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Tail dependence of bivariate skew normal triangular array with varying correlation coefficients

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ABSTRACT

The tail dependence coefficient measures extremal dependence between two random variables. In this note, we investigate the tail dependence of a bivariate skew normal triangular array with equal skewness and varying correlation coefficients $\{\rho_n, n \geq 1\}$ satisfying the Hüsler-Reiss condition via a redefined sequential tail dependence coefficient. For more detailed insights, the convergence rate to the sequential tail dependence coefficients is also established under a refined Hüsler-Reiss condition. Numerical experiments are conducted to illustrate the theoretical results.

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1. Introduction

Quantifying and understanding tail dependence of a model is important in statistic modelling of multivariate extremes and finds significant application in risk measurement (Embrechts et al., 2002). For a bivariate random vector (X, Y), a natural and classical measure of extremal dependence is the tail dependence coefficient (Hult & Lindskog, 2002, Definition 2.3). Assuming that (X, Y) has marginal distribution functions $F_1(x)$ and $F_2(y)$, the upper tail dependence coefficient χ_U of (X, Y) is defined as

$$\chi_U = \lim_{u \uparrow 1} \mathrm{P}\left(Y > F_2^{\leftarrow}(u) \,|\, X > F_1^{\leftarrow}(u)\right)$$

provided the limit exists, where $F^{\leftarrow}(u) := \inf\{s \in \mathbb{R} \,|\, F(x) \geq u\}$ for all $u \in (0, 1)$ is the generalized inverse of F. Similarly,

$$\chi_L = \lim_{u \downarrow 0} \mathrm{P}\left(Y \leq F_2^{\leftarrow}(u) \,|\, X \leq F_1^{\leftarrow}(u)\right)$$

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is referred to as the lower tail dependence coefficient. Under the assumption that F_1 and F_2 are continuous, they are equivalent to

$$\chi_U = \lim_{t\downarrow 0} \chi_t^U := \lim_{t\downarrow 0} \frac{1}{t} \mathrm{P}\left(1 - F_1(X) \le t, 1 - F_2(Y) \le t\right), \tag{1}$$

and

$$\chi_L = \lim_{t\downarrow 0} \chi_t^L := \lim_{t\downarrow 0} \frac{1}{t} \mathrm{P}\left(F_1(X) \le t, F_2(Y) \le t\right)$$

respectively. The tail dependence coefficient, dating back to Sibuya (1960), is also known as χ dependence measure in Coles et al. (1999). The upper tail dependence coefficient measures the probability that one variable is extremely large given that another random variable is large. It captures the ability of distributions to generate joint extremes. Generally, (X, Y) is said to exhibit asymptotic upper tail dependence if $\chi_U > 0$. The trivial values $\chi_U = 0$ and $\chi_U = 1$ indicate, respectively, asymptotic independence and full dependence in the upper tail. Concepts corresponding to the lower tail are defined analogously.

The derivations and analytic computations of tail dependence coefficients for specific families of distributions have received much attention in recent years. Sibuya (1960) appears as an incipient work in this field which investigated the asymptotic independence property of bivariate normal distributions with constant correlation coefficients. Following this work, the closed-form expression of tail dependence coefficients of many classical or constructed distribution families has been established in the literature, including Aleiyouka et al. (2017), Banachewicz and van Der Vaart (2008), Engelke et al. (2019), Fung and Seneta (2010), Hult and Lindskog (2002), Hammerstein (2016), Ling and Peng (2015), Padoan (2011), Schmidt (2002), and Sepanski (2020). Although the tail dependence coefficient can indicate whether a distribution is asymptotically dependent, practical applications may require more than just knowing this limit. For more detailed insights, it is often necessary to further investigate the convergence rate of χ_t^U to its limit χ_U. This issue was first addressed by Ledford and Tawn (1996), where they examined the asymptotic convergence behaviour of the tails in bivariate extreme value distributions, including Clayton and Morgenstern distributions. Since this pioneering work, subsequent studies have investigated the decay rate for various distribution families. For example, see Engelke et al. (2019), Heffernan (2000), Hashorva (2010, 2012), and Lao et al. (2023).

The above studies investigated the tail properties of distributions with static and constant distribution parameters. However, many previous studies have provided empirical evidence of time-varying dependence structures among financial assets, for example, Christoffersen et al. (2018) and Guegan and Zhang (2010). Data in finance and insurance often cover a long time period and the tail dependence structure of economic factors may be time-varying. Using dynamic models whose tail dependence depends on the level of the model's parameters can be one effective mean to capture these characteristics, and become a popular approach for tail risk modelling in asset investment. For recent work, see Fortin et al. (2023), Ito and Yoshiba (2025), and Zhang (2021). To study the tail dependence property of models with varying tail dependence structure is a key prerequisite for correct application of models. Recently, Hu et al. (2022) considered the asymptotic tail dependence property of a bivariate Hüsler-Reiss model with parameter depending on n in terms of a tail dependence function. Specifically, for the bivariate normal triangular array $\{(\xi_{ni}, \eta_{ni}), 1 \le i \le, n \ge 1\}$ with standard normal marginal distribution functions $\Phi(x)$ and correlation coefficient $\{\rho_n, n \ge 1\}$ of

(ξ_{ni}, η_{ni}), $1 \le i \le n$ satisfying the so-called Hüsler-Reiss condition (Hüsler & Reiss, 1989)

$$\lambda_n = (1 - \rho_n)\log n \to \lambda \in [0, \infty] \quad \text{as } n \to \infty, \tag{2}$$

they obtained the limit upper tail dependence function of the sequence $(\xi_{n1}, \eta_{n1}), n \ge 1$ given by

$$\chi_U(x, y) = \lim_{n\to\infty} n\mathrm{P}\left(1 - \Phi(\xi_{n1}) \le \frac{x}{n}, 1 - \Phi(\eta_{n1}) \le \frac{y}{n}\right), \quad x, y > 0, \tag{3}$$

and further established the convergence rates to the limit. The quantity in (3) allows to investigate the tail dependence properties of random vectors with parameters that depend on n, whereas χ_t^U defined in (1) is not applicable in this scenario.

The bivariate skew normal distributions introduced in Azzalini and Valle (1996) have received considerable attention both in theoretical studies and applied studies for its ability to model asymmetry in risk measurement. A random vector (X, Y) follows a bivariate skew normal distribution $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R})$ if it has the density

$$f(\boldsymbol{x}) = 2\phi_2(\boldsymbol{x}, \boldsymbol{R})\Phi(\boldsymbol{\alpha}^\top \boldsymbol{x}), \quad \boldsymbol{x} = (x, y)^\top \in \mathbb{R}^2,$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^\top \in \mathbb{R}^2$ is the skewness parameter vector (the symbol $\top$ denotes the transpose operator of a vector), and $\phi_2(\boldsymbol{x}, \boldsymbol{R})$ denotes the density function of a bivariate normal distribution with mean $\boldsymbol{0}$ and correlation matrix

$$\boldsymbol{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad -1 < \rho < 1.$$

The asymptotic tail dependence properties of bivariate skew normal distributions were discussed in Bortot (2010). Fung and Seneta (2016) showed that the bivariate skew normal distribution with equal skewness parameters is tail independent and χ_t^L admits a regularly varying behaviour. Recently, Beranger et al. (2019) proved that the extended skew normal distribution, which includes the bivariate skew normal distribution as a special case, is also asymptotic independent in the upper tail. Moreover, following Hüsler and Reiss (1989), they considered an independent bivariate skew normal distributed triangular array $\{(X_{ni}, Y_{ni}), 1 \le i \le n, n \ge 1\}$ with correlation coefficients ρ_n and skewness parameters depending on n. Under the Hüsler-Reiss condition and an assumption on the increasing rate of asymmetry, Beranger et al. (2019) established the limit distribution of normalized maxima of $\{(X_{ni}, Y_{ni}), 1 \le i \le n, n \ge 1\}$ and formulated the upper tail dependence coefficient through the limit extremal distribution.

In this paper, we are interested in the sequential tail dependence coefficients of bivariate skew normal triangular array with correlation coefficients $\{\rho_n, n \ge 1\}$ satisfying the Hüsler-Reiss condition. Similarly to Beranger et al. (2019) but particularly, for fixed n, we assume the joint density function of (X_{ni}, Y_{ni}), $1 \le i \le n$ is given by

$$f_n(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{R}_n) = 2\phi_2(\boldsymbol{x}, \boldsymbol{R}_n)\Phi(\boldsymbol{\alpha}^\top \boldsymbol{x}), \quad \boldsymbol{x} = (x, y)^\top \in \mathbb{R}^2, \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha, \alpha)^\top \in \mathbb{R}^2$ and

$$\boldsymbol{R}_n = \begin{pmatrix} 1 & \rho_n \\ \rho_n & 1 \end{pmatrix}, \quad -1 < \rho_n < 1.$$

Supposing the correlation coefficient $\{\rho_n, n \ge 1\}$ satisfies the Hüsler-Reiss condition (2), we investigate the asymptotic tail dependence of $\{(X_{ni}, Y_{ni}), 1 \le i \le n, n \ge 1\}$ through the sequential tail dependence coefficient defined similarly to (3). Since the index i does not play a role in the asymptotic analysis of the tail dependence structure, without loss of generality, we omit the sub-index i in the following and write (X_n, Y_n) to represent a generic pair from the n-th row of the array. Hence, we define the sequential upper tail dependence coefficient as

$$\chi_U = \lim_{n\to\infty} \chi_n^U := \lim_{n\to\infty} n\mathrm{P}\left(1 - F_{\omega_n}(X_n) \le \frac{1}{n}, 1 - F_{\omega_n}(Y_n) \le \frac{1}{n}\right), \tag{5}$$

and the sequential lower tail dependence coefficient by

$$\chi_L = \lim_{n\to\infty} \chi_n^L := \lim_{n\to\infty} n\mathrm{P}\left(F_{\omega_n}(X_n) \ge \frac{1}{n}, F_{\omega_n}(Y_n) \ge \frac{1}{n}\right),$$

if the limits exist. Here F_{ω_n} represents the common marginal distribution function of X_n and Y_n, which follows a univariate skew-normal distribution characterized by the skewness parameter ω_n (see (6)). The specific expressions of χ_U and χ_L are derived, and the convergence rates of χ_n^U to χ_U are established under the refined Hüsler-Reiss conditions. Compared with the asymptotically independent bivariate skew normal distribution with equal skewness and constant correlation coefficients ρ, this new model has a wider region of tail dependence which improves the ability of bivariate skew normal distribution to model tail dependence. We remark that we only require an assumption on the correlation coefficient $\{\rho_n, n \ge 1\}$ in our model, differently from the model considered in Beranger et al. (2019), where the absolute values of skewness parameters are assumed to tend to infinity additionally.

The organization of this paper is as follows. In Section 2 we present the main results. A numerical study provided in Section 3 illustrates the asymptotic behaviour of χ_n^U. All auxiliary lemmas and proofs are deferred to Section 4.

2. Main results

In the following, let $\overline{\Phi}(x) = 1 - \Phi(x)$ and $\phi(x)$ denote the survival function and density function of standard normal distribution, respectively. Recall that for each $n \ge 1$, (X_n, Y_n) denotes a typical observation from the n-th row of a triangular array of bivariate skew normal random variables with varying correlation ρ_n. Thus, while we use the notation $\{(X_n, Y_n), n \ge 1\}$ for simplicity, the underlying structure is a triangular array. We now state the main results of this paper.

Theorem 2.1: *Let $\{(X_n, Y_n), n \ge 1\}$ be a bivariate skew-normal distributed random vector sequence with density function given by (4), where $\{\rho_n, n \ge 1\}$ satisfies the Hüsler-Reiss condition (2). Assume further that $\{\rho_n, n \ge 1\}$ is bounded away from -1 when $\lambda = \infty$. Then*

$$\chi_U = \begin{cases} 2\overline{\Phi}(\sqrt{\lambda}), & \alpha > 0, \\ 2\overline{\Phi}\left(\sqrt{\lambda(1+4\alpha^2)}\right), & \alpha < 0. \end{cases}$$

Remark 2.1: In Theorems 2.1 and 2.5, we require the sequence $\{\rho_n, n \ge 1\}$ to be bounded away from -1 when $\lambda = \infty$ to ensure that the skewness parameters $\{\omega_n, n \ge 1\}$ of the marginal skew normal distributions of $\{(X_n, Y_n), n \ge 1\}$ stay bounded away from zero. This is essential for applying the expansion of the marginal tail distribution $\{\overline{F}_{\omega_n}(\cdot), n \ge 1\}$ from

Lemma 4.2. The boundedness condition guarantees the correct comparison of convergence orders in the expansion and underpins the validity of the analysis.

Remark 2.2: Let $\{(X_{ni}, Y_{ni}), 1 \le i \le n, n \ge 1\}$ be an independent bivariate skew-normal triangular array with the density function given in (4), and it follows from Theorem 2.1 and the equivalence between the tail dependence function and normalized sample maxima (cf. Resnick, 2008, Chapter 5) that

$$\mathrm{P}\left(\max_{1\le i\le n} \frac{1}{1-F_{\omega_n}(X_{ni})} \le n, \max_{1\le i\le n} \frac{1}{1-F_{\omega_n}(Y_{ni})} \le n\right) \to -2-\chi_U,$$

as $n \to \infty$. This indicates that, under the Hüsler-Reiss condition, the extremes in the triangular array may exhibit asymptotic extremal dependence.

Remark 2.3: Fung and Seneta (2016) show that the tail of bivariate skew-normal distribution with identical skewness coefficients and a constant correlation coefficient $\rho \in (0, 1)$ is asymptotically independent. Theorem 2.1 demonstrates that, by introducing the Hüsler-Reiss condition, the bivariate skew normal model with n-varying parameters can capture a wider range of tail dependence structures. Specifically, depending on the value of λ, the model can exhibit either asymptotic dependence or asymptotic independence in the tails.

The lower sequential tail dependence coefficient is determined by the upper sequential tail dependence coefficient, noting that $f_n(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{R}_n) = f_n(-\boldsymbol{x}, -\boldsymbol{\alpha}, \boldsymbol{R}_n)$ for the density function of (X_n, Y_n) given by (4). The corresponding result for the lower tail is formalized in the following theorem, and its proof, being analogous to that of Theorem 2.1, is omitted.

Theorem 2.2: *Assuming the conditions of Theorem 2.1 hold, then*

$$\chi_L = \begin{cases} 2\overline{\Phi}\left(\sqrt{\lambda(1+4\alpha^2)}\right), & \alpha > 0, \\ 2\overline{\Phi}(\sqrt{\lambda}), & \alpha < 0. \end{cases}$$

Remark 2.4: If $\rho_n \equiv \rho \in (-1, 1)$, then the Hüsler-Reiss condition (2) holds with $\lambda = \infty$, and it hence follows from Theorem 2.1 that $\chi_U = 0$, coinciding with the result provided by Bortot (2010).

Remark 2.5: The sequential upper tail dependence coefficient of bivariate skew normal distribution with $\alpha > 0$ is identical to that of the Hüsler-Reiss distribution (Hu et al., 2022, Theorem 2.1), i.e., the bivariate Gaussian distribution with correlation coefficient ρ_n. It does not rely on the value of α in this case.

Theorem 2.3: *For $\lambda \in (0, \infty)$, assume the conditions of Theorem 2.1 hold.*

(1) If $\alpha > 0$, assuming further that $\frac{\log n}{\log\log n}(\lambda_n - \lambda) \to \gamma \in \mathbb{R}$ as $n \to \infty$, then

$$\frac{\log n}{\log\log n}(\chi_n^U - \chi_U) \to \left(\frac{\sqrt{\lambda}}{2} - \frac{\gamma}{\sqrt{\lambda}}\right)\phi\left(\sqrt{\lambda}\right).$$

(2) *If* $\alpha < 0$, *assuming* $\frac{\log n}{(\log\log n)^2}(\lambda_n - \lambda) \to \gamma \in \mathbb{R}$ *as* $n \to \infty$, *then*

$$\frac{\log n}{(\log\log n)^2}(\chi_n^U - \chi_U) \to \frac{1}{4}\overline{\Phi}\left(\sqrt{\lambda(1+4\alpha^2)}\right) - \frac{\gamma}{\sqrt{\lambda}}\sqrt{1+4\alpha^2}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right),$$

where χ_U *is given by Theorem 2.1.*

Theorem 2.4: *For* $\lambda = 0$, *assuming the conditions of Theorem 2.1 hold and further,* $\lambda_n \log\log n \to \infty$ *as* $n \to \infty$, *then for sufficiently large* n,

$$\chi_n^U = \chi_U - \sqrt{\frac{2\lambda_n}{\pi}}\,[1+o(1)]$$

when $\alpha > 0$, *and*

$$\chi_n^U = \chi_U - \sqrt{\frac{2}{\pi}}\sqrt{(1+4\alpha^2)\lambda_n}\,[1+o(1)]$$

when $\alpha < 0$, *where* $\chi_U = 1$ *as given by Theorem 2.1.*

Theorem 2.5: *For* $\lambda = \infty$, *assuming the conditions of Theorem 2.1 hold and further,* $\lambda_n/\log\log n \to 0$ *as* $n \to \infty$, *then for sufficiently large* n,

$$\chi_n^U = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)[1+o(1)]$$

when $\alpha > 0$, *and*

$$\chi_n^U = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(1+4\alpha^2)\lambda_n}}\exp\left[-\frac{(1+4\alpha^2)\lambda_n}{2}\right][1+o(1)]$$

when $\alpha < 0$.

Remark 2.6: When $\rho_n \equiv \rho$, the random vector (X_n, Y_n) satisfies the Hüsler-Reiss condition (2) for $\lambda = \infty$. However, the assumption of Theorem 2.5 does not hold in this case, and therefore the expansions of χ_n^U for $\rho_n \equiv \rho$ can not be obtained from Theorem 2.5. Instead, the convergence rate for χ_n^U can be derived from Fung and Seneta (2016), as given in the following.

(1) If $\alpha > 0$, we have that as $n \to \infty$,

$$\chi_n^U \sim n^{-\frac{1-\rho_n}{1+\rho_n}}\frac{1+\rho_n}{2}\sqrt{\frac{1+\rho_n}{1+\rho_n}}(\pi\log n)^{-\frac{\rho_n}{1+\rho_n}}.$$

(2) If $\alpha < 0$, we have that as $n \to \infty$,

$$\chi_n^U \sim -n^{-\beta_n^2}\frac{(-2\pi\omega_n)^{\beta_n^2}}{\sqrt{\pi}\beta_n(1+\beta_n^2)^2}(\log n)^{\beta_n^2-\frac{1}{2}},$$

where

$$\omega_n = \frac{\alpha(1+\rho_n)}{\sqrt{1+\alpha^2(1-\rho_n^2)}}, \quad \beta_n = -\sqrt{\frac{(1-\rho_n)[1+2\alpha^2(1+\rho_n)]}{1+\rho_n}}. \tag{6}$$

Remark 2.7: Theorems 2.3–2.5 and Remark 2.6 show that the positive skewness parameter $\alpha > 0$ has little impact on the convergence rate, while under otherwise identical conditions, the convergence of χ_n^U to χ_U is faster when $\alpha < 0$.

Remark 2.8: The parameter λ in the Hüsler-Reiss condition characterizes the rate at which the correlation coefficients $\{\rho_n, n \geq 1\}$ converge to 1, and this convergence rate determines the form of the tail dependence. Specifically, we have the following results.

(1) When $\lambda \in (0, \infty)$, we are in the general Hüsler-Reiss regime of intermediate or partial extremal dependence. The components are neither perfectly dependent nor completely independent in the tails. Instead, they exhibit a non-trivial probability of joint extremes.
(2) When $\lambda = 0$, it means that $\rho_n \to 1$ sufficiently fast as $n \to \infty$, indicating perfect sequential tail dependence, that is the sequential tail dependence coefficient equals 0.
(3) When $\lambda = \infty$, the sequence of correlations $\{\rho_n, n \geq 1\}$ is either constant or still tends to 1, but does so at a slow rate. In this case, the limiting distribution behaves as if the components are asymptotically independent in the extremes, resulting in a tail dependence coefficient of $\chi_U = 0$.

Moreover, the rate that λ_n converges to λ controls the rate at which the scaled joint tail probability χ_n^U converges to its limit χ_U, as seen in Theorems 2.3–2.5.

3. Numerical analysis

In this section, we present a numerical study, which is done in R, to illustrate the behaviour of χ_n^U in the framework of bivariate n-varying skew normal distribution defined in (4), with ρ_n satisfying the Hüsler-Reiss condition. To show the convergence behaviour of χ_n^U to its limit χ_U, we calculate the values of χ_n^U and χ_U of $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ with finite n under the following three settings and observe the difference between χ_n^U and χ_U:

(i) (a) $\rho_n = 1 - \lambda/\log n$, $\lambda_n = \lambda$ with $\lambda \in (0, \infty)$;
 (b) $\rho_n = 1 - \lambda/\log n - 1/[2(\log n)^2]$, $\lambda_n = \lambda + 1/(2\log n)$ with $\lambda \in (0, \infty)$;
(ii) (a) $\rho_n = 1 - 1/(\log n)^2$, $\lambda_n = 1/\log n$ with $\lambda = 0$;
 (b) $\rho_n = 1 - 1/(n \log n)$, $\lambda_n = 1/n$ with $\lambda = 0$;
 (c) $\rho_n = 1 - 1/(\exp(n)\log n)$, $\lambda_n = \exp(-n)$ with $\lambda = 0$;
(iii) $\rho_n \equiv \rho \in (-1, 1)$ implying $\lambda = \infty$.

For case (i) with $\lambda = 0.5, 1, 2$, Figures 1–2 present the values of χ_n^U and χ_U for the skew-normal distribution $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R}_n)$, where the skewness parameter α takes the value 0.8, 0.5, 0.25 and $-0.8, -0.5, -0.25$. The figures show that smaller λ leads to stronger tail dependence. When $\alpha > 0$ and other parameters are fixed, despite the skewness parameter differs, the model's χ_n^U converges to the same limit. In contrast, when $\alpha < 0$, the parameter α has a significant impact on the value of χ_n^U. This is consistent with the conclusion in Theorem 2.1: when $\alpha > 0$, χ_U is independent of α, whereas when $\alpha < 0$, χ_U is a function of α and it decreases when α decreases. Given $\alpha > 0$, the faster λ_n converges to λ, the faster χ_n^U converges to χ_U. As illustrated in the figure, the convergence is faster when $\lambda_n = 1 - \lambda/\log n$. On the contrary, when $\alpha < 0$, the convergence rate of χ_n^U is insensitive to the changes in λ_n in Figure 2. In addition, for $\alpha < 0$, the convergence rate of χ_n^U is faster when α is smaller,

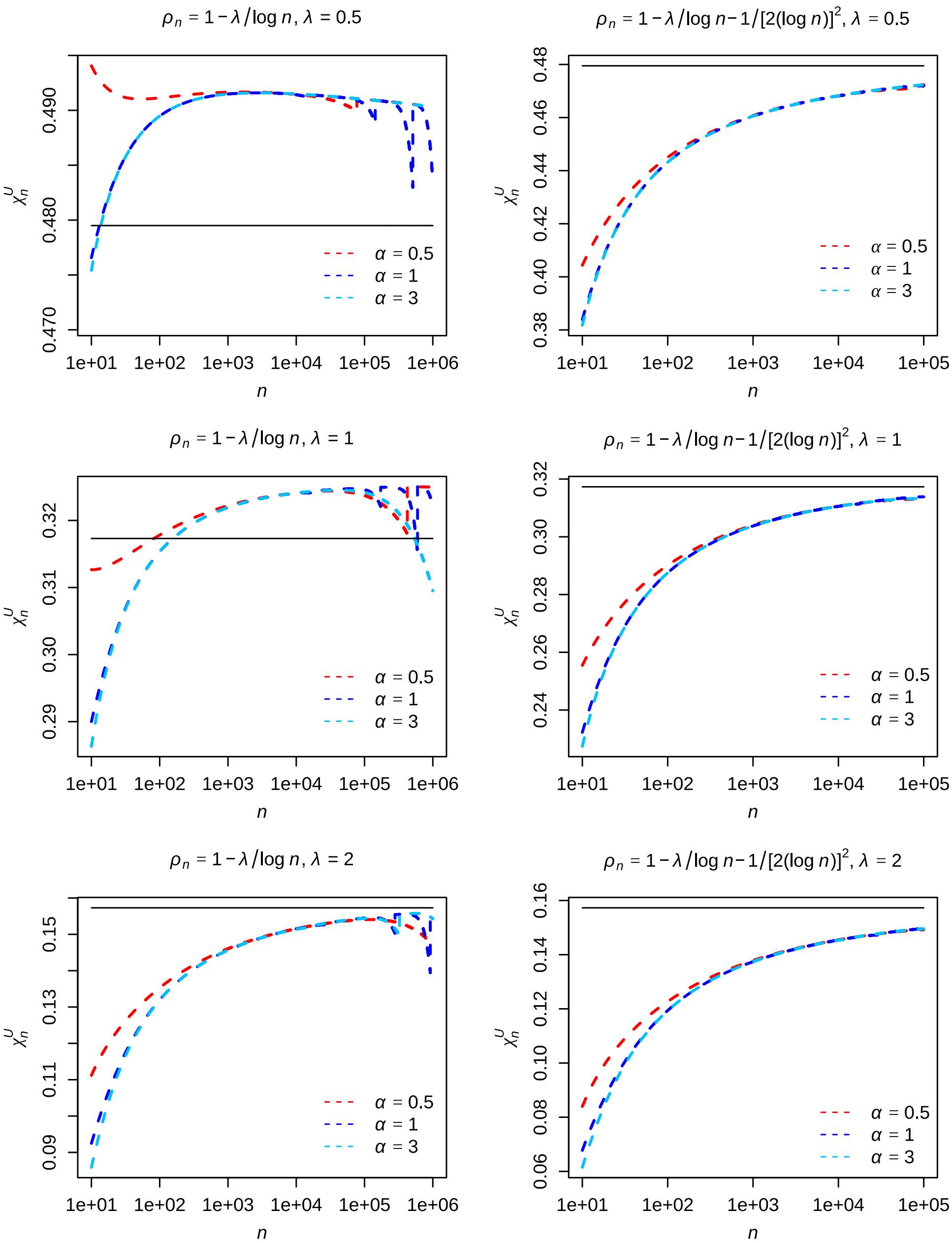

Figure 1. The values of χ_n^U for $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ with $\alpha = 0.5, 1, 3$ are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 10^6$ (left panel) and $n = 10, 11, \ldots, 10^5$ (right panel). Different rows display results for varying values of λ. The left and right panels respectively illustrate the results for $\rho_n = 1 - \lambda/\log n$ and $\rho_n = 1 - \lambda/\log n - 1/[2(\log n)^2]$. The horizontal lines represent the true limiting values of $\chi_U = 2\overline{\Phi}(\sqrt{\lambda})$, which are, from the top row to the bottom row, 0.4795, 0.3173, and 0.1573, respectively.

supporting the conclusion in Theorem 2.3. Moreover, it is observed that when $\alpha < 0$ and n is large, χ_n^U oscillates as it converges to the limit. This occurs because χ_n^U is a sequence, and when n is sufficiently large, the quantiles of the univariate skew-normal distribution at $1 - 1/n$ behave as a step function.

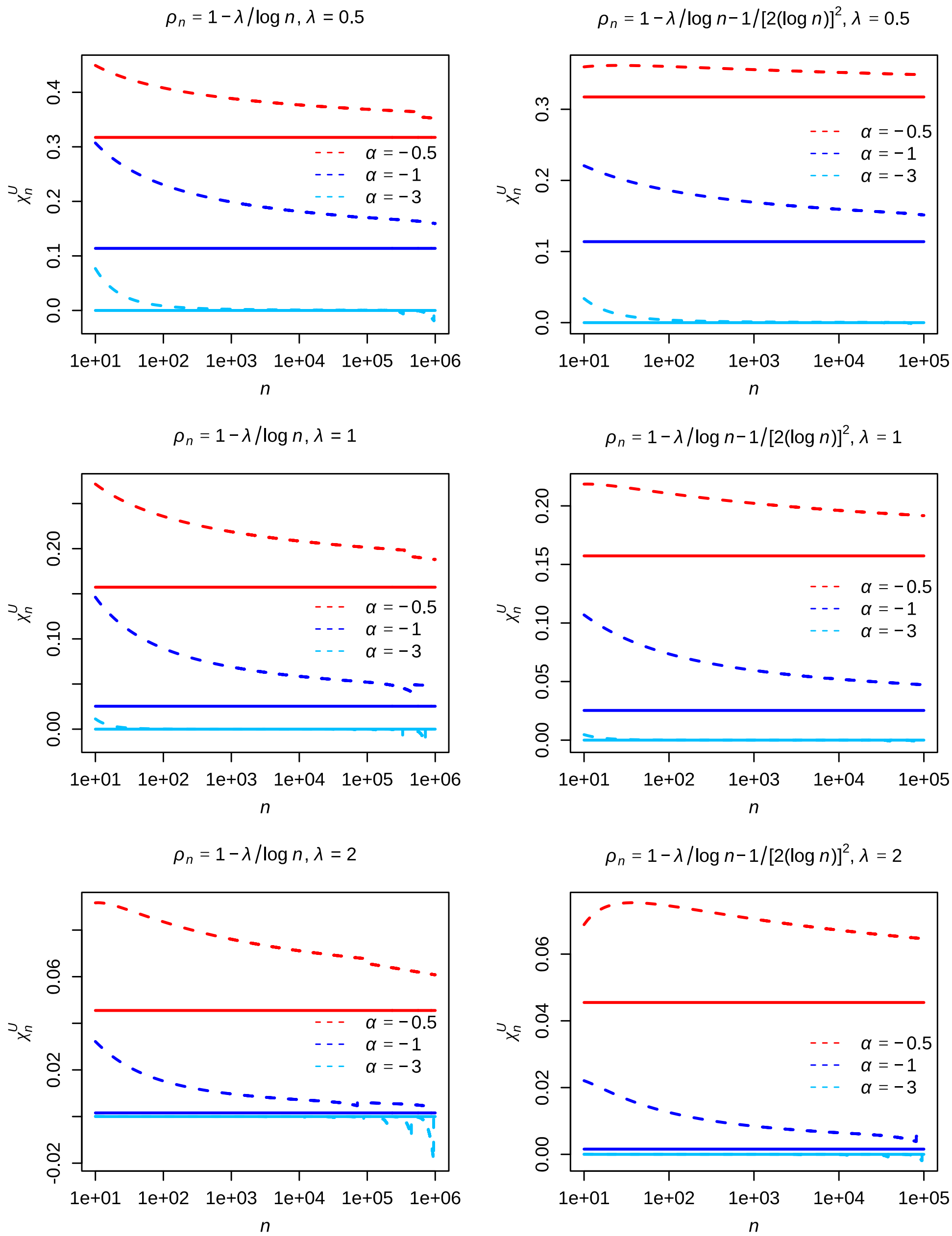

Figure 2. The values of χ_n^U for SN($\boldsymbol{\alpha}, \boldsymbol{R}_n$) with $\alpha = -0.5, -1, -3$ are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 10^6$ (left panel) and $n = 10, 11, \ldots, 10^5$ (right panel). Different rows display results for varying values of λ. The left and right panels respectively illustrate the results for $\rho_n = 1 - \lambda/\log n$ and $\rho_n = 1 - \lambda/\log n - 1/[2(\log n)^2]$. The horizontal lines represent the true limiting values of $\chi_U = 2\overline{\Phi}(\sqrt{\lambda(1 + 4\alpha^2)})$, which are, from the top row to the bottom row, (0.3173, 0.1138, 0), (0.1573, 0.0253, 0), (0.0455, 0.0016, 0) for $\alpha = (-0.5, -1, -3)$, respectively.

For case (ii), Figure 3 presents the values of χ_n^U and χ_U with different ρ_n, which converge to one with various rates. Clearly, the convergence rate of χ_n^U relies on the convergence rate of ρ_n, where faster convergence rate of ρ_n results in faster convergence rate of χ_n^U. In addition, when $\lambda = 0$, the convergence speed of χ_n^U decreases as α decreases, which is consistent with the statement in Theorem 2.4.

For case (iii), we take $\rho_n \equiv \rho = \pm 0.8, \pm 0.5, \pm 0.25$. The simulation results for the skew-normal distribution $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ with positive skewness parameter $\alpha = 0.8, 0.5, 0.25$ and negative skewness parameter $\alpha = -0.8, -0.5, -0.25$ are shown in Figures 4 and 5. With ρ_n fixed to be a constant, the model coincides with the bivariate skew normal distribution considered in Fung and Seneta (2016). Figures 4-5 indicate that the decay rate of χ_n^U increases for decreasing value of ρ, in line with the results given in Fung and Seneta (2016). Furthermore, compared with the models having positive ρ_n, the value of χ_n^U for models possessing negative ρ_n converges more quickly.

Let $\overline{\chi}_n^U$ be the second-order approximation of χ_n^U. Specifically,

(1) for $\lambda \in (0, \infty)$, when $\alpha > 0$,

$$\overline{\chi}_n^U = \chi_U + \frac{\log\log n}{\log n}\left(\frac{\sqrt{\lambda}}{2} - \frac{\gamma}{\sqrt{\lambda}}\right)\phi\left(\sqrt{\lambda}\right),$$

and when $\alpha < 0$,

$$\overline{\chi}_n^U = \chi_U + \frac{(\log\log n)^2}{\log n}\left[\frac{1}{4}\overline{\Phi}\left(\sqrt{\lambda(1+4\alpha^2)}\right) - \frac{\gamma\sqrt{1+4\alpha^2}}{\sqrt{\lambda}}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right)\right];$$

(2) for $\lambda = 0$,

$$\overline{\chi}_n^U = \begin{cases} \chi_U - \sqrt{\dfrac{2\lambda_n}{\pi}}, & \alpha > 0, \\ \chi_U - \sqrt{\dfrac{2}{\pi}}\sqrt{(1+4\alpha^2)\lambda_n}, & \alpha < 0; \end{cases}$$

(3) for $\lambda = \infty$,

$$\overline{\chi}_n^U = \begin{cases} \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\sqrt{\lambda_n}}\exp\left(-\dfrac{\lambda_n}{2}\right), & \alpha > 0, \\ \sqrt{\dfrac{2}{\pi}}\dfrac{1}{\sqrt{(1+4\alpha^2)\lambda_n}}\exp\left[-\dfrac{(1+4\alpha^2)\lambda_n}{2}\right], & \alpha < 0. \end{cases}$$

To validate the convergence rates of $\chi_n^U - \chi_U$ to zero in Theorems 2.3–2.5, we calculate the value of $\chi_n^U, \overline{\chi}_n^U$ and χ_U at $n = 10, 11, \ldots, 10^5$ under the following setting:

(iv) (a) $\lambda = 1, \rho_n = 1 - \lambda/\log n - \log\log n/(\log n)^2, \lambda_n = \lambda + \log\log n/\log n$;
(b) $\rho_n = 1 - 1/[(\log n)(\log\log n)^{0.8}], \lambda_n = 1/(\log\log n)^{0.8}$, implying $\lambda = 0$;
(c) $\rho_n = 1 - (\log\log n)^{0.8}/\log n$, implying $\lambda = \infty$.

Note that the assumptions of Theorems 2.3–2.5 are satisfied respectively under this setting.

The performance of the first-order approximation χ_U and the second-order approximation $\overline{\chi}_n^U$ is presented in Figure 6. As seen in the figure, in most cases, the second-order

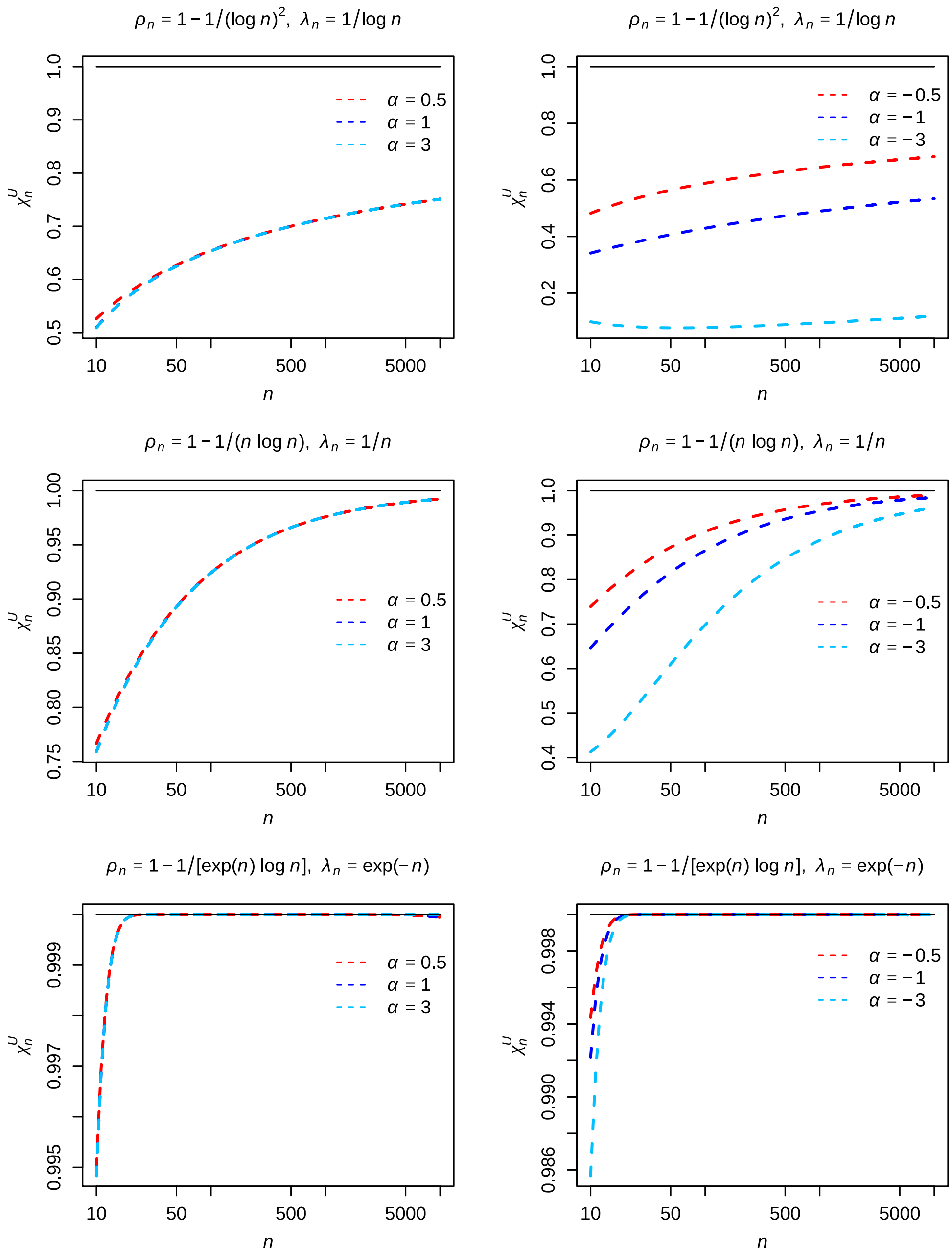

Figure 3. The values of χ_n^U for $\mathrm{SN}(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ with $\alpha = \pm 0.5, \pm 1, \pm 3$ are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 10^4$. Different rows display results for varying values of ρ_n, which is, from top to bottom, $\rho_n = 1 - 1/(\log n)^2$, $\rho_n = 1 - 1/(n \log n)$ and $\rho_n = 1 - 1/(\exp(n) \log n)$. The left and right panels illustrate the result for positive and negative α, respectively. The horizontal lines show the true limiting values of χ_U, specifically, $\chi_U = 1$.

approximation is closer to the value of χ_n^U than the first-order approximation. However, when n is relatively small, $\overline{\chi}_n^U$ may exhibit a larger difference from χ_n^U, as shown in the subplot in the lower right corner of Figure 6. This is because the second-order approximation is derived

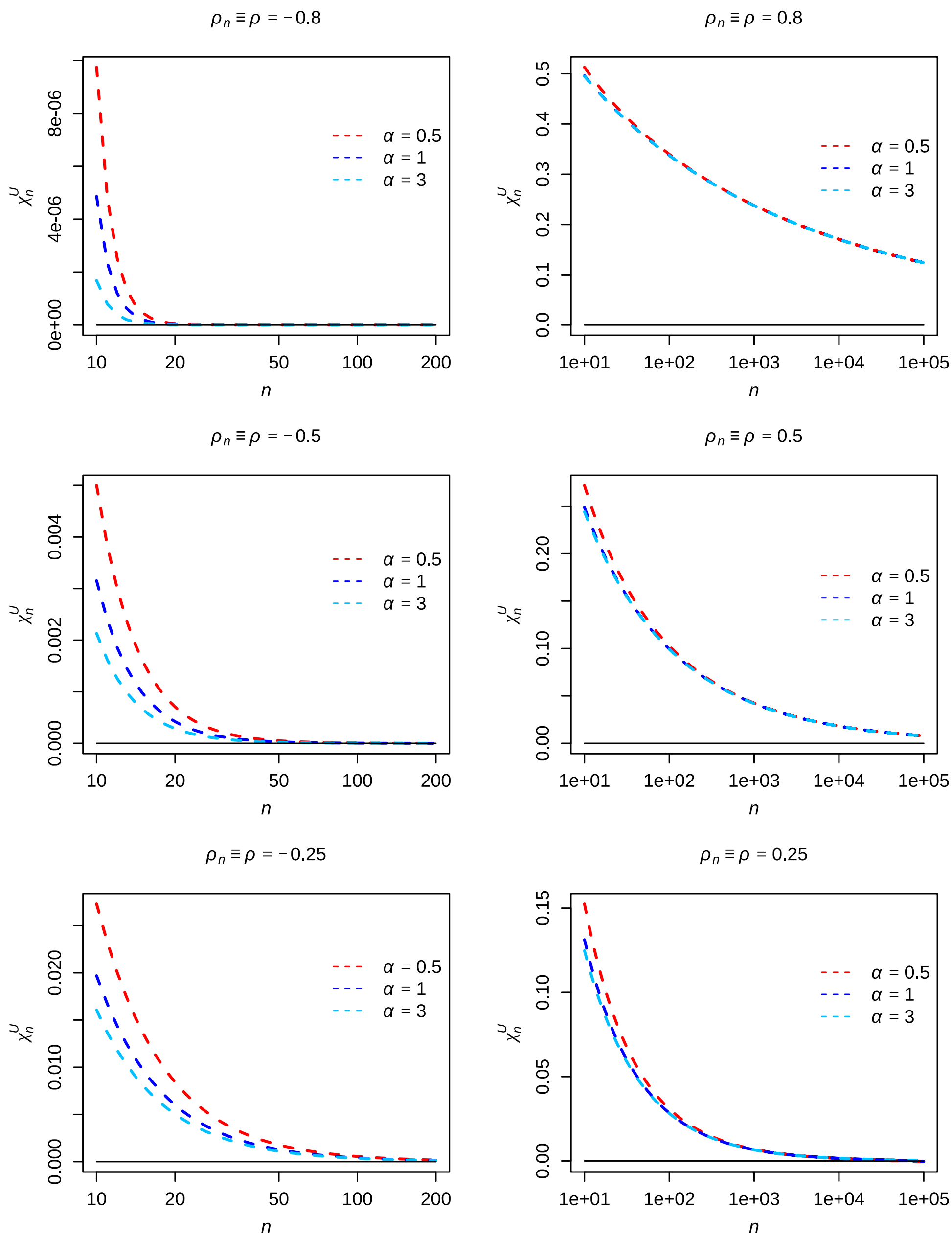

Figure 4. The values of χ_n^U for SN($\boldsymbol{\alpha}, \boldsymbol{R}_n$) with $\alpha = 0.5, 1, 3$ and $\rho_n \equiv \rho = \pm 0.8, \pm 0.5, \pm 0.25$ (implying $\lambda = \infty$) are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 200$ (left panel) and $n = 10, 11, \ldots, 10^5$ (right panel). The left and right panels illustrate the result for negative and positive ρ_n, respectively. The horizontal lines show the true limiting values of χ_U, specifically, $\chi_U = 0$.

under the condition that n approaches infinity. When the convergence rate of λ_n is slow and n is small, the error might be significant, but as n increases, it tends to converge to the limit.

The simulation results illustrate that the bivariate skew-normal distribution with varying correlation coefficients exhibits richer tail behaviour. Therefore, in practical applications, it can more accurately describe the tail characteristics of the data.

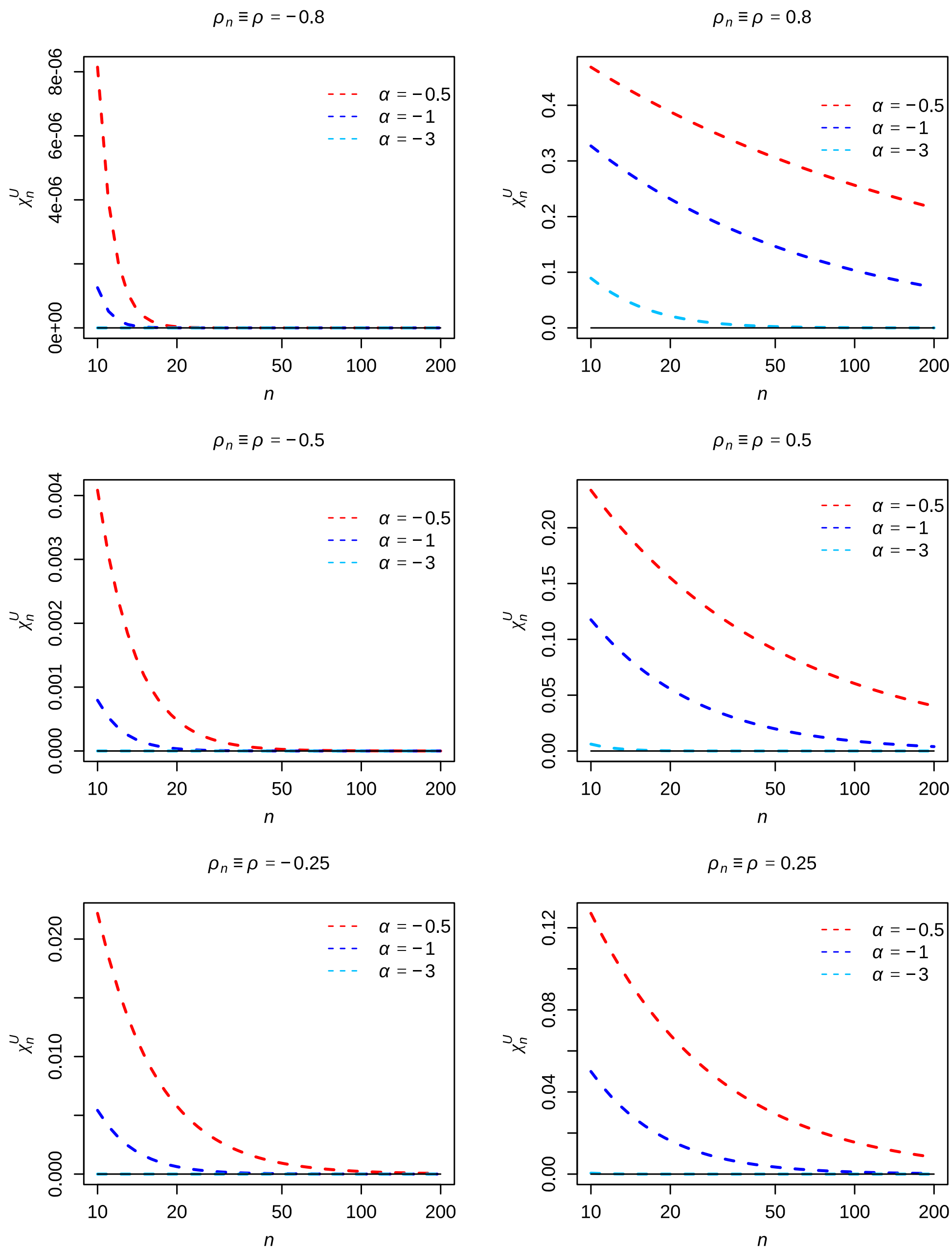

Figure 5. The values of χ_n^U for SN$(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ with $\alpha = -0.5, -1, -3$ and $\rho_n \equiv \rho = \pm 0.8, \pm 0.5, \pm 0.25$ (implying $\lambda = \infty$) are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 200$. The left and right panels illustrate the result for negative and positive ρ_n, respectively. The horizontal lines show the true limiting values of χ_U, specifically, $\chi_U = 0$.

4. Proofs

Let F_ω and f_ω denote the distribution function and density function of a univariate skew normal distribution with skewness parameter ω (shorted by SN(ω)). From (4) we know that $X_n \stackrel{d}{=} Y_n$ and X_n follows univariate skew normal distribution with skewness parameter ω_n,

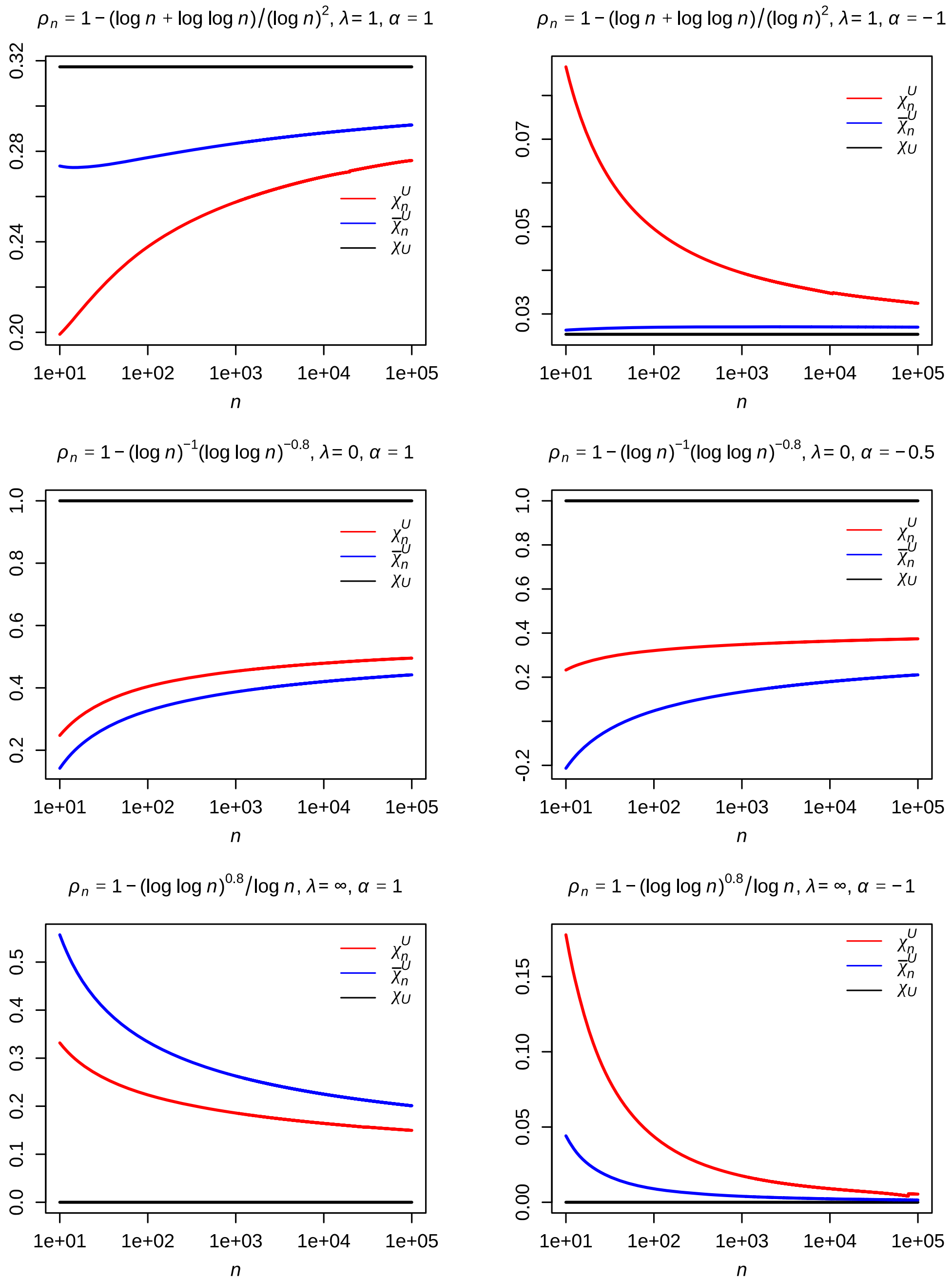

Figure 6. The values of χ_n^U, $\overline{\chi}_n^U$, and χ_U for the SN$(\boldsymbol{\alpha}, \boldsymbol{R}_n)$ are plotted against n on a logarithmic scale for $n = 10, 11, \ldots, 10^5$. Different rows display results for varying values of ρ_n, which are, from top to bottom, $\rho_n = 1 - \lambda/\log n - \log\log n/(\log n)^2$, $\rho_n = 1 - (\log n)^{-1}(\log\log n)^{-0.8}$, $\rho_n = 1 - (\log\log n)^{0.8}/\log n$. The left and right panels illustrate the result for positive and negative α respectively. The horizontal lines show the true limiting values of χ_U.

with its density function given by

$$f_{\omega_n}(x) = 2\phi(x)\Phi(\omega_n x). \tag{7}$$

To prove Theorem 2.1, we need to utilize the asymptotic expansion of the quantile function of a univariate skew-normal distribution with a n-varying correlation coefficient ρ_n. The difficulty is that the quantile function of the skew-normal distribution does not have a closed form. In Sections 3 and 4.2 (Eq. 16) of Fung and Seneta (2018), the authors derived the asymptotic expansion of the quantile function for a skew-normal distribution with a constant correlation coefficient, using regularly varying functions. In the following, we use the similar method to prove the asymptotic expansion of the upper quantile of a skew-normal distribution with a n-varying correlation coefficient.

We emphasize that the distinction in our lemma lies in the fact that our correlation coefficient ρ_n is variable, so as n changes, the distribution function of this univariate skew-normal distribution also varies with n. Therefore, what we actually obtain is the asymptotic expansion of the quantile function for a sequence of skew-normal distribution functions. Let $\{x_n, n \geq 1\}$ be a sequence of positive numbers such that $x_n \to \infty$ as $n \to \infty$. Next, we will first prove a result similar to Corollary 1 in Fung and Seneta (2018), which gives an asymptotic closed form expression of the quantile functions.

Lemma 4.1: *Consider the sequence $\{G_n(x), n \geq 1\}$, where each $G_n(x)$ is a positive, continuous, and strictly increasing distribution function defined on $[A, \infty)$, with $A > 0$. Let $\{u_n, n \geq 1\}$ be a sequence of positive numbers such that $u_n \to 1^-$ as $n \to \infty$. Additionally, assume that there exists a sequence of functions $\{y_n(x), n \geq 1\}$ for which $y_n(u_n) \to \infty$ and*

$$y_n(G_n(x_n)) = x_n\,[1 + O(\eta(x_n))] \tag{8}$$

holds for large n. Here,

$$\eta(x) = x^{-\rho}L(1/x),$$

with $\rho \geq 0$ being a constant and $L(x)$, $x > 0$, a function that varies slowly at 0. Consequently, $\eta(x_n) \to 0$ as $n \to \infty$.

If $H_n(x)$ is the inverse function of $G_n(\cdot)$, then

$$H_n(u_n) = y_n(u_n)[1 + O(\eta(y_n(u_n)))]$$

for large n.

Proof: Recall that $H_n(x)$ is the inverse function of $G_n(\cdot)$. For any $n \geq 1$ we have

$$G_n(H_n(u_n)) = u_n,$$

and thus

$$y_n(G_n(H_n(u_n))) = y_n(u_n). \tag{9}$$

Moreover, note that $u_n \to 1^-$ implying $H_n(u_n) \to \infty$ as $n \to \infty$. By (8) and (9) we get

$$H_n(u_n)[1 + O(\eta(H_n(u_n)))] = y_n(u_n) \tag{10}$$

for large n. It follows that $\frac{H_n(u_n)}{y_n(u_n)} \to 1$ as $n \to \infty$ and hence

$$\lim_{n\to\infty} \frac{\eta(H_n(u_n))}{\eta(y_n(u_n))} = 1 \tag{11}$$

by the uniform convergence theorem of slowly varying function, cf., Proposition 0.5 of Resnick (2008). Consequently, combining (10) and (11) leads to the desired conclusion. The proof is complete. ■

In the following lemma, we provide the asymptotic expansion of the tail distribution function for a univariate skew-normal distribution with a n-varying skewness parameter. This conclusion will be used later in the proof of the quantiles of the univariate skew-normal distributions with n-varying skewness parameter in Lemma 4.3 and in the subsequent proofs.

Lemma 4.2: *Let $\{\alpha_n, n \geq 1\}$ be a sequence of real numbers and $[a, b]$ be a fixed finite interval bounded away from zero. For the distribution function $F_{\alpha_n}(x)$ of $\mathrm{SN}(\alpha_n)$, we have the following results:*

(1) if $\alpha_n > 0$ and $\alpha_n \in [a, b] \subset (0, \infty)$, then

$$\overline{F}_{\alpha_n}(x_n) = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-\frac{x_n^2}{2}}}{x_n} \left[1 + O\left(x_n^{-2}\right)\right]$$

for large n;

(2) if $\alpha_n < 0$ and $\alpha_n \in [a, b] \subset (-\infty, 0)$, then

$$\overline{F}_{\alpha_n}(x_n) = \frac{\mathrm{e}^{-\frac{(1+\alpha_n^2)x_n^2}{2}}}{-\pi\alpha_n(1+\alpha_n^2)x_n^2} \left[1 + O\left(x_n^{-2}\right)\right]$$

for large n.

Proof: By integration by parts,

$$\begin{aligned}\overline{F}_{\alpha_n}(x_n) = &\frac{2\phi(x_n)\Phi(\alpha_n x_n)}{x_n} \left[1 + \frac{\alpha_n}{1+\alpha_n^2} \frac{\phi(\alpha_n x_n)}{\Phi(\alpha_n x_n)} x_n^{-1} - x_n^{-2}\right] \\ &- \frac{2\alpha_n(3+\alpha_n^2)}{1+\alpha_n^2} \int_{x_n}^{\infty} s^{-3}\phi(s)\phi(\alpha_n s)\,\mathrm{d}s + 6\int_{x_n}^{\infty} s^{-4}\phi(s)\Phi(\alpha_n s)\,\mathrm{d}s.\end{aligned} \tag{12}$$

For $x_n > 0$, we have

$$0 < \int_{x_n}^{\infty} s^{-3}\phi(s)\phi(\alpha_n s)\,\mathrm{d}s < \frac{1}{1+\alpha_n^2}\phi(x_n)\phi(\alpha_n x_n)x_n^{-4} \tag{13}$$

and

$$0 < \int_{x_n}^{\infty} s^{-4}\phi(s)\Phi(\alpha_n s)\,\mathrm{d}s < \phi(x_n)\Phi(\alpha_n x_n)x_n^{-5} + \frac{|\alpha_n|}{1+\alpha_n^2}\phi(x_n)\phi(\alpha_n x_n)x_n^{-6}. \tag{14}$$

Recall that

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\left[1 - x^{-2} + 3x^{-4} - 15x^{-6} + O(x^{-8})\right] \tag{15}$$

for large x, cf. Castro (1987). In the case of $\alpha_n \in [a, b] \subset (0, \infty)$, we have $x_n\phi(\alpha_n x_n) \to 0$ and $\Phi(\alpha_n x_n) \to 1$ as $n \to \infty$. Hence, it follows from (12)–(15) that for large n,

$$\overline{F}_{\alpha_n}(x_n) = \frac{2\phi(x_n)\Phi(\alpha_n x_n)}{x_n}\left[1 - x_n^{-2} + o(x_n^{-2})\right] = \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\frac{x_n^2}{2}}}{x_n}\left[1 + O\left(x_n^{-2}\right)\right].$$

Similarly, in the case of $\alpha_n \in [a, b] \subset (-\infty, 0)$, we have $\alpha_n x_n \to -\infty$, and thus $\Phi(\alpha_n x_n) \to 0$ as $n \to \infty$. By (12)–(15) we obtain

$$\begin{aligned}\overline{F}_{\alpha_n}(x_n) &= \frac{2\phi(x_n)\Phi(\alpha_n x_n)}{x_n}\left[1 + \frac{\alpha_n}{1+\alpha_n^2}\frac{\phi(\alpha_n x_n)}{\Phi(\alpha_n x_n)}x_n^{-1} + O(x_n^{-2})\right]\\ &= \frac{\mathrm{e}^{-\frac{(1+\alpha_n^2)x_n^2}{2}}}{-\pi\alpha_n(1+\alpha_n^2)x_n^2}\left[1 + O\left(x_n^{-2}\right)\right]\end{aligned}$$

for large n. The proof is complete. ∎

Lemma 4.3: *Let $F_{\alpha_n}^{\leftarrow}(x)$ denote the quantile function of $\mathrm{SN}(\alpha_n)$, where $\{\alpha_n, n \geq 1\}$ is a sequence of real numbers, and $[a, b]$ be a finite interval bounded away from zero for large n. We have the following results:*

(1) if $\alpha_n > 0$ and $\alpha_n \in [a, b] \subset (0, \infty)$, then for large n

$$\begin{aligned}F_{\alpha_n}^{\leftarrow}\left(1 - \frac{t}{n}\right) = \sqrt{2\log n}\Bigg[1 &- \frac{2\log t + \log\pi + \log\log n}{4\log n}\\ &- \frac{(\log\log n)^2}{32(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^2}\right)\Bigg]\end{aligned}$$

uniformly for $t \in [\varepsilon, 1]$ with $\varepsilon > 0$;

(2) if $\alpha_n < 0$ and $\alpha_n \in [a, b] \subset (-\infty, 0)$, then for large n

$$\begin{aligned}F_{\alpha_n}^{\leftarrow}\left(1 - \frac{t}{n}\right) = \sqrt{\frac{2\log n}{1+\alpha_n^2}}\Bigg[1 &- \frac{\log t + \log(-2\pi\alpha_n) + \log\log n}{2\log n}\\ &- \frac{(\log\log n)^2}{8(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^2}\right)\Bigg]\end{aligned}$$

uniformly for $t \in [\varepsilon, 1]$ with $\varepsilon > 0$.

Proof: To show this lemma, we shall use Lemmas 4.1, 4.2 and strategy similar to that used in Section 3 of Fung and Seneta (2018).

For $\alpha_n > 0$, assume α_n is bounded in the finite interval $[a, b] \subset (0, \infty)$. Similarly to Fung and Seneta (2018), we take

$$y_n(u) = \left\{-\log[-\pi(1-u)^2 \log(1-u)]\right\}^{1/2}, \quad u \in (0,1) \tag{16}$$

for $n \geq 1$ to approximate the quantile function. Recall that $x_n \to \infty$ as $n \to \infty$, and by Lemma 4.2 and (16) we have

$$y_n\left(F_{\alpha_n}(x_n)\right) = x_n\left[1 + O\left(\frac{\log x_n}{x_n^4}\right)\right]. \tag{17}$$

Let $u_n(t) = 1 - t/n$, $t \in [\varepsilon, 1]$ for $\varepsilon > 0$. Hence, $u_n(t) \to 1^-$ uniformly on $[\varepsilon, 1]$ as $n \to \infty$. By (17) and Lemma 4.1 we have

$$\begin{aligned} F_{\alpha_n}^{\leftarrow}\left(1 - \frac{t}{n}\right) &= y_n\left(1 - \frac{t}{n}\right)\left[1 + O\left(\frac{\log\log n}{(\log n)^2}\right)\right] \\ &= \sqrt{2\log n}\left[1 - \frac{2\log t + \log \pi + \log\log n}{4\log n}\right. \\ &\quad \left. - \frac{(\log\log n)^2}{32(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^2}\right)\right] \end{aligned} \tag{18}$$

uniformly for $t \in [\varepsilon, 1]$ as $n \to \infty$.

Similarly, for $\alpha_n < 0$ and $\alpha_n \in [a, b] \subset (-\infty, 0)$, set

$$y_n(u) = \left\{\frac{2}{1+\alpha_n^2}\left[-\log\left(-2\pi\alpha_n(1-u)\left|\log\frac{2(1-u)}{1+\alpha_n^2}\right|\right)\right]\right\}^{1/2}, \quad u \in (0,1).$$

Using Lemma 4.2, we can verify that (17) holds for sufficiently large n such that $\overline{F}_{\alpha_n}(x_n) < (1+b^2)/2$, which is feasible since $\overline{F}_{\alpha_n}(x_n) \to 0$ as $n \to \infty$. Consequently, by arguments similar to (18) we can show that the expansion of $F_{\alpha_n}^{\leftarrow}(1 - \frac{t}{n})$ in the lemma for $\alpha < 0$ holds uniformly for $t \in [\varepsilon, 1]$ and large n. The proof is complete. ∎

Proof of Theorem 2.1: Assume first that $\lambda \in [0, \infty)$. We will focus on the case where $\lambda \in (0, \infty)$, while the result for $\lambda = 0$ can be verified in a similar manner.

For $\lambda \in (0, \infty)$ and $\alpha > 0$, set $q_n(t) = F_{\omega_n}^{\leftarrow}(1 - \frac{t}{n})$ and write $q_n = q_n(1)$ when there is no ambiguity. Since $\omega_n \to 2\alpha$ and $1 - 1/n \to 1$ in this case, we have $q_n \to \infty$ as $n \to \infty$. Hence, we can find an integer $N \geq 1$ such that $q_n \geq 1$ for $n \geq N$. For sake of simplicity, assume $q_n \geq 1$ for all $n \geq 1$, otherwise we can set $q_n = \max(q_n, 1)$. This is reasonable because we are more concerned with the limiting behaviour of the joint tail with sufficiently large n. Recall that $X_n \stackrel{d}{=} Y_n \sim \mathrm{SN}(\omega_n)$. From the definition of χ_n^U we have

$$\begin{aligned} \chi_n^U &= n\mathrm{P}\left(X_n \geq q_n, Y_n \geq q_n\right) \\ &= n\int_{q_n}^{\infty} \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = s\right) \mathrm{d}F_{\omega_n}(s) \\ &= \int_0^1 \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right) \mathrm{d}s. \end{aligned} \tag{19}$$

With the density functions given by (4) and (7), we can write

$$\mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right) = \int_{q_n}^{\infty} \frac{1}{\sqrt{1-\rho_n^2}} \phi\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \frac{\Phi(\alpha(t+q_n(s)))}{\Phi(\omega_n q_n(s))}\,\mathrm{d}t. \tag{20}$$

Hence,

$$\overline{\Phi}\left(\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \frac{\Phi(\alpha(q_n+q_n(s)))}{\Phi(\omega_n q_n(s))} \leq \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right) \leq \frac{\overline{\Phi}\left(\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)}{\Phi(\omega_n q_n(s))}. \tag{21}$$

Combining (19) and the bounds in (21) yields

$$\int_0^1 \overline{\Phi}\left(\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \frac{\Phi(\alpha(q_n+q_n(s)))}{\Phi(\omega_n q_n(s))}\,\mathrm{d}s \leq \chi_n^U \leq \int_0^1 \frac{\overline{\Phi}\left(\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)}{\Phi(\omega_n q_n(s))}\,\mathrm{d}s.$$

In the following we show that the upper bound and lower bound converge to the same limit.

Recall that for $\lambda \in (0,\infty)$, we have $\rho_n \to 1$ and $\omega_n \to 2\alpha > 0$ as $n \to \infty$. Thus ω_n is bounded away from zero for large n. Hence, it follows from Lemma 4.3 that

$$\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}} \to \frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}, \quad \text{as } n \to \infty$$

uniformly on $[\varepsilon, 1]$ with $\varepsilon > 0$. Since $\omega_n q_n(\varepsilon) > 0$, we have

$$\sup_{s\in[\varepsilon,1]} \frac{\overline{\Phi}\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)}{\Phi(\omega_n q_n(s))} \leq \frac{1}{\Phi(\omega_n q_n(\varepsilon))} < 2,$$

and applying the Lebesgue dominated convergence theorem gives

$$\begin{aligned}\lim_{n\to\infty} \int_\varepsilon^1 \frac{\overline{\Phi}\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)}{\Phi(\omega_n q_n(s))}\,\mathrm{d}s &= \int_\varepsilon^1 \overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\mathrm{d}s \\ &= 2\overline{\Phi}(\sqrt{\lambda}) - \varepsilon\overline{\Phi}\left(\frac{\log\varepsilon}{2\sqrt{\lambda}} + \sqrt{\lambda}\right) - \Phi\left(\frac{\log\varepsilon}{2\sqrt{\lambda}} - \sqrt{\lambda}\right). \end{aligned} \tag{22}$$

Also, since $\Phi(\omega_n q_n(s) > 1/2$ holds uniformly for $0 < s < 1$ when n is sufficiently large, it follows that for any $0 < \varepsilon < 1$, we have

$$\int_0^\varepsilon \frac{\overline{\Phi}\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)}{\Phi(\omega_n q_n(s))}\,\mathrm{d}s \leq 2\int_0^\varepsilon \overline{\Phi}\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\mathrm{d}s \leq 2\varepsilon. \tag{23}$$

Combing (19)–(23) and by the arbitrariness of ε, we get

$$\limsup_{n\to\infty} \chi_n^U \leq 2\overline{\Phi}(\sqrt{\lambda}).$$

Analogously, by arguments similar to (22) and (23), it can be shown that

$$\liminf_{n\to\infty} \chi_n^U \geq 2\overline{\Phi}(\sqrt{\lambda}),$$

which, combined with the limsup statement, proves the desired result.

Now we consider the case $\lambda \in (0, \infty)$ and $\alpha < 0$. Let V be a half normal distributed random variable with density function

$$f_V(v) = \sqrt{2/\pi}\mathrm{e}^{-v^2/2}, \quad v > 0.$$

For given $\boldsymbol{\alpha}$ and $\boldsymbol{R}_n$, it was demonstrated in Azzalini and Valle (1996) that (X_n, Y_n) can be represented as

$$(X_n, Y_n)^\top \stackrel{d}{=} \boldsymbol{\delta}_n V + \sqrt{1-\delta_n^2}\boldsymbol{Z}_n \tag{24}$$

with

$$\boldsymbol{\delta}_n = (\delta_n, \delta_n)^\top = \left(\frac{\alpha(1+\rho_n)}{\sqrt{1+2\alpha^2(1+\rho_n)}}, \frac{\alpha(1+\rho_n)}{\sqrt{1+2\alpha^2(1+\rho_n)}}\right)^\top,$$

where $\boldsymbol{Z}_n = (Z_{n1}, Z_{n2})^\top$ is a bivariate normal distributed random vector with mean zero and correlation matrix

$$\Psi_n = \begin{pmatrix} 1 & \dfrac{\rho_n - \alpha^2(1-\rho_n^2)}{1+\alpha^2(1-\rho_n^2)} \\ \dfrac{\rho_n - \alpha^2(1-\rho_n^2)}{1+\alpha^2(1-\rho_n^2)} & 1 \end{pmatrix}.$$

Moreover, the random vector $\boldsymbol{Z}_n$ is independent of V. By Theorem 2.3 in Loperfido (2002) we know that $\min(Z_{n1}, Z_{n2}) \sim \mathrm{SN}(\beta_n)$ with

$$\beta_n = -\sqrt{\frac{(1-\rho_n)(1+2\alpha^2(1+\rho_n))}{1+\rho_n}},$$

and thus it follows from (24) and the definition of χ_n^U in (5) that,

$$\begin{aligned}
\chi_n^U &= n\mathrm{P}\left(\delta_n V + \sqrt{1-\delta_n^2}Z_{n1} \geq q_n, \delta_n V + \sqrt{1-\delta_n^2}Z_{n2} \geq q_n\right) \\
&= n\mathrm{E}\left[\mathrm{P}\left(Z_{n1} \geq \frac{q_n - \delta_n V}{\sqrt{1-\delta_n^2}}, Z_{n2} \geq \frac{q_n - \delta_n V}{\sqrt{1-\delta_n^2}}\right)\right] \\
&= n\mathrm{E}\left[\mathrm{P}\left(\min(Z_{n1}, Z_{n2}) \geq \frac{q_n - \delta_n V}{\sqrt{1-\delta_n^2}}\right)\right] \\
&= n\mathrm{E}\left[\overline{F}_{\beta_n}\left(\frac{q_n - \delta_n V}{\sqrt{1-\delta_n^2}}\right)\right].
\end{aligned} \tag{25}$$

Noting that $\beta_n < 0$, with arguments similar to those in the proof of Lemma 4.2, we can show that for $x > 0$,

$$2\phi(x)\Phi(\beta_n x)x^{-1} + \frac{2\beta_n}{1+\beta_n^2}\phi(x)\phi(\beta_n x)x^{-2} - 2\phi(x)\Phi(\beta_n x)x^{-3} < \overline{F}_{\beta_n}(x) < 2\phi(x)\Phi(\beta_n x)x^{-1}. \tag{26}$$

Denote $z_n(v) = \frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}$. Since $z_n(v) > 0$ for all $v \in (0,\infty)$ provided $q_n > 0$, by (25) and (26) we have

$$n(I_{n,1} + I_{n,2} - I_{n,3}) \le \chi_n^U \le nI_{n,1}, \tag{27}$$

where

$$I_{n,1} = 2\int_0^\infty \phi(z_n(v))\Phi(\beta_n z_n(v))z_n(v)^{-1} f_V(v)\,\mathrm{d}v,$$

$$I_{n,2} = \frac{2\beta_n}{1+\beta_n^2}\int_0^\infty \phi(z_n(v))\phi(\beta_n z_n(v))z_n(v)^{-2} f_V(v)\,\mathrm{d}v$$

and

$$I_{n,3} = 2\int_0^\infty \phi(z_n(v))\Phi(\beta_n z_n(v))z_n(v)^{-3} f_V(v)\,\mathrm{d}v.$$

Next, we will calculate $I_{n,1}$, $I_{n,2}$ and $I_{n,3}$ separately. Using integration by parts gives

$$I_{n,1} = -\frac{\sqrt{2}(1-\delta_n^2)^{3/2}}{\sqrt{\pi}\delta_n} q_n^{-2} \exp\left(-\frac{q_n^2}{2(1-\delta_n^2)}\right)\left[\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) f_V(0) + \int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\exp\left(-\frac{\delta_n^2 v^2}{2(1-\delta_n^2)}\right)\left(1-\frac{\delta_n v}{q_n}\right)^{-1} k_n(v)\,\mathrm{d}v\right], \tag{28}$$

where

$$k_n(v) = f_V(v)\left\{\left[\frac{\delta_n}{q_n}\left(1-\frac{\delta_n v}{q_n}\right)^{-1} - \frac{v}{1-\delta_n^2}\right]\Phi(\beta_n z_n(v)) - \frac{\beta_n\delta_n}{\sqrt{1-\delta_n^2}}\phi(\beta_n z_n(v))\right\}.$$

Considering that

$$\frac{1}{1-\delta_n^2} < 1+4\alpha^2, \quad \text{and} \quad |\delta_n| + \frac{\beta_n\delta_n\phi(0)}{\sqrt{1-\delta_n^2}} < -\alpha\left[\sqrt{(1+4\alpha^2)/2\pi}+2\right]$$

for large n such that $q_n > 1$, we have that for $v \in (0,\infty)$,

$$|k_n(v)| < f_V(v)\left\{(1+4\alpha^2)v - \alpha\left[\sqrt{(1+4\alpha^2)/2\pi}+2\right]\right\}.$$

Thus with the fact that $\delta < 0$, we obtain

$$\lim_{n\to\infty}\int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\exp\left(-\frac{\delta_n^2 v^2}{2(1-\delta_n^2)}\right)\left(1-\frac{\delta_n v}{q_n}\right)^{-1} k_n(v)\,\mathrm{d}v \le \lim_{n\to\infty}\int_0^\infty f_V(v)\left[(1+4\alpha^2)v - \alpha(\sqrt{(1+4\alpha^2)/2\pi}+2)\right]\mathrm{d}v \le \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\exp\left(-\frac{\delta_n^2 v^2}{2(1-\delta_n^2)}\right)\left(1-\frac{\delta_n v}{q_n}\right)^{-1} k_n(v)\,\mathrm{d}v = 0. \tag{29}$$

Recall that $\omega_n \to 2\alpha < 0$ for $\lambda \in (0,\infty)$ and thus ω_n is bounded away from zero for large n. Hence, using (28), (29), Lemma 4.3 and $(1-\delta_n^2)(1+\omega_n^2) = 1$, we have

$$\begin{aligned} I_{n,1} &= -\frac{2(1-\delta_n^2)^{3/2}}{\pi\delta_n}\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) q_n^{-2}\exp\left(-\frac{q_n^2}{2(1-\delta_n^2)}\right)(1+o(1)) \\ &= 2n^{-1}\left[\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right)+o(1)\right]. \end{aligned} \tag{30}$$

Similarly,

$$I_{n,2} = \frac{\beta_n}{\sqrt{\pi}(1+\beta_n^2)^2} n^{-1}(\log n)^{-1/2}\exp\left[-\frac{\beta_n^2 q_n^2}{2(1-\delta_n^2)}\right][1+o(1)] \tag{31}$$

and

$$I_{n,3} = (n\log n)^{-1}\left[\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right)+o(1)\right]. \tag{32}$$

Since $\delta_n \to 2\alpha/\sqrt{1+4\alpha^2}$, $\beta_n \to 0$ and $\beta_n q_n \to -\sqrt{\lambda}$ as $n \to \infty$ by Lemma 4.3, combining (27), (30)–(32) gives the desired result.

Now we consider the case where $\lambda = \infty$. Supposing $\alpha > 0$ and ρ_n is bounded away from -1, then $0 < \omega_n < 2\alpha$ and ω_n is bounded away from zero for large n. Hence, for any $\varepsilon > 0$, it follows from Lemma 4.3 that $\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}} \to \infty$ uniformly for $s \in [\varepsilon, 1]$ as $n \to \infty$. Consequently, by (17) and arguments similar to (22)–(23), we conclude that $\chi_n^U \to 0$ as $n \to \infty$.

For $\alpha < 0$, we have $\beta_n < 0$, and it was shown in Capitanio (2010) that

$$\begin{aligned} &\frac{\sqrt{2}}{\sqrt{\pi}|\beta_n|(1+\beta_n^2)}\phi\left(x\sqrt{1+\beta_n^2}\right)x^{-2} - \frac{\sqrt{2}(1+3\beta_n^2)}{\sqrt{\pi}|\beta_n|^3(1+\beta_n^2)^2}\phi\left(x\sqrt{1+\beta_n^2}\right)x^{-4} \\ &\quad < \overline{F}_{\beta_n}(x) < \frac{\sqrt{2}}{\sqrt{\pi}|\beta_n|(1+\beta_n^2)}\phi\left(x\sqrt{1+\beta_n^2}\right)x^{-2}, \end{aligned} \tag{33}$$

for $x > 0$. Note that $\delta_n < 0$ and $q_n \to \infty$ as $n \to \infty$, implying $z_n(v) > 0$ for $v \in (0,\infty)$ and large n. By arguments similar to those used in the proof of the case $\lambda \in (0,\infty)$ and $\alpha < 0$, (19), (33) and Lemma 4.3 we have

$$n(J_{n,1} - J_{n,2}) < \chi_n^U < nJ_{n,1}, \tag{34}$$

where

$$\begin{aligned} J_{n,1} &= \frac{\sqrt{2}}{\sqrt{\pi}|\beta_n|(1+\beta_n^2)}\int_0^\infty \phi\left(z_n(v)\sqrt{1+\beta_n^2}\right)z_n(v)^{-2} f_V(v)\,\mathrm{d}v \\ &= \frac{\sqrt{2}(1-\delta_n^2)^{1/2}}{\sqrt{\pi}(1+\beta_n^2)^2}\frac{1}{|\beta_n|q_n}\exp\left(-\frac{\beta_n^2 q_n^2}{2(1-\delta_n^2)}\right)n^{-1}[1+o(1)] \end{aligned} \tag{35}$$

and

$$J_{n,2} = \frac{\sqrt{2}(1+3\beta_n^2)}{\sqrt{\pi}|\beta_n|^3(1+\beta_n^2)^2} \int_0^\infty \phi\left(z_n(v)\sqrt{1+\beta_n^2}\right) z_n(v)^{-4} f_V(v)\, \mathrm{d}v$$
$$= \frac{\sqrt{2}(1+3\beta_n^2)(1-\delta_n^2)^{3/2}}{\sqrt{\pi}(1+\beta_n^2)^3} \frac{1}{(|\beta_n|q_n)^3} \exp\left(-\frac{\beta_n^2 q_n^2}{2(1-\delta_n^2)}\right) n^{-1}[1+o(1)]. \tag{36}$$

Note that β_n and δ_n are bounded and $|\beta_n|q_n \to \infty$ as $n \to \infty$. The desired result follows from (34)–(36). The proof is complete. ■

The proof of Theorems 2.3–2.5 requires higher-order expansions of the quantile of the skew-normal distribution $\mathrm{SN}(\alpha_n)$. The following lemma provides a higher-order expansion of the quantile function in Lemma 4.3. The proof is similar to that of Lemma 3.1 of Hu et al. (2022) and is therefore omitted.

Lemma 4.4: *Let $F_{\alpha_n}^{\leftarrow}(x)$ be the quantile function of the univariate skew normal distribution $\mathrm{SN}(\alpha_n)$, and $\{\alpha_n, n \geq 1\}$ be a sequence of real numbers.*

(1) If there exists a positive integer N such that $\alpha_n \in [a,b] \subset (0,\infty)$ when $n > N$, then as $n \to \infty$,

$$F_{\alpha_n}^{\leftarrow}\left(1-\frac{t}{n}\right) = \sqrt{2\log n}\left[1 - \frac{\log(\pi\log n)}{4\log n} + \frac{\log(\pi\log n)-2}{8\left(\log n\right)^2} - \frac{\left(\log(\pi\log n)\right)^2}{32\left(\log n\right)^2}\right]$$
$$- \frac{\log t}{\sqrt{2\log n}}\left[1 + \frac{\log t + \log(\pi\log n) - 2}{4\log n}\right] + o\left(\left(\log n\right)^{-\frac{3}{2}}\right)$$

holds uniformly on $t \in [1/\log n, 1]$.

(2) If there exists a positive integer N such that $\alpha_n \in [a,b] \subset (-\infty,0)$ when $n > N$, then as $n \to \infty$,

$$F_{\alpha_n}^{\leftarrow}\left(1-\frac{t}{n}\right) = \sqrt{\frac{2\log n}{1+\alpha_n^2}}\left\{1 - \frac{\log(-2\pi\alpha_n\log n)}{2\log n} + \frac{\log(-2\pi\alpha_n\log n)}{2\left(\log n\right)^2}\right.$$
$$\left. - \frac{\left[\log(-2\pi\alpha_n\log n)\right]^2}{8\left(\log n\right)^2} + \frac{3\alpha_n^2+1}{4\alpha_n^2}\frac{1}{(\log n)^2}\right\}$$
$$- \frac{1}{\sqrt{1+\alpha_n^2}}\frac{\log t}{\sqrt{2\log n}}\left\{1 + \frac{\log t + 2[\log(-2\pi\alpha_n\log n) - 2]}{4\log n}\right\}$$
$$+ o\left(\left(\log n\right)^{-\frac{3}{2}}\right)$$

holds uniformly on $t \in [1/\log n, 1]$.

Proof of Theorem 2.3: Consider the case $\alpha > 0$. For $s \in [0,1]$, by (20) we have

$$\begin{aligned}
&\mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right) \\
&= \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\left[\Phi(\omega_n q_n(s))\right]^{-1} \\
&\quad - \frac{1}{\sqrt{1-\rho_n^2}}\frac{1}{\Phi(\omega_n q_n(s))}\int_{q_n}^{\infty}\phi\left(\frac{t-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\overline{\Phi}(\alpha(t+q_n(s)))\,\mathrm{d}t \\
&:= K_{n,1}(s) + K_{n,2}(s).
\end{aligned} \tag{37}$$

We will address these two parts separately in the following.

When $\alpha > 0$, it follows from Lemma 4.4 that

$$\begin{aligned}
\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}} &= \frac{\log s}{2\sqrt{\lambda_n}} + \sqrt{\lambda_n} + \frac{\log\log n}{4\log n}\left(\frac{\log s}{2\sqrt{\lambda_n}} - \sqrt{\lambda_n}\right) \\
&\quad + \frac{(\log s)^2 + (\log \pi - 3\lambda_n - 2)\log s}{8\sqrt{\lambda_n}\log n} + \frac{\sqrt{\lambda_n}(\lambda_n - \log\pi)}{4\log n} + o\left(\frac{1}{\log n}\right)
\end{aligned}$$

holds uniformly on $s \in [1/\log n, 1]$. Since $\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}} \to \frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}$ as $n \to \infty$, expanding $\overline{\Phi}(\frac{q_n-\rho_n q_n(s)}{\sqrt{1-\rho_n^2}})$ at the point $\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}$ using the Taylor series with the Peano remainder, and by the assumption that $\frac{\log n}{\log\log n}(\lambda_n - \lambda) \to \gamma$ we have

$$\begin{aligned}
\overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) &= \overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right) - \phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\left[\left(\sqrt{\lambda_n} - \sqrt{\lambda}\right)\right. \\
&\quad + \frac{\log s}{2}\left(\frac{1}{\sqrt{\lambda_n}} - \frac{1}{\sqrt{\lambda}}\right) + \frac{\log\log n}{4\log n}\left(\frac{\log s}{2\sqrt{\lambda_n}} - \sqrt{\lambda_n}\right) \\
&\quad + \frac{\log s}{2\sqrt{\lambda_n}}\frac{\log\pi - 3\lambda_n - 2 + \log s}{4\log n} + \frac{\sqrt{\lambda_n}(\lambda_n - \log\pi)}{4\log n} \\
&\quad \left. + o\left(\frac{1}{\log n}\right)\right]
\end{aligned} \tag{38}$$

holds uniformly on $s \in [1/\log n, 1]$.

Since $\omega_n q_n(s) \to \infty$ holds uniformly on $s \in [0,1]$ as $n \to \infty$, from (15) we know that

$$\begin{aligned}
\overline{\Phi}\left(\omega_n q_n(s)\right) &= \frac{1}{4\alpha}\exp\left[4\alpha^2(1+2\alpha^2)\lambda_n\right] n^{-4\alpha^2}(\pi\log n)^{2\alpha^2-\frac{1}{2}} s^{4\alpha^2}(1+o(1)) \\
&= o\left(\phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\frac{1}{\log n}\right)
\end{aligned} \tag{39}$$

holds uniformly on $s \in [1/\log n, x]$. Thus combining (38) with (39) gives

$$
\begin{aligned}
K_{n,1}(s) &= \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\left[1 - \overline{\Phi}(\omega_n q_n(s)) + o\left(\overline{\Phi}(\omega_n q_n(s))\right)\right]^{-1} \\
&= \overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right) - \phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right) \\
&\quad \times \left\{\left(\sqrt{\lambda_n} - \sqrt{\lambda}\right) - \frac{\sqrt{\lambda_n}\left[\log(\pi \log n) - \lambda_n\right]}{4\log n}\right\} \\
&\quad - (\log s)\phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\left[\left(\frac{1}{2\sqrt{\lambda_n}} - \frac{1}{2\sqrt{\lambda}}\right) + \frac{\log(\pi \log n) - 3\lambda_n - 2}{8\sqrt{\lambda_n}\log n}\right] \\
&\quad - \frac{(\log s)^2}{8\sqrt{\lambda_n}\log n}\phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right) + o\left(\frac{1}{\log n}\phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\right)
\end{aligned}
$$

uniformly for $s \in [1/\log n, 1]$.

For sufficiently large n, applying Mill's inequality to $K_{n,2}(s)$ we have $\overline{\Phi}(x) \le \phi(x)/x$ for $x > 0$. Note that for $s \in [1/\log n, 1]$ we have

$$
\exp\left[-\alpha^2 q_n(s)t\right] < 1, \quad \exp\left(-\frac{1}{2}\alpha^2 t^2\right) < \exp\left(-\frac{1}{2}\alpha^2 q_n^2\right), \quad \frac{1}{1 + t/q_n(s)} < 1.
$$

For $t > q_n$, using Lemma 4.4 we get

$$
\begin{aligned}
|K_{n,2}(s)| &\le \left|\frac{1}{\sqrt{1-\rho_n^2}}\frac{1}{\Phi(\omega_n q_n(s))}\int_{q_n}^{\infty}\phi\left(\frac{t - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\frac{\phi(\alpha(t + q_n(s)))}{\alpha(t + q_n(s))}\,\mathrm{d}t\right| \\
&= \frac{1}{\alpha\sqrt{2\pi}\sqrt{1-\rho_n^2}}\frac{1}{\Phi(\omega_n q_n(s))}\exp\left[-\frac{1}{2}\alpha^2 q_n^2(s)\right]q_n^{-1}(s) \\
&\quad \times \int_{q_n}^{\infty}\phi\left(\frac{t - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\exp\left[-\frac{1}{2}\alpha^2 t^2 - \alpha^2 q_n(s)t\right]\left[1 + \frac{t}{q_n(s)}\right]^{-1}\,\mathrm{d}t \\
&\le \frac{1}{\alpha\sqrt{2\pi}}\frac{1}{\Phi(\omega_n q_n(s))}\exp\left\{-\frac{1}{2}\alpha^2\left[q_n^2(s) + q_n^2\right]\right\}q_n^{-1}(s) \\
&= O\left(n^{-2\alpha^2}(\log n)^{\alpha^2 - 1/2}\right) \qquad (40)
\end{aligned}
$$

uniformly on $s \in [1/\log n, 1]$.

In view of the expressions of $K_{n,1}(s)$ and $K_{n,2}(s)$ we obtain

$$
\begin{aligned}
&\int_{\frac{1}{\log n}}^{1} \mathrm{P}\left(Y_n \ge q_n \mid X_n = q_n(s)\right)\mathrm{d}s - \int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\mathrm{d}s \\
&= \int_{\frac{1}{\log n}}^{1}\left[K_{n,1}(s) - \overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda}\right)\right]\mathrm{d}s + \int_{\frac{1}{\log n}}^{1} K_{n,2}(s)\,\mathrm{d}s \\
&= -\left\{\left(\sqrt{\lambda_n} - \sqrt{\lambda}\right) - \frac{\sqrt{\lambda_n}\left[\log(\pi \log n) - \lambda_n\right]}{4\log n}\right\}r_{n,0}(\lambda)
\end{aligned}
$$

$$- \left[\left(\frac{1}{2\sqrt{\lambda_n}} - \frac{1}{2\sqrt{\lambda}} \right) + \frac{\log(\pi \log n) - 3\lambda_n - 2}{8\sqrt{\lambda_n} \log n} \right] r_{n,1}(\lambda)$$
$$- \frac{1}{8\sqrt{\lambda_n} \log n} r_{n,2}(\lambda) + O\left(n^{-2\alpha^2} (\log n)^{\alpha^2 - 1/2} \right),$$

where $r_{n,k}(\lambda)$, $k = 0, 1, 2$ are given by

(i)

$$r_{n,0}(\lambda) = \int_{\frac{1}{\log n}}^{1} \phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda} \right) \mathrm{d}s = 2\sqrt{\lambda} \left[\Phi\left(-\sqrt{\lambda} \right) - \Phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right) \right]$$
$$\to 2\sqrt{\lambda}\Phi\left(-\sqrt{\lambda} \right);$$

(ii)

$$r_{n,1}(\lambda) = \int_{\frac{1}{\log n}}^{1} \phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda} \right) \log s \, \mathrm{d}s$$
$$= 4\lambda \left[\sqrt{\lambda}\Phi(-\sqrt{\lambda}) - \sqrt{\lambda}\Phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right) \right.$$
$$\left. -\phi(\sqrt{\lambda}) + \phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right) \right]$$
$$\to 4\lambda \left[\sqrt{\lambda}\Phi(-\sqrt{\lambda}) - \phi(\sqrt{\lambda}) \right];$$

(iii)

$$r_{n,2}(\lambda) = \int_{\frac{1}{\log n}}^{1} \phi\left(\frac{\log s}{2\sqrt{\lambda}} + \sqrt{\lambda} \right) (\log s)^2 \, \mathrm{d}s$$
$$= 8\lambda^{3/2}(2 - \lambda)\Phi(-\sqrt{\lambda}) - 8\lambda^2\phi\left(\sqrt{\lambda} \right)$$
$$- 4\lambda\phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right) \log\log n$$
$$+ 8\lambda^2\phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right) - 8\lambda^{5/2}\Phi\left(\frac{-\log\log n}{2\sqrt{\lambda}} - \sqrt{\lambda} \right)$$
$$\to 8\lambda^{3/2}(1 - \lambda)\Phi(-\sqrt{\lambda}) - 8\lambda^2\phi\left(\sqrt{\lambda} \right).$$

Since as $n \to \infty$,

$$\frac{\log n}{\log\log n} \int_0^{\frac{1}{\log n}} \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s) \right) \mathrm{d}s \leq \frac{1}{\log\log n} \to 0$$

and

$$\begin{aligned}
&\frac{\log n}{\log\log n}\left[\int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}}+\sqrt{\lambda}\right)\mathrm{d}s-\chi_U\right]\\
&=\frac{\log n}{\log\log n}\left[-\frac{1}{\log n}\overline{\Phi}\left(\frac{-\log\log n}{2\sqrt{\lambda}}+\sqrt{\lambda}\right)-\overline{\Phi}\left(\frac{-\log\log n}{2\sqrt{\lambda}}+\sqrt{\lambda}\right)\right]\\
&\to 0,
\end{aligned}$$

by assumption $\frac{\log n}{\log\log n}(\lambda_n-\lambda)\to\gamma$, we have

$$c_n\left(\sqrt{\lambda_n}-\sqrt{\lambda}\right)=c_n\left[\frac{1}{2}\lambda^{-\frac{1}{2}}\left(\lambda_n-\lambda\right)+o\left(c_n^{-1}\right)\right]\to\frac{\gamma}{2\sqrt{\lambda}}$$

and

$$c_n\left(\frac{1}{\sqrt{\lambda_n}}-\frac{1}{\sqrt{\lambda}}\right)=c_n\left[-\frac{1}{2}\lambda^{-\frac{3}{2}}\left(\lambda_n-\lambda\right)+o\left(c_n^{-1}\right)\right]\to-\frac{\gamma}{2\lambda^{\frac{3}{2}}}.$$

Hence, with (19) we know that as $n\to\infty$,

$$\begin{aligned}
&\frac{\log n}{\log\log n}\left(\chi_n^U-\chi_U\right)\\
&=\frac{\log n}{\log\log n}\left\{\int_{\frac{1}{\log n}}^{1}\left[\mathrm{P}\left(Y_n\geq q_n\,|\,X_n=q_n(s)\right)-\overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}}+\sqrt{\lambda}\right)\right]\mathrm{d}s\right\}\\
&\quad+\frac{\log n}{\log\log n}\left[\int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{\log s}{2\sqrt{\lambda}}+\sqrt{\lambda}\right)\mathrm{d}s-\chi_U\right]\\
&\quad+\frac{\log n}{\log\log n}\int_0^{\frac{1}{\log n}}\mathrm{P}\left(Y_n\geq q_n\,|\,X_n=q_n(s)\right)\mathrm{d}s\\
&\to\left(\frac{\sqrt{\lambda}}{2}-\frac{\gamma}{\sqrt{\lambda}}\right)\phi\left(\sqrt{\lambda}\right).
\end{aligned}$$

The proof for $\alpha>0$ is finished.

Now we consider the case where $\alpha<0$. For any $\nu\in(0,\infty)$, we have as $n\to\infty$,

$$\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}>\frac{q_n}{\sqrt{1-\delta_n^2}}\to\infty.$$

Similarly to Lemma 4.2 it can be shown that

$$\begin{aligned}
\overline{F}_{\beta_n}\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)&=2\phi\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\Phi\left(\beta_n\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)^{-1}\\
&\quad+\frac{2\beta_n}{1+\beta_n^2}\phi\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\phi\left(\beta_n\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)^{-2}[1+o(1)]\\
&\quad-2\phi\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\phi\left(\beta_n\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)\left(\frac{q_n-\delta_n\nu}{\sqrt{1-\delta_n^2}}\right)^{-3}[1+o(1)]
\end{aligned}$$

holds uniformly on $v \in (0, \infty)$. Hence, by (25) we have

$$\begin{aligned}
\chi_n^U &= 2n \int_0^\infty \phi\left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \Phi\left(\beta_n \frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right)^{-1} f_V(v)\,\mathrm{d}v \\
&\quad + \frac{2n\beta_n}{1+\beta_n^2} \int_0^\infty \phi\left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \phi\left(\beta_n \frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right)^{-2} f_V(v)\,\mathrm{d}v[1+o(1)] \\
&\quad - 2n \int_0^\infty \phi\left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \Phi\left(\beta_n \frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) \left(\frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right)^{-3} f_V(v)\,\mathrm{d}v[1+o(1)] \\
&:= L_{n,1} + L_{n,2} + L_{n,3}.
\end{aligned} \tag{41}$$

We compute each term of $L_{n,1}$, $L_{n,2}$ and $L_{n,3}$ in turn. Noting that $\beta_n < 0$ and $\delta_n < 0$, by applying integration by parts to the first part, we obtain

$$\begin{aligned}
L_{n,1} &= \frac{2n}{\sqrt{2\pi}} \frac{(1-\delta_n^2)^{3/2}}{\delta_n} \exp\left[-\frac{q_n^2}{2(1-\delta_n^2)}\right] q_n^{-2} \int_0^\infty \bar{\kappa}_n(v)\,\mathrm{d}\left[\exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\right] \\
&= -\frac{2n}{\pi} \frac{(1-\delta_n^2)^{3/2}}{\delta_n} \Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) q_n^{-2} \exp\left[-\frac{q_n^2}{2(1-\delta_n^2)}\right] \\
&\quad - \frac{2n}{\sqrt{2\pi}} \frac{(1-\delta_n^2)^{3/2}}{\delta_n} \exp\left[-\frac{q_n^2}{2(1-\delta_n^2)}\right] q_n^{-2} \int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right) \mathrm{d}\bar{\kappa}_n(v) \\
&:= L_{n,11} + L_{n,12},
\end{aligned} \tag{42}$$

where

$$\bar{\kappa}_n(v) = \exp\left[-\frac{\delta_n^2 v^2}{2(1-\delta_n^2)}\right] \Phi\left(\beta_n \frac{q_n - \delta_n v}{\sqrt{1-\delta_n^2}}\right) f_V(v) \left(1 - \frac{\delta_n v}{q_n}\right)^{-1}.$$

We firstly consider the term $L_{n,11}$. According to the definitions of β_n in (6) and λ_n in (2) we know that for $\alpha < 0$,

$$\begin{aligned}
\beta_n = -\sqrt{\frac{\lambda_n(1+4\alpha^2)}{2\log n}} &\left\{1 + \left(\frac{1}{4} - \frac{\alpha^2}{1+4\alpha^2}\right)\frac{\lambda_n}{\log n}\right. \\
&\left. + \left[\frac{3}{32} - \frac{\alpha^4}{2(1+4\alpha^2)^2}\right] \frac{\lambda_n^2}{(\log n)^2} + O\left(\frac{\lambda_n^3}{(\log n)^3}\right)\right\}.
\end{aligned} \tag{43}$$

Substituting the expressions for q_n, β_n and δ_n in (43) gives

$$\begin{aligned}\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} = -\sqrt{\lambda_n(1+4\alpha^2)}\Bigg\{1 &+ \left(\frac{1}{4}-\frac{\alpha^2}{1+4\alpha^2}\right)\frac{\lambda_n}{\log n} \\ &- \frac{\log\left(-2\pi\omega_n\log n\right)}{2\log n} + \left[\frac{3}{32}-\frac{\alpha^4}{2(1+4\alpha^2)}\right]\frac{\lambda_n^2}{(\log n)^2} \\ &- \left(\frac{1}{4}-\frac{\alpha^2}{1+4\alpha^2}\right)\frac{\lambda_n}{\log n}\frac{\log\left(-2\pi\omega_n\log n\right)}{2\log n} + \frac{\log\left(-2\pi\omega_n\log n\right)}{2(\log n)^2} \\ &- \frac{[\log\left(-2\pi\omega_n\log n\right)]^2}{8(\log n)^2} + \frac{3\omega_n^2+1}{4\omega_n^2(\log n)^2} + o\left(\frac{1}{(\log n)^2}\right)\Bigg\}.\end{aligned}$$

Hence, as $n\to\infty$,

$$\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} \to -\sqrt{\lambda(1+4\alpha^2)}.$$

Applying the condition $\frac{\log}{\log\log n}(\lambda_n-\lambda)\to\gamma\in\mathbb{R}$ and expanding $\Phi(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}})$ at the point $-\sqrt{\lambda(1+4\alpha^2)}$ by Taylor expansion we have

$$\begin{aligned}\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) &= \Phi\left(-\sqrt{\lambda(1+4\alpha^2)}\right) + \sqrt{1+4\alpha^2}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right)\left(\sqrt{\lambda_n}-\sqrt{\lambda}\right) \\ &\quad - \sqrt{\lambda_n(1+4\alpha^2)}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right)\Bigg[\left(\frac{1}{4}-\frac{\alpha^2}{1+4\alpha^2}\right)\frac{\lambda_n}{\log n} \\ &\quad - \frac{\log\left(-2\pi\omega_n\log n\right)}{2\log n}\Bigg] + o\left(\frac{1}{\log n}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}L_{n,11} &= 2\Phi\left(-\sqrt{\lambda(1+4\alpha^2)}\right)\left\{1+\frac{\left[\log\left(-2\pi\omega_n\log n\right)\right]^2}{8\log n}+\frac{\log\left(-2\pi\omega_n\log n\right)}{\log n}\right\} \\ &\quad - 2\sqrt{1+4\alpha^2}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right)\left(\sqrt{\lambda_n}-\sqrt{\lambda}\right) \\ &\quad + 2\sqrt{\lambda_n(1+4\alpha^2)}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right)\frac{\log\left(-2\pi\omega_n\log n\right)}{\log n} + O\left(\frac{1}{\log n}\right). \qquad (44)\end{aligned}$$

Now we consider $L_{n,12}$. Using integration by parts again implies

$$\begin{aligned}\int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\mathrm{d}\bar{\kappa}_n(v) &= \sqrt{\frac{2}{\pi}}\frac{1-\delta_n^2}{\delta_n q_n}\left[\frac{\delta_n}{q_n}\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right)-\frac{\beta_n\delta_n}{\sqrt{1-\delta_n^2}}\phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right)\right] \\ &\quad - \frac{1-\delta_n^2}{\delta_n q_n}\int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right)\mathrm{d}^2\bar{\kappa}_n(v).\end{aligned}$$

Since $\delta_n < 0$ and $\beta_n < 0$, we have that for $v \in (0, \infty)$, $\exp(\frac{\delta_n q_n v}{1-\delta_n^2}) < 1$, $\exp[-\frac{\delta_n^2 v^2}{2(1-\delta_n^2)}] < 1$ and $(1 - \delta_n v/q_n)^{-1} < 1$. Direct calculation gives

$$\left| \int_0^\infty \exp\left(\frac{\delta_n q_n v}{1-\delta_n^2}\right) \mathrm{d}^2 \bar{\kappa}_n(v) \right| \le O(q_n^{-1}).$$

Due to the fact that $q_n \to \infty$, $\beta_n \to 0$ and $\delta_n \to 2\alpha/\sqrt{1-4\alpha^2}$ as $n \to \infty$,

$$|L_{n,12}| \le \frac{2n}{\pi} \frac{(1-\delta_n^2)^{5/2}}{\delta_n^2 q_n^4} \exp\left[-\frac{q_n^2}{2(1-\delta_n^2)}\right] O(1) \le O\left(\frac{1}{\log n}\right). \tag{45}$$

Using similar arguments to $L_{n,2}$ we have that for sufficiently large n,

$$\begin{aligned} L_{n,2} &= -2n \frac{\beta_n}{1+\beta_n^2} \frac{(1-\delta_n^2)^2}{\pi \delta_n} q_n^{-3} \phi\left(\frac{\sqrt{1+\beta_n^2} q_n}{\sqrt{1-\delta_n^2}}\right)[1+o(1)] \\ &= -\sqrt{\lambda_n(1+4\alpha^2)} \phi\left(\sqrt{\lambda_n(1+4\alpha^2)}\right) \frac{1}{\log n}[1+o(1)] \end{aligned} \tag{46}$$

and

$$\begin{aligned} L_{n,3} &= 2n \frac{(1-\delta_n^2)^{5/2}}{\pi \delta_n} q_n^{-4} \Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) \exp\left[-\frac{(1+\beta_n^2) q_n^2}{2(1-\delta_n^2)}\right][1+o(1)] \\ &= -\Phi\left(-\sqrt{\lambda(1+4\alpha^2)}\right) \frac{1}{\log n}[1+o(1)]. \end{aligned} \tag{47}$$

Hence, by (41) and the expansions of $L_{n,11}$, $L_{n,12}$, $L_{n,2}$ and $L_{n,3}$ in (44)–(47), it follows from the assumption $\frac{\log n}{(\log\log n)^2}(\sqrt{\lambda_n} - \sqrt{\lambda}) \to \gamma$ that

$$\begin{aligned} \frac{\log n}{(\log\log n)^2}(\chi_n^U - \chi_U) &= \frac{\log n}{(\log\log n)^2}(L_{n,1} + L_{n,2} + L_{n,3} - \chi_U) \\ &\to \frac{1}{4}\Phi\left(-\sqrt{\lambda(1+4\alpha^2)}\right) - \frac{\gamma}{\sqrt{\lambda}}\sqrt{1+4\alpha^2}\phi\left(\sqrt{\lambda(1+4\alpha^2)}\right) \end{aligned}$$

as $n \to \infty$. The proof is complete. ■

Proof of Theorem 2.4: For $\alpha > 0$, noting that $\lambda_n \to 0$, $\lambda_n \log\log n \to \infty$ as $n \to \infty$, it follows from Lemma 4.4 that

$$\begin{aligned} \frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}} &= \frac{\log s}{2\sqrt{\lambda_n}} + \sqrt{\lambda_n} + \frac{(\log\log n)^{\frac{3}{2}}}{4\log n}\left(\frac{\log s}{2\sqrt{\lambda_n \log\log n}} - \frac{\sqrt{\lambda_n}}{\sqrt{\log\log n}}\right) \\ &\quad + \frac{\sqrt{\log\log n}}{\log n} \frac{(\log s)^2 + (\log\pi - 3\lambda_n - 2)\log s}{8\sqrt{\lambda_n \log\log n}} + o\left(\frac{1}{\sqrt{\lambda_n}\log n}\right) \end{aligned}$$

uniformly on $s \in [1/\log n, 1]$ and tends to infinity uniformly. Thus, similarly to the proof of Theorem 2.3 in Hu et al. (2022), we know that for sufficiently large n

$$\int_{\frac{1}{\log n}}^{1} \Phi\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \mathrm{d}s = \Phi\left(\sqrt{\frac{1-\rho_n}{1+\rho_n}} q_n\right) - \frac{1}{\log n} \Phi\left(\frac{q_n - \rho_n q_n\left(\frac{1}{\log n}\right)}{\sqrt{1-\rho_n^2}}\right)$$

$$- \int_{\frac{1}{\log n}}^{1} s \,\mathrm{d}\Phi\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)$$

$$= \sqrt{\frac{2\lambda_n}{\pi}} (1 + o(1)). \tag{48}$$

Analogously, by the definition of ω_n in (6), we have

$$\omega_n = 2\alpha \left[1 - \left(\frac{1}{2} + \alpha^2\right) \frac{\lambda_n}{\log n} + o\left(\frac{\lambda_n}{\log n}\right)\right] \tag{49}$$

as $n \to \infty$. Hence, further by Lemma 4.4, we get

$$\omega_n q_n(s) = 2\alpha\sqrt{2\log n}\left[1 - \frac{\log \pi + \log\log n}{4\log n} - \frac{\log s}{2\log n} + o\left(\frac{1}{\log n}\right)\right].$$

Applying (15) yields

$$\overline{\Phi}\left(\omega_n q_n(s)\right) = \frac{1}{4\alpha} (\pi \log n)^{2\alpha^2 - \frac{1}{2}} \left(\frac{s}{n}\right)^{4\alpha^2} [1 + o(1)] \tag{50}$$

uniformly on $s \in [1/\log n, 1]$. Integrating the right end of the above expression from $1/\log n$ to 1 implies

$$\int_{\frac{1}{\log n}}^{1} \overline{\Phi}\left(\omega_n q_n(s)\right) \mathrm{d}s = \frac{1}{4\alpha(1+4\alpha^2)} n^{-4\alpha^2} (\pi \log n)^{2\alpha^2 - \frac{1}{2}} [1 + o(1)]. \tag{51}$$

From the proof of Theorem 2.3 we know that $\mathrm{P}(Y_n \geq q_n \,|\, X_n = q_n(s))$ has the expression (37). By (48), (50) and (51) we have that for sufficiently large n

$$\int_{\frac{1}{\log n}}^{1} K_{n,1}(s) \,\mathrm{d}s = \int_{\frac{1}{\log n}}^{1} \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) [1 - \overline{\Phi}(\omega_n q_n(s))]^{-1} \mathrm{d}s$$

$$= \int_{\frac{1}{\log n}}^{1} \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \left[1 - \overline{\Phi}(\omega_n q_n(s)) + O\left(\overline{\Phi}^2(\omega_n q_n(s))\right)\right] \mathrm{d}s$$

$$= 1 - \frac{1}{\log n} - \int_{\frac{1}{\log n}}^{1} \Phi\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \mathrm{d}s$$

$$- \int_{\frac{1}{\log n}}^{1} \overline{\Phi}(\omega_n q_n(s)) \,\mathrm{d}s + O\left((\log n)^{4\alpha^2 - 1} n^{-8\alpha^2}\right)$$

$$= 1 - \sqrt{\frac{2\lambda_n}{\pi}} [1 + o(1)].$$

It hence follows from (40) that

$$\int_{\frac{1}{\log n}}^{1} K_{n,2}(s)\,\mathrm{d}s = O\left(n^{-2\alpha^2}(\log n)^{\alpha^2-1/2}\right) = o\left(\sqrt{\lambda_n}\right).$$

Since

$$\int_0^{\frac{1}{\log n}} \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right)\mathrm{d}s \leq \frac{1}{\log n} = o\left(\sqrt{\lambda_n}\right),$$

combining (37) with (19) gives

$$\begin{aligned}\chi_n^U - \chi_U &= \int_{\frac{1}{\log n}}^{1} [K_{n,1}(s) + K_{n,2}(s)]\,\mathrm{d}s + \int_0^{\frac{1}{\log n}} \mathrm{P}\left(Y_n \geq q_n \,|\, X_n = q_n(s)\right)\mathrm{d}s - \chi_U \\ &= -\sqrt{\frac{2\lambda_n}{\pi}}[1 + o(1)].\end{aligned}$$

The proof for $\alpha > 0$ is finished.

If $\alpha < 0$, χ_n^U has the expression (41). By the expansion of β_n in (43) and Lemma 4.4,

$$\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} = -\sqrt{\lambda_n(1+4\alpha^2)}\left[1 - \frac{\log\left(-2\pi\omega_n \log n\right)}{2\log n} + o\left(\frac{1}{\log n}\right)\right].$$

Hence, $\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} \to 0$ as $n \to \infty$. Applying Taylor's expansion to $\Phi(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}})$ at the point zero gives

$$\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\sqrt{(1+4\alpha^2)\lambda_n} + O\left(\lambda_n^{3/2}\right).$$

Similarly to the proof of Theorem 2.3, substituting the above equation and the expression of q_n in Lemma 4.4 in $L_{n,11}$ gives

$$L_{n,11} = 1 - \sqrt{\frac{2}{\pi}}\sqrt{(1+4\alpha^2)\lambda_n} + O\left(\lambda_n^{3/2}\right).$$

Similarly to the proof of (45)–(47), we can show that under the assumption (45) of the theorem,

$$L_{n,2} = -\sqrt{(1+4\alpha^2)\lambda_n}\,\phi\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)\frac{1}{\log n}[1+o(1)]$$

and

$$L_{n,3} = -\frac{1}{2}\exp\left[-\frac{(1+4\alpha^2)\lambda_n}{2}\right]\frac{1}{\log n}[1+o(1)].$$

Thus, with (45) we have

$$\chi_n^U - \chi_U = L_{n,1} + L_{n,2} + L_{n,3} - \chi_U = -\sqrt{\frac{2}{\pi}}\sqrt{(1+4\alpha^2)\lambda_n}[1+o(1)].$$

The proof is complete. ∎

Proof of Theorem 2.5: Consider first the case where $\alpha > 0$. By assumption $\frac{\lambda_n}{\log\log n} \to 0$ and Lemma 4.4 we have that as $n \to \infty$,

$$\begin{aligned}
&\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}} \\
&\quad = \frac{\log s}{2\sqrt{\lambda_n}} + \sqrt{\lambda_n} + \frac{(\log\log n)^{\frac{3}{2}}}{4\log n}\left[\frac{\log s}{2\sqrt{\lambda_n}\log\log n} - \frac{\sqrt{\lambda_n}}{\sqrt{\log\log n}} + \frac{(\lambda_n)^{\frac{3}{2}}}{(\log\log n)^{\frac{3}{2}}}\right] \\
&\qquad + \frac{(\log s)^2 + (\log \pi - 2)\log s}{8\sqrt{\lambda_n}\log n} - \frac{3\sqrt{\lambda_n}\log s}{8\log n} + o\left(\frac{\lambda_n^{\frac{3}{2}}}{\log n}\right)
\end{aligned}$$

uniformly on $s \in [1/\log n, 1]$. Hence,

$$\int_{\frac{1}{\log n}}^{1} \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right) \mathrm{d}s = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)[1+o(1)].$$

By (49) and Lemma 4.4 we obtain

$$\omega_n q_n(s) = 2\alpha\sqrt{2\log n}\left[1 - \left(\frac{1}{2}+\alpha^2\right)\frac{\lambda_n}{\log n} - \frac{\log(\pi\log n)}{4\log n} - \frac{\log s}{2\log n} + o\left(\frac{1}{\log n}\right)\right]$$

and

$$\overline{\Phi}\left(\omega_n q_n(s)\right) = \frac{1}{4\alpha}\exp\left[4\alpha^2(1+2\alpha^2)\lambda_n\right](\pi\log n)^{2\alpha^2-\frac{1}{2}}\left(\frac{s}{n}\right)^{4\alpha^2}[1+o(1)]$$

uniformly on $s \in [1/\log n, 1]$. Thus,

$$\begin{aligned}
\int_{\frac{1}{\log n}}^{1} \overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\overline{\Phi}\left(\omega_n q_n(s)\right)\mathrm{d}s &\le \int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\omega_n q_n(s)\right)\mathrm{d}s \\
&= O\left(n^{-4\alpha^2}(\log n)^{2\alpha^2-\frac{1}{2}}\exp\left[4\alpha^2(1+2\alpha^2)\lambda_n\right]\right) \\
&= o\left(\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)\right).
\end{aligned}$$

And furthermore,

$$\begin{aligned}
\int_{\frac{1}{\log n}}^{1} K_{n,1}(s)\,\mathrm{d}s &= \int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\left[1 + O\left(\overline{\Phi}\left(\omega_n q_n(s)\right)\right)\right]\mathrm{d}s \\
&= \int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\mathrm{d}s \\
&\quad + \int_{\frac{1}{\log n}}^{1}\overline{\Phi}\left(\frac{q_n - \rho_n q_n(s)}{\sqrt{1-\rho_n^2}}\right)\overline{\Phi}\left(\omega_n q_n(s)\right)\mathrm{d}s\,O(1) \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)[1+o(1)].
\end{aligned}$$

Since in the same manner to (40) we can show that

$$\left|\int_{\frac{1}{\log n}}^{1} K_{n,2}(s)\,\mathrm{d}s\right| \le O\left(n^{-2\alpha^2}(\log n)^{\alpha^2-1/2}\right) = o\left(\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)\right),$$

and

$$\int_0^{\frac{1}{\log n}} \mathrm{P}\left(Y_n \ge q_n \mid X_n = q_n(s)\right)\mathrm{d}s \le \frac{1}{\log n} = o\left(\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)\right).$$

Therefore, with (37) we have

$$\begin{aligned}\chi_n^U &= \int_{\frac{1}{\log n}}^{1} \mathrm{P}\left(Y_n \ge q_n \mid X_n = q_n(s)\right)\mathrm{d}s + \int_0^{\frac{1}{\log n}} \mathrm{P}\left(Y_n \ge q_n \mid X_n = q_n(s)\right)\mathrm{d}s \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\lambda_n}}\exp\left(-\frac{\lambda_n}{2}\right)[1+o(1)].\end{aligned}$$

The desired result for $\alpha > 0$ is proved.

If $\alpha < 0$, we have $\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} \to -\infty$ as $n \to \infty$. By Lemma 4.4 and the assumption $\frac{\lambda_n}{\log\log n} \to 0$ we know that

$$\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}} = -\sqrt{\lambda_n(1+4\alpha^2)}\left[1 + \frac{\lambda_n}{4(1+4\alpha^2)\log n} - \frac{\log\left(-2\pi\omega_n\log n\right)}{2\log n} + o\left(\frac{1}{\log n}\right)\right]$$

as $n \to \infty$. Thus, it follows from (15) that

$$\Phi\left(\frac{\beta_n q_n}{\sqrt{1-\delta_n^2}}\right) = \frac{\phi\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)}{\sqrt{(1+4\alpha^2)\lambda_n}}\left[1 - \frac{1}{(1+4\alpha^2)\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right)\right].$$

Hence, with (42) we get

$$L_{n,11} = \frac{2\phi\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)}{\sqrt{(1+4\alpha^2)\lambda_n}}\left[1 - \frac{1}{(1+4\alpha^2)\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right)\right].$$

In the same way, we can obtain the expression (45) of $L_{n,12}$. Moreover, using the same arguments as that of (46)–(47) to $L_{n,2}$ and $L_{n,3}$ yields

$$L_{n,2} = -\sqrt{(1+4\alpha^2)\lambda_n}\phi\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)\frac{1}{\log n}[1+o(1)]$$

and

$$L_{n,3} = -\frac{\sqrt{2\pi}}{\sqrt{(1+4\alpha^2)\lambda_n}}\phi^2\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)\frac{1}{\log n}[1+o(1)].$$

Hence,

$$\chi_n^U = L_{n,1} + L_{n,2} + L_{n,3} = \frac{2\phi\left(\sqrt{(1+4\alpha^2)\lambda_n}\right)}{\sqrt{(1+4\alpha^2)\lambda_n}}\left[1 - \frac{1}{(1+4\alpha^2)\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right)\right].$$

The proof is complete. ∎

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