, and thus is omitted.

■

Proof of Proposition 3.3:

(i) Assume first that $j_1 = j_2 = 1$. Then the number of times that (z_1, z_2, z_3) appears as a row in the three columns of D_0 is the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad Xl_{i_2} = z_2, \quad X_1^2 + \alpha_{j_3 - s - 1}X_1 + Xl_{i_3} = z_3, \tag{A17}$$

plus the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad Xl_{i_2} = z_2, \quad \alpha\left(X_1^2 + \alpha_{j_3 - s - 1}X_1\right) + Xl_{i_3} + \gamma_{j_3 - s - 1} = z_3. \tag{A18}$$

Since $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$, (A17) will not have a solution unless

$$X_1^2 + \alpha_{j3 - s - 1}X_1 = z_3 - \beta_1 z_1 - \beta_2 z_2. \tag{A19}$$

For every X_1 satisfying (A19), there are s^{n-3} solutions for X in (A17). Similarly, we find from (A18) that X_1 must satisfy

$$\alpha\left(X_1^2 + \alpha_{j_3 - s - 1}X_1\right) = z_3 - \beta_1 z_1 - \beta_2 z_2 - \gamma_{j_3 - s - 1}. \tag{A20}$$

Thus it suffices to show that there are precisely two values of X_1 that satisfy (A19) or (A20). It is easily seen that (A19) has 0, 1 or 2 solutions depending on whether $\Delta_1 = \alpha_{j_3 - s - 1}^2 + 4(z_3 - \beta_1 z_1 - \beta_2 z_2)$ is respectively a nonresidue, zero, or a quadratic residue. Similarly, (A20) has 0, 1 or 2 solutions depending on whether $\Delta_2 = \alpha\Delta_1$ is respectively a nonresidue, zero, or a quadratic residue. The total number of solutions is always 2, as required. This completes the verification for Proposition 3.3(i).

(ii) Next, consider the case where $j_1 = 1$ and $2 \leq j_2 \leq s$. Then the number of times that (z_1, z_2, z_3) appears is the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2 - 1}Xl_{i_2} = z_2, \quad X_1^2 + \alpha_{j_3 - s - 1}X_1 + Xl_{i_3} = z_3, \tag{A21}$$

plus the number of pairs (X_1, X) such that

$$\begin{aligned} Xl_{i_1} = z_1, \quad X_1 + \alpha_{j_2 - 1}Xl_{i_2} + \eta_{j_2 - 1} = z_2, \\ \alpha\left(X_1^2 + \alpha_{j_3 - s - 1}X_1\right) + Xl_{i_3} + \gamma_{j_3 - s - 1} = z_3. \end{aligned} \tag{A22}$$

Since $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$, (A21) will not have a solution unless

$$X_1^2 + \left(\alpha_{j_3 - s - 1} - \beta_2\alpha_{j_2 - 1}^{-1}\right)X_1 = z_3 - \beta_1 z_1 - \beta_2\alpha_{j_2 - 1}^{-1}z_2. \tag{A23}$$

For every X_1 satisfying (A23), there are s^{n-3} solutions for X in (A21). It is easily seen that (A23) has 0, 1 or 2 solutions depending on whether $\Delta_1 = \alpha_{j_3 - s - 1}^2 + 4(z_3 - \beta_1 z_1 - \beta_2 z_2)$ is, respectively, a nonresidue, zero, or a quadratic residue. Similarly, (A22) will not have a solution unless

$$\begin{aligned} \alpha X_1^2 + \left(\alpha\alpha_{j_3 - s - 1} - \beta_2\alpha_{j_2 - 1}^{-1}\right)X_1 \\ = \left(z_3 - \beta_1 z_1 - \beta_2\alpha_{j_2 - 1}^{-1}z_2\right) - \left(\gamma_{j_3 - s - 1} - \beta_2\alpha_{j_2 - 1}^{-1}\eta_{j_2 - 1}\right), \end{aligned} \tag{A24}$$

and for every X_1 satisfying (A24), there are s^{n-3} solutions for X in (A22). Moreover, (A24) has 0, 1 or 2 solutions depending on whether

$$\Delta_2 = \alpha\Delta_1 + \beta_2\left(1 - \beta_2\right)\left(\alpha - 1\right)/\alpha_{j_2 - 1}^2 \tag{A25}$$

is, respectively, a nonresidue, zero, or a quadratic residue. If $\beta_2 = 1$, then $\Delta_2 = \alpha\Delta_1$. There are precisely two values of X_1 that satisfy (A23) or (A24), as required. If $\beta_2 \neq 1$, then there must be 3-tuples (z_1, z_2, z_3) satisfying $\Delta_1 = 0$, which means $\Delta_2 \neq 0$ by (A25). For such (z_1, z_2, z_3), the total number of solutions to (A23) and (A24) is one or three, which is not desired. This completes the verification for Proposition 3.3(ii).

(iii) At last, we turn to the case where $2 \leq j_1 \leq j_2 \leq s$. The corresponding verification can be obtained by utilizing the same technique as in the case of $j_1 = 1$ and $2 \leq j_2 \leq s$, and thus is omitted.

∎

Proof of Proposition 3.4:

(a) : (i) and (ii). Assume that $j_1 = 1$. Then the number of times that (z_1, z_2, z_3) appears as a row in the three columns of D_0 is the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad X_1^2 + \alpha_{j_2-s-1}X_1 + Xl_{i_2} = z_2, \quad X_1^2 + \alpha_{j_3-s-1}X_1 + Xl_{i_3} = z_3, \quad \text{(A26)}$$

plus the number of pairs (X_1, X) such that

$$Xl_{i_1} = z_1, \quad \alpha\left(X_1^2 + \alpha_{j_2-s-1}X_1\right) + Xl_{i_2} + \gamma_{j_2-s-1} = z_2,$$
$$\alpha\left(X_1^2 + \alpha_{j_3-s-1}X_1\right) + Xl_{i_3} + \gamma_{j_3-s-1} = z_3. \quad \text{(A27)}$$

Since $l_{i_3} = \beta_1 l_{i_1} + \beta_2 l_{i_2}$, (A26) will not have a solution unless

$$(1 - \beta_2)X_1^2 + \left(\alpha_{j_3-s-1} - \beta_2\alpha_{j_2-s-1}\right)X_1 = z_3 - \beta_1 z_1 - \beta_2 z_2. \quad \text{(A28)}$$

For every X_1 satisfying (A28), there are s^{n-3} solutions for X in (A26). Similarly, (A27) will not have a solution unless

$$\alpha\left[(1 - \beta_2)X_1^2 + \left(\alpha_{j_3-s-1} - \beta_2\alpha_{j_2-s-1}\right)X_1\right]$$
$$= (z_3 - \beta_1 z_1 - \beta_2 z_2) + \left(\beta_2\gamma_{j_2-s-1} - \gamma_{j_3-s-1}\right), \quad \text{(A29)}$$

and for every X_1 satisfying (A29), there are s^{n-3} solutions for X in (A27).

Suppose first that $\beta_2 = 1$. If $j_2 = j_3$, which means the left-hand sides of (A28) and (A29) are both zero, then there must be 3-tuples (z_1, z_2, z_3) not satisfying any of (A28) and (A29), and for such (z_1, z_2, z_3), neither (A26) nor (A27) will have a solution, which is not desired. If $j_2 \neq j_3$, then there are precisely two values of X_1 satisfying (A28) or (A29), as required.

Next, consider the case where $\beta_2 \neq 1$. Then (A28) and (A29) are both quadratic equations in X_1, and at this time, (A25) becomes

$$\Delta_2 = \alpha\Delta_1 + \beta_2\alpha(\alpha - 1)\left(\alpha_{j_2-s-1} - \alpha_{j_3-s-1}\right)^2. \quad \text{(A30)}$$

Then similar arguments as in the proof of Proposition 3.3(ii) lead to the desired result.

(b) : (iii) and (iv). The verification for the case of $j_1 > 1$ is similar to the one for the case of $j_1 = 1$. When $j_1 > 1$, (A28), (A29) and (A30) become

$$(1 - \beta_2)X_1^2 + \left(\alpha_{j_3-s-1} - \beta_1\alpha_{j_1-1}^{-1} - \beta_2\alpha_{j_2-s-1}\right)X_1 = z_3 - \beta_1\alpha_{j_1-1}^{-1}z_1 - \beta_2 z_2,$$

$$(1 - \beta_2)\alpha X_1^2 + \left(\alpha\alpha_{j_3-s-1} - \beta_1\alpha_{j_1-1}^{-1} - \beta_2\alpha\alpha_{j_2-s-1}\right)X_1$$
$$= z_3 - \beta_1\alpha_{j_1-1}^{-1}z_1 - \beta_2 z_2 - \gamma_{j_3-s-1} + \beta_1\alpha_{j_1-1}^{-1}\eta_{j_1-1} + \beta_2\gamma_{j_2-s-1},$$

and

$$\Delta_2 = \alpha\Delta_1 + \beta_2(\alpha - 1)\alpha_{j_1-1}^{-2}\left[\alpha_{2j_1-1}\left(\alpha_{j_2-s-1} - \alpha_{j_3-s-1}\right)^2 - \beta_1\beta_2^{-1}(\beta_1 + \beta_2 - 1)\right],$$

respectively. By adopting the same technique as in the case of $j_1 = 1$, the verification is immediately obtained.

∎

Proof of Proposition 3.5: The proof of Proposition 3.5 is similar to that of Proposition 3.4. For proving Proposition 3.5, (A28), (A29) and (A30) become

$$(1 - \beta_1 - \beta_2)X_1^2 + \left(\alpha_{j_3-s-1} - \beta_1\alpha_{j_1-s-1} - \beta_2\alpha_{j_2-s-1}\right)X_1 = z_3 - \beta_1 z_1 - \beta_2 z_2,$$

$$(1 - \beta_1 - \beta_2)\alpha X_1^2 + \left(\alpha_{j_3-s-1} - \beta_1\alpha_{j_1-s-1} - \beta_2\alpha_{j_2-s-1}\right)\alpha X_1$$
$$= z_3 - \beta_1 z_1 - \beta_2 z_2 - \gamma_{j_3-s-1} - \beta_1\gamma_{j_1-s-1} - \beta_2\gamma_{j_2-s-1},$$

and

$$\Delta_2 = \alpha\,\Delta_1 + \alpha(\alpha-1)\Big[\,\big(\alpha_{j_3-s-1} - \beta_1\alpha_{j_1-s-1} - \beta_2\alpha_{j_2-s-1}\big)^2$$
$$-\,(1-\beta_1-\beta_2)\left(\alpha_{j_3-s-1}^2 - \beta_1\alpha_{j_1-s-1}^2 - \beta_2\alpha_{j_2-s-1}^2\right)\Big],$$

respectively. By adopting the same technique as in the proof of Proposition 3.4, the desired result immediately follows. ∎

Proof of Theorem 4.1: By Proposition 4.1 and Theorem 3.1 of He et al. (2022), A' is an OA$(sN, 2m, s, 3)$ and B' is an OA$(sN, 2m, s, 2)$.

Assume first that $1 \le i \le m$, $1 \le j \le m$ and $i \ne j$. Since (a_i, a_j, b_i) is an OA(3), by the definition of A' and B',

$$\left(a'_i, a'_j, b'_i\right) = \begin{pmatrix} a_i & a_j & b_i \\ a_i & a_j & b_i \\ \vdots & \vdots & \vdots \\ a_i & a_j & b_i \end{pmatrix}$$

is an OA(3).

Next, consider the case where $1 \le i \le m$, $m+1 \le j \le 2m$. Let $j = k+m$, and then $1 \le k \le m$. We have

$$\left(a'_i, a'_j, b'_i\right) = \begin{pmatrix} a_i & a_k & b_i \\ a_i & a_k+1 & b_i \\ \vdots & \vdots & \vdots \\ a_i & a_k+s-1 & b_i \end{pmatrix}.$$

If $k \ne i$, (a_i, a_k, b_i) is an OA(3). Further, (a_i, a_k+u, b_i) is an OA(3) for any $u \in \{0, 1, \ldots, s-1\}$. Hence, (a'_i, a'_j, b'_i) is an OA(3). If $k = i$, for any $z_1, z_2 \in \{0, 1, \ldots, s-1\}$, the number of times that (z_1, z_2) appears as a row in the columns (a_i, b_i) is N/s^2. So, the number of times that (z_1, z_2, z_1+u) appears as a row in the columns (a_i, b_i, a_i+u) is N/s^2 for any $u \in \{0, 1, \ldots, s-1\}$. Further, for any $z_3 \in \{0, 1, \ldots, s-1\}$, the number of times that (z_1, z_2, z_3) appears as a row in the columns (a'_i, b'_i, a'_j) is N/s^2. By the definition of OA(3), (a'_i, a'_j, b'_i) is an OA(3).

Then, consider the case where $m+1 \le i \le 2m$, $m+1 \le j \le 2m$, $i \ne j$. Let $i = l+m$ and $j = k+m$, and then $1 \le l \le m$, $1 \le k \le m$, $l \ne k$. Since (a_l, a_k, b_l) is an OA(3), (a_l+u, a_k+u, b_l+u) is an OA(3) for any $u \in \{0, 1, \ldots, s-1\}$. Hence,

$$\left(a'_i, a'_j, b'_i\right) = \begin{pmatrix} a_l & a_k & b_l \\ a_l+1 & a_k+1 & b_l+1 \\ \vdots & \vdots & \vdots \\ a_l+s-1 & a_k+s-1 & b_l+s-1 \end{pmatrix}$$

is an OA(3).

At last, we turn to the case where $m+1 \le i \le 2m$, $1 \le j \le m$. Let $i = l+m$, then $1 \le l \le m$ and

$$\left(a'_i, a'_j, b'_i\right) = \begin{pmatrix} a_l & a_j & b_l \\ a_l+1 & a_j & b_l+1 \\ \vdots & \vdots & \vdots \\ a_l+s-1 & a_j & b_l+s-1 \end{pmatrix}.$$

If $l \ne j$, (a_l, a_j, b_l) is an OA(3). Further, (a_l+u, a_j, b_l+u) is an OA(3) for any $u \in \{0, 1, \ldots, s-1\}$. Hence, (a'_i, a'_j, b'_i) is an OA(3). If $l = j$, for any $z_1, z_2 \in \{0, 1, \ldots, s-1\}$ and $u \in \{0, 1, \ldots, s-1\}$, the number of times that $(z_1 - u, z_2 - u)$ appears as a row in the columns (a_l, b_l) is N/s^2. So, the number of times that $(z_1, z_2, z_1 - u)$ appears as a row in the columns (a_l+u, b_l+u, a_l) is N/s^2. Further, for any $z_3 \in \{0, 1, \ldots, s-1\}$, the number of times that (z_1, z_2, z_3) appears as a row in the columns (a'_i, b'_i, a'_j) is N/s^2. Therefore, (a'_i, a'_j, b'_i) is an OA(3) by definition. ∎