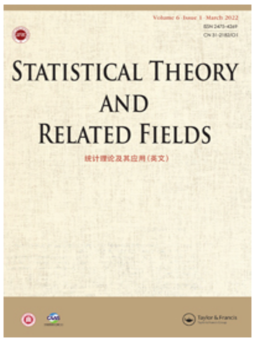

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Yiying Zhang

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Stochastic comparisons on total capacity of weighted *k*-out-of-*n* systems with heterogeneous components

Yiying Zhang

School of Statistics and Data Science, LPMC and KLMDASR, Nankai University, Tianjin, People's Republic of China

ABSTRACT
This paper carries out stochastic comparisons on the total capacity of weighted k-out-of-n systems with heterogeneous components. The expectation order, the increasing convex/concave order and the usual stochastic order are employed to investigate stochastic behaviours of system capacity. Sufficient conditions are established in terms of majorisation-type orders between the vectors of component lifetime distribution parameters and the vectors of weights. Some examples are also provided as illustrations.

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1. Introduction

The weighted k-out-of-n system might be first introduced in Wu and Chen (1994) describing the situation where the components may have different contributions to the system reliability/performance, and the system functions if and only if the total capacity (or weight) of the working components is no less than k. Under appropriate certain conditions imposed on the weights, it can be shown that the weighted k-out-of-n systems are equivalent to coherent systems (cf. Samaniego & Shaked, 2008).

Consider a weighted k-out-of-n system with n components having lifetimes $X_1, \ldots, X_n$ and the weight/capacity of component i is w_i, $i = 1, \ldots, n$. Then, the system works if and only if the total contribution of working components is no less than the threshold value $k > 0$. Let $\psi(t; \boldsymbol{w}, \boldsymbol{X})$ be the instantaneous system capacity at time $t \in \mathbb{R}_+$, that is, $\psi(t; \boldsymbol{w}, \boldsymbol{X}) := \sum_{i=1}^{n} w_i I(X_i > t)$. Then the failure time of this system can be expressed as

$$T(k; \boldsymbol{w}, \boldsymbol{X}) = \inf\{t : \psi(t; \boldsymbol{w}, \boldsymbol{X}) < k\},$$

from which we have

$$\mathbb{P}(T(k; \boldsymbol{w}, \boldsymbol{X}) > t) = \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X}) \geq k), \quad t \in \mathbb{R}_+. \tag{1}$$

The study on reliability analysis of weighted k-out-of-n system has attracted many researchers' attention in the past few years. For example, Rahmani et al. (2016b) discussed the influence of components lifetimes and weights on the system's total capacity under the independent case in the sense of the hazard rate ordering and the usual stochastic ordering. When it is allowed to allocate components lifetimes to the weights, they also presented the optimal allocation policy so as to maximise the system's total capacity. It is found that larger weight should be accompanied with good component with higher reliability to enhance the system performance. Li et al. (2016) studied the ordering properties of weighted k-out-of-n system with statistically dependent component lifetimes when the component weight vector varies according to some majorisation-type orders. Recently, Zhang et al. (2018) investigated performance levels of k-out-of-n systems with random weights (cf. Eryilmaz, 2013, 2015) and obtained optimal assembling policies by means of maximising the system's total capacity according to some stochastic orders. For more studies on other interesting topics such as (joint) importance measures of components and weights, redundancies allocation and system assembly, interested readers may refer to Meshkat and Mahmoudi (2017); Rahmani et al. (2016a); Zhang (2018, 2021).

To the best of the author's knowledge, it is still absent on the study of the heterogeneity among components lifetimes on the performance levels of weighted k-out-of-n systems. In this paper, we investigate ordering properties of the total capacity of weighted k-out-of-n systems with heterogeneous components in

CONTACT Yiying Zhang yyzhang@nankai.edu.cn School of Statistics and Data Science, LPMC and KLMDASR, Nankai University, Tianjin 300071, People's Republic of China

accordance with the expectation ordering, the increasing convex/concave ordering and the usual stochastic ordering. Sufficient conditions will be established by means of some majorisation-type orders.

The rest of the paper is organised as follows. Section 2 collects some useful definitions, notions and lemmas used in the sequel. Section 3 presents the comparison results of the total capacity of weighted k-out-of-n systems with heterogeneous components when the vector of distribution parameters and vector of weights change according to some majorisation-type orders. Some examples on the conditions are also provided as illustrations. Section 4 concludes the paper.

2. Preliminaries

Throughout, the terms 'increasing' and 'decreasing' are used in a non-strict sense. All expectations are well defined whenever they appear. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathcal{I}_n = \{\boldsymbol{x} : x_1 \le x_2 \le \ldots \le x_n\}$, and $\mathcal{D}_n = \{\boldsymbol{x} : x_1 \ge x_2 \ge \ldots \ge x_n\}$. Let $\boldsymbol{x}^{\{i,j\}}$ be the sub-vector of $\boldsymbol{x}$ with its ith and jth elements deleted. We use '$\overset{\text{sign}}{=}$' to denote that both sides of the equality have the same sign, and '$\mathbf{1}_n$' to denote an n-dimensional vector with all of its components equalling to 1.

Stochastic orders is a very helpful tool to compare the magnitude or variability of different random variables arising from many research areas.

Definition 2.1: For two random variables X and Y with distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, density functions f and g, hazard rate functions $h_F = f/\bar{F}$ and $h_G = g/\bar{G}$, and reversed hazard rate functions $r_F = f/F$ and $r_G = g/G$, respectively, X is said to be smaller than Y in the hazard rate order (denoted by $X \le_{\text{hr}} Y$) if $h_F(x) \ge h_G(x)$ for all $x \in \mathbb{R}$; the reversed hazard rate order (denoted by $X \le_{\text{rh}} Y$) if $r_F(x) \le r_G(x)$ for all $x \in \mathbb{R}$; the usual stochastic order (denoted by $X \le_{\text{st}} Y$) if $\bar{F}(x) \le \bar{G}(x)$ for all $x \in \mathbb{R}$; the increasing convex [concave] order (denoted by $X \le_{\text{icx[icv]}} Y$) if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for all increasing and convex [concave] function $\phi : \mathbb{R} \mapsto \mathbb{R}$; and the expectation order (denoted by $X \le_{\text{e}} Y$) if $\mathbb{E}[X] \le \mathbb{E}[Y]$.

The usual stochastic order, which is often termed as the *first stochastic dominance* in economics and finance, implies both of the increasing convex and concave orders, which in turn imply the expectation order. Interested readers may refer to the excellent monographs by Shaked and Shanthikumar (2007) for more details on the properties and applications of these stochastic orders.

The notion of majorisation is quite useful in establishing various inequalities stemming from reliability theory, applied probability, actuarial science, and so on. Let $x_{1:n} \le \cdots \le x_{n:n}$ be the increasing arrangement of the components of $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Definition 2.2: A real-valued vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ is said to be larger than another real-valued vector $\boldsymbol{y} = (y_1, \ldots, y_n)$ in the sense of the majorisation order, written as $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{i:n} \le \sum_{i=1}^{j} y_{i:n}$ for $j = 1, \ldots, n-1$, and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$; the supermajorisation order, written as $\boldsymbol{x} \overset{\text{w}}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{i:n} \le \sum_{i=1}^{j} y_{i:n}$ for $j = 1, \ldots, n$; and the submajorisation order, written as $\boldsymbol{x} \succeq_{\text{w}} \boldsymbol{y}$, if $\sum_{j=i}^{n} x_{j:n} \ge \sum_{j=i}^{n} y_{j:n}$ for $i = 1, \ldots, n$.

It is obvious that the majorisation order implies both the supermajorisation order and the submajorisation order, while the reverse statement is not true in general.

The following lemma plays a key role in deriving our main results.

Lemma 2.1 (Marshall et al. (2011)): *Let ϕ be a real-valued function, defined and continuous on $\mathcal{I}_n$ (w.r.t. $\mathcal{D}_n$) and continuously differentiable on the interior of $\mathcal{I}_n$ (w.r.t. $\mathcal{D}_n$). Denote the partial derivative of ϕ with respect to its kth argument by $\phi_{(k)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial z_k$, for $k = 1, \ldots, n$. Then, $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ whenever $\boldsymbol{x} \overset{\text{w}}{\preceq} \boldsymbol{y}$ (w.r.t. $\boldsymbol{x} \preceq_{\text{w}} \boldsymbol{y}$) on $\mathcal{I}_n$ (w.r.t. $\mathcal{D}_n$) if and only if $\phi_{(1)}(\boldsymbol{z}) \le \phi_{(2)}(\boldsymbol{z}) \le \cdots \le \phi_{(n)}(\boldsymbol{z}) \le 0$ (w.r.t. $0 \le \phi_{(1)}(\boldsymbol{z}) \le \phi_{(2)}(\boldsymbol{z}) \le \cdots \le \phi_{(n)}(\boldsymbol{z})$), for all $\boldsymbol{z}$ in the interior of $\mathcal{I}_n$ (w.r.t. $\mathcal{D}_n$).*

For more details on majorisation-type orders and their applications, one may refer to Marshall et al. (2011).

Next, let us introduce the definition of copula.

Definition 2.3: For a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with joint distribution function H and univariate marginal distribution functions $F_1, \ldots, F_n$, its copula is a distribution function $C : [0,1]^n \mapsto [0,1]$, satisfying

$$H(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_n(x_n)), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Similarly, a survival copula is a distribution function $\hat{C} : [0,1]^n \mapsto [0,1]$, satisfying

$$\begin{aligned}\bar{H}(\boldsymbol{x}) &= \mathbb{P}(X_1 > x_1, \ldots, X_n > x_n)\\ &= \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)), \quad \boldsymbol{x} \in \mathbb{R}^n,\end{aligned}$$

where $\bar{H}$ is the joint survival function of $\boldsymbol{X}$.

Copulas impose an easy addressable dependence structure on the marginal distributions of random vector. In some results of the present paper, we shall assume that the components lifetimes have a symmetric survival copula. For more detailed discussions on copulas, interested readers may refer to Nelsen (2007).

Definition 2.4: A real-valued bivariate function $g(x_1, x_2)$ is said to be arrangement increasing (AI) if $g(x_1, x_2) \ge g(x_2, x_1)$, for any $x_1 \le x_2$. The function

g is said to be arrangement decreasing (AD) when the inequality is reversed.

The notion AI [AD] means that the function achieves larger [smaller] value when (x_1, x_2) and the locations $(1, 2)$ are similarly ordered. They are very useful in establishing various inequalities arising from different areas. Readers can refer to Boland and Proschan (1988); Hollander et al. (1977) for more discussions.

3. Main results

In this section, we establish sufficient conditions for the expectation order, the increasing convex [concave] order and the usual stochastic order between two weighted k-out-of-n systems composed of heterogeneous components.

Henceforth, it is assumed that $w_1 \leq w_2 \leq \cdots \leq w_n$. For a weighted k-out-of-n system, let X_{λ_i} be the lifetime of the ith component, and the reliability function of X_{λ_i} is denoted by $\bar{F}(\cdot;\lambda_i)$, $i = 1, \ldots, n$. Denote $\boldsymbol{X_\lambda} = (X_{\lambda_1}, \ldots, X_{\lambda_n})$. First, we make the following assumption throughout the paper:

Assumption 3.1: $\bar{F}(t;\lambda)$ is decreasing in $\lambda > 0$, for all $t \in \mathbb{R}_+$.

According to Rahmani et al. (2016b); Zhang et al. (2018), larger weight should be allocated with component with higher reliability. Then, under Assumption 3.1, the best assembly of components and weights can be achieved by setting $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and the worst assembly is obtained if $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. That is, $\boldsymbol{\lambda} \in \mathcal{D}_n$ and $\boldsymbol{\lambda} \in \mathcal{I}_n$, respectively.

First, we study the expectation order. The following assumption is needed for deriving the first main result.

Assumption 3.2: $\bar{F}(t;\lambda)$ is convex in $\lambda > 0$, for all $t \in \mathbb{R}_+$.

Theorem 3.1: *Under Assumptions* 3.1 *and* 3.2*, if* $\boldsymbol{\lambda} \in \mathcal{D}_n$ *and* $\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu}$*, then* $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\mathrm{e}} \psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$*, for all* $t \in \mathbb{R}_+$.

Proof: The desired result is equivalent to showing that $\mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda})] \geq \mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\mu})]$, i.e., $\sum_{i=1}^n w_i\bar{F}(t;\lambda_i) \geq \sum_{i=1}^n w_i\bar{F}(t;\mu_i)$, for all $t \in \mathbb{R}_+$.

We first assume that $\boldsymbol{\mu} \in \mathcal{D}_n$. Denote by $\phi_1(\boldsymbol{\lambda}) := \sum_{i=1}^n w_i\bar{F}(t;\lambda_i)$. Then $\frac{\partial\phi_1(\boldsymbol{\lambda})}{\partial\lambda_i} = w_i\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i} \leq 0$, $i = 1, \ldots, n$, and

$$\begin{aligned}\frac{\partial\phi_1(\boldsymbol{\lambda})}{\partial\lambda_i} - \frac{\partial\phi_1(\boldsymbol{\lambda})}{\partial\lambda_j} &= w_i\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i} - w_j\frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\\ &\geq w_j\left(\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i} - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\right)\\ &\geq 0,\end{aligned}$$

for $1 \leq i < j \leq n$. Thus the proof is finished by applying Lemma 2.1.

Now, let us consider any permutation of $\boldsymbol{\mu}$, say $\boldsymbol{\mu_\pi} = (\mu_{\pi_1}, \ldots, \mu_{\pi_n})$, where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ is a permutation of $(1, 2, \ldots, n)$. Without loss of generality, we set

$$\begin{aligned}\boldsymbol{\mu_\pi} = (&\mu_1, \ldots, \mu_{i-1}, \mu_j, \mu_{i+1}, \ldots, \mu_{j-1},\\ &\mu_i, \mu_{j+1}, \ldots, \mu_n), \quad 1 \leq i < j \leq n.\end{aligned}$$

Then, it can be verified that

$$\begin{aligned}&\mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\mu})] - \mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_{\mu_\pi}})]\\ &\quad = w_i\bar{F}(t;\mu_i) + w_j\bar{F}(t;\mu_j) - w_i\bar{F}(t;\mu_j) - w_j\bar{F}(t;\mu_i)\\ &\quad = (w_i - w_j)[\bar{F}(t;\mu_i) - \bar{F}(t;\mu_j)] \geq 0,\end{aligned}$$

where the inequality follows from Assumption 3.1. To sum up, the proof is finished. ■

Remark 3.1: As per Theorem 3.1, it indicates that, under appropriate conditions, more heterogeneity among the components leads to larger expected system capacity. It should be noted that the components lifetimes can be dependent in Theorem 3.1, which does not affect the expectation ordering.

Combining Theorem 3.1 with Theorem 3.4 in Li et al. (2016), the following result can be obtained immediately.

Theorem 3.2: *Under Assumptions* 3.1 *and* 3.2*, if* $\boldsymbol{w}, \boldsymbol{v} \in \mathcal{I}_n$*,* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{D}_n$*,* $\boldsymbol{w} \geq_{\mathrm{w}} \boldsymbol{v}$ *and* $\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu}$*, then* $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\mathrm{e}} \psi(t;\boldsymbol{v},\boldsymbol{X_\mu})$*, for all* $t \in \mathbb{R}_+$.

Proof: From Theorem 3.1, it holds that $\mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda})] \geq \mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\mu})]$, for all $t \in \mathbb{R}_+$. On the other hand, the result of Theorem 3.4 in Li et al. (2016) implies that $\mathbb{E}[\psi(t;\boldsymbol{w},\boldsymbol{X_\mu})] \geq \mathbb{E}[\psi(t;\boldsymbol{v},\boldsymbol{X_\mu})]$, for all $t \in \mathbb{R}_+$. Thus the desired result follows. ■

Next, we establish sufficient conditions for the increasing convex ordering between $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda})$ and $\psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$, for $t \in \mathbb{R}_+$. We need the following independence assumption on the lifetimes of components.

Assumption 3.3: The components lifetimes in the weighted k-out-of-n system are independent.

Besides, we need an additional assumption imposed on the reliability function $\bar{F}(\cdot;\lambda)$ with respect to the parameter $\lambda > 0$.

Assumption 3.4: The function $\frac{\partial\ln\bar{F}(t;\lambda)}{\partial\lambda}$ is increasing in $\lambda > 0$, for all $t \in \mathbb{R}_+$.

Theorem 3.3: *Under Assumptions* 3.1–3.4*, if* $\boldsymbol{\lambda} \in \mathcal{D}_n$ *and* $\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\mu}$*, then* $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\mathrm{icx}} \psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$*, for all* $t \in \mathbb{R}_+$.

Proof: We first prove the result when $\boldsymbol{\mu} \in \mathcal{D}_n$. The desired result is equivalent to showing that $\mathbb{E}[u(\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}))] \geq \mathbb{E}[u(\psi(t;\boldsymbol{w},\boldsymbol{X_\mu}))]$, for all increasing and convex function u and all $t \in \mathbb{R}_+$. Let $\Delta_{i,j}(t) := \sum_{r\neq i,j}^{n} w_r I(X_{\lambda_r} > t)$. Note that, for any $1 \leq i < j \leq n$,

$$
\begin{aligned}
\phi_2(\boldsymbol{\lambda}) &:= \mathbb{E}[u(\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}))] \\
&= \bar{F}(t;\lambda_i)\bar{F}(t;\lambda_j)\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))] \\
&\quad + \bar{F}(t;\lambda_i)[1-\bar{F}(t;\lambda_j)]\mathbb{E}[u(w_i + \Delta_{i,j}(t))] \\
&\quad + \bar{F}(t;\lambda_j)[1-\bar{F}(t;\lambda_i)]\mathbb{E}[u(w_j + \Delta_{i,j}(t))] \\
&\quad + [1-\bar{F}(t;\lambda_i)][1-\bar{F}(t;\lambda_j)]\mathbb{E}[u(\Delta_{i,j}(t))].
\end{aligned}
$$

Taking the derivative of $\phi_2(\boldsymbol{\lambda})$ with respective to λ_i, we have

$$
\begin{aligned}
&\frac{\partial\phi_2(\boldsymbol{\lambda})}{\partial\lambda_i} \\
&= \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j)\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))] \\
&\quad + \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}[1-\bar{F}(t;\lambda_j)]\mathbb{E}[u(w_i + \Delta_{i,j}(t))] \\
&\quad - \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j)\mathbb{E}[u(w_j + \Delta_{i,j}(t))] \\
&\quad - \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}[1-\bar{F}(t;\lambda_j)]\mathbb{E}[u(\Delta_{i,j}(t))] \\
&= \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j)\left\{\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))]\right. \\
&\quad \left. - \mathbb{E}[u(w_j + \Delta_{i,j}(t))]\right\} \\
&\quad + \frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}[1-\bar{F}(t;\lambda_j)]\left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t))]\right. \\
&\quad \left. - \mathbb{E}[u(\Delta_{i,j}(t))]\right\}.
\end{aligned}
$$

Since u is increasing, the Assumption 3.1 implies that $\frac{\partial\phi_2(\boldsymbol{\lambda})}{\partial\lambda_i} \leq 0$, for $i = 1,\ldots,n$ and $t \in \mathbb{R}_+$. Since $\lambda_i \geq \lambda_j$, $w_i \leq w_j$, and u is increasing and convex, from Assumptions 3.1, 3.2 and 3.4, we can obtain $\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i} \geq \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}$, $\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j) \geq \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\bar{F}(t;\lambda_i)$, and

$$
\begin{aligned}
&\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))] - \mathbb{E}[u(w_j + \Delta_{i,j}(t))] \\
&\quad = \mathbb{E}\left\{\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))\right. \\
&\quad\quad \left. - u(w_j + \Delta_{i,j}(t))|\boldsymbol{X}_{\boldsymbol{\lambda}}^{\{i,j\}}]\right\} \\
&\quad \geq \mathbb{E}\left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t)) - u(\Delta_{i,j}(t))|\boldsymbol{X}_{\boldsymbol{\lambda}}^{\{i,j\}}]\right\} \\
&\quad = \mathbb{E}[u(w_i + \Delta_{i,j}(t))] - \mathbb{E}[u(\Delta_{i,j}(t))].
\end{aligned}
$$

Then, one can observe that

$$
\begin{aligned}
&\frac{\partial\phi_2(\boldsymbol{\lambda})}{\partial\lambda_i} - \frac{\partial\phi_2(\boldsymbol{\lambda})}{\partial\lambda_j} \\
&= \left[\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j) - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\bar{F}(t;\lambda_i)\right] \\
&\quad \times \left\{\mathbb{E}[u(w_i + w_j + \Delta_{i,j}(t))]\right. \\
&\quad \left. - \mathbb{E}[u(w_j + \Delta_{i,j}(t))]\right\} \\
&\quad + \left[\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}[1-\bar{F}(t;\lambda_j)]\right. \\
&\quad \left. - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}[1-\bar{F}(t;\lambda_i)]\right] \\
&\quad \times \left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t))] - \mathbb{E}[u(\Delta_{i,j}(t))]\right\} \\
&\geq \left[\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}\bar{F}(t;\lambda_j) - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\bar{F}(t;\lambda_i)\right] \\
&\quad \times \left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t))] - \mathbb{E}[u(\Delta_{i,j}(t))]\right\} \\
&\quad + \left[\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i}[1-\bar{F}(t;\lambda_j)]\right. \\
&\quad \left. - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}[1-\bar{F}(t;\lambda_i)]\right] \\
&\quad \times \left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t))] - \mathbb{E}[u(\Delta_{i,j}(t))]\right\} \\
&= \left(\frac{\partial\bar{F}(t;\lambda_i)}{\partial\lambda_i} - \frac{\partial\bar{F}(t;\lambda_j)}{\partial\lambda_j}\right)\left\{\mathbb{E}[u(w_i + \Delta_{i,j}(t))]\right. \\
&\quad \left. - \mathbb{E}[u(\Delta_{i,j}(t))]\right\} \\
&\geq 0.
\end{aligned}
$$

Hence, the proof is finished by applying Lemma 2.1.

On the other hand, it follows from Theorem 2.2 of Rahmani et al. (2016b) that $\psi(t;\boldsymbol{w},\boldsymbol{X_\mu}) \geq_{\text{st}} \psi(t;\boldsymbol{w},\boldsymbol{X}_{\boldsymbol{\mu_\pi}})$, where $\boldsymbol{\mu_\pi}$ is a permutation of $\boldsymbol{\mu}$ with a permutation vector of indexes $\boldsymbol{\pi}$. Since the usual stochastic ordering implies the increasing convex ordering, it must hold that $\psi(t;\boldsymbol{w},\boldsymbol{X_\mu}) \geq_{\text{icx}} \psi(t;\boldsymbol{w},\boldsymbol{X}_{\boldsymbol{\mu_\pi}})$. To sum up, the desired result follows. ∎

Remark 3.2: In the same spirit with Theorem 3.1, the result of Theorem 3.3 states that, under suitable conditions (larger weight accompanied with 'good' component), more heterogeneity among components lifetimes results in greater system capacity in the sense of the increase convex ordering. Since the increasing convex ordering is stronger than the expectation ordering, it is natural to note that some more restrictive conditions are needed.

In order to study the effects of both of the heterogeneity among weights and components on the system capacity, we strengthen Assumption 3.1 as follows.

Assumption 3.5: For $\lambda_1 \geq \lambda_2$, it holds that $X_{\lambda_1} \leq_{\text{hr}} X_{\lambda_2}$.

Assumption 3.6: For $\lambda_1 \geq \lambda_2$, it holds that $X_{\lambda_1} \leq_{\text{rh}} X_{\lambda_2}$.

Theorem 3.4: *Under Assumptions* 3.2–3.5, *if* $\boldsymbol{w},\boldsymbol{v} \in \mathcal{I}_n$, $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathcal{D}_n$, $\boldsymbol{w} \geq_{\text{w}} \boldsymbol{v}$ *and* $\boldsymbol{\lambda} \overset{\text{w}}{\succeq} \boldsymbol{\mu}$, *then* $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\text{icx}} \psi(t;\boldsymbol{v},\boldsymbol{X_\mu})$, *for all* $t \in \mathbb{R}_+$.

Proof: Since Assumption 3.5 implies Assumption 3.1, by using Theorem 3.3 we have that $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\text{icx}} \psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$. On the other hand, by combining Theorem 3.7 of Li et al. (2016) with Proposition 5.4 of Cai and Wei (2014), it holds that $\psi(t;\boldsymbol{w},\boldsymbol{X_\mu}) \geq_{\text{icx}} \psi(t;\boldsymbol{v},\boldsymbol{X_\mu})$. Thus the desired result follows. ■

In many practical scenarios, it often occurs that the weighted k-out-of-n system might be consisting of two types of components and each type of component has the common weight (cf. Eryilmaz & Kan, 2020; Eryilmaz & Sarikaya, 2014). The following corollary can be obtained from Theorem 3.4 for this special weighted k-out-of-n system.

Corollary 3.1: *Consider two weighted k-out-of-n systems with weights $(w_1\mathbf{1}_{n_1}, w_2\mathbf{1}_{n_2})$ and $(v_1\mathbf{1}_{n_1}, v_2\mathbf{1}_{n_2})$, and component lifetimes $(X_{\lambda_1}\mathbf{1}_{n_1}, X_{\lambda_2}\mathbf{1}_{n_2})$ and $(X_{\mu_1}\mathbf{1}_{n_1}, X_{\mu_2}\mathbf{1}_{n_2})$, respectively, where $n_1 + n_2 = n$. Suppose that $w_1 \leq w_2$, $v_1 \leq v_2$, $\lambda_1 \geq \lambda_2$, and $\mu_1 \geq \mu_2$. Under Assumptions 3.2–3.5, if $(w_1\mathbf{1}_{n_1}, w_2\mathbf{1}_{n_2}) \succeq_{\text{w}} (v_1\mathbf{1}_{n_1}, v_2\mathbf{1}_{n_2})$ and $(\lambda_1\mathbf{1}_{n_1}, \lambda_2\mathbf{1}_{n_2}) \overset{\text{w}}{\succeq} (\mu_1\mathbf{1}_{n_1}, \mu_2\mathbf{1}_{n_2})$, then $\psi(t; w_1, w_2, \lambda_1, \lambda_2) \geq_{\text{icx}} \psi(t; v_1, v_2, \mu_1, \mu_2)$, for all $t \in \mathbb{R}_+$.*

By adopting a similar proof in Theorem 3.3, the following result presents sufficient conditions for the increasing concave ordering when the larger weight is accompanied with 'bad' component.

Theorem 3.5: *Under Assumptions 3.1–3.4, if $\boldsymbol{\lambda} \in \mathcal{I}_n$ and $\boldsymbol{\lambda} \succeq_{\text{w}} \boldsymbol{\mu}$, then $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \leq_{\text{icv}} \psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$, for all $t \in \mathbb{R}_+$.*

The next result can be obtained from Theorem 3.8 of Li et al. (2016) together with Proposition 4.1 of Cai and Wei (2015), whose proof is similar with that of Theorem 3.4 and thus omitted.

Theorem 3.6: *Under Assumptions 3.2, 3.3, 3.4, and 3.6, if $\boldsymbol{w}, \boldsymbol{v} \in \mathcal{I}_n$, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{I}_n$, $\boldsymbol{w} \overset{\text{w}}{\succeq} \boldsymbol{v}$ and $\boldsymbol{\lambda} \succeq_{\text{w}} \boldsymbol{\mu}$, then $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \leq_{\text{icv}} \psi(t;\boldsymbol{v},\boldsymbol{X_\mu})$, for all $t \in \mathbb{R}_+$.*

The next corollary can be derived from Theorem 3.6 immediately.

Corollary 3.2: *Under the setting of Corollary 3.1, suppose that $w_1 \leq w_2$, $v_1 \leq v_2$, $\lambda_1 \leq \lambda_2$, and $\mu_1 \leq \mu_2$. Under Assumptions 3.2, 3.3, 3.4, and 3.6, if $(w_1\mathbf{1}_{n_1}, w_2\mathbf{1}_{n_2}) \overset{\text{w}}{\succeq} (v_1\mathbf{1}_{n_1}, v_2\mathbf{1}_{n_2})$ and $(\lambda_1\mathbf{1}_{n_1}, \lambda_2\mathbf{1}_{n_2}) \succeq_{\text{w}} (\mu_1\mathbf{1}_{n_1}, \mu_2\mathbf{1}_{n_2})$, then $\psi(t; w_1, w_2, \lambda_1, \lambda_2) \leq_{\text{icv}} \psi(t; v_1, v_2, \mu_1, \mu_2)$, for all $t \in \mathbb{R}_+$.*

As an illustration, we present two distribution families satisfying Assumptions 3.1, 3.2, 3.4, 3.5 and 3.6.

Example 3.1: We consider the scale and proportional hazard rates (PHR) distribution families as follows:

(a) Scale family: For this case, we have $\bar{F}(t;\lambda) = \bar{F}(\lambda t)$, $t \in \mathbb{R}_+$. Thus Assumption 3.1 holds naturally, and Assumption 3.2 reduces to the condition that the underlying density function f is decreasing (e.g., the gamma and Weibull density functions with shape parameters less than 1). Assumption 3.4 can be simplified into that the underlying hazard rate function $h_F(t)$ is decreasing in $t \in \mathbb{R}_+$. Further, Assumption 3.5 is equivalent to saying that $th_F(t)$ is increasing in $t \in \mathbb{R}_+$, and Assumption 3.6 is equivalent to saying that $tr_F(t)$ is decreasing in $t \in \mathbb{R}_+$. These simplified conditions are fulfilled by some well-known distributions within the scale family; see Ding et al. (2017); Zhang et al. (2019).

(b) PHR family: In this case, we have $\bar{F}(t;\lambda) = \bar{F}^\lambda(t)$, $t \in \mathbb{R}_+$. It is easy to check that Assumptions 3.1, 3.2, 3.4, and 3.5 hold naturally.

Now, let us verify Assumption 3.6. Let $\phi_3(\lambda) = \frac{\lambda}{x^{-\lambda}-1}$, where $x \in (0, 1]$ and $\lambda > 0$. Observe that

$$\phi_3'(\lambda) \overset{\text{sign}}{=} x^{-\lambda} - 1 + \lambda x^{-\lambda} \ln x =: \phi_4(\lambda).$$

Since $\phi_4'(\lambda) = -\lambda x^{-\lambda}(\ln x)^2 \leq 0$, it follows that $\phi_4(\lambda)$ is decreasing in $\lambda > 0$. Thus, $\phi_4(\lambda) \leq \lim_{\lambda\to 0_+} \phi_3(\lambda) = 0$, which in turn implies that $\phi_3(\lambda)$ is decreasing in $\lambda > 0$. Since the reversed hazard rate function of X_{λ_1} can be written as

$$r_{X_{\lambda_1}}(t) = \frac{\lambda_1 \bar{F}^{\lambda_1 - 1}(t) f(t)}{1 - \bar{F}^{\lambda_1}(t)} = h_F(t)\frac{\lambda_1}{\bar{F}^{-\lambda_1}(t) - 1}.$$

By using the decreasing property of $\phi_3(\lambda)$, we then have

$$\begin{aligned} r_{X_{\lambda_1}}(t) &= h_F(t)\frac{\lambda_1}{\bar{F}^{-\lambda_1}(t) - 1} \\ &\leq h_F(t)\frac{\lambda_2}{\bar{F}^{-\lambda_2}(t) - 1} = r_{X_{\lambda_2}}(t), \quad \lambda_1 \geq \lambda_2, \end{aligned}$$

which implies Assumption 3.6.

The previous results both in Theorems 3.3 and 3.5 are established based on Assumption 3.3, that is, the components in the system are assumed to be independent. As a generalisation, the following theorem studies the increasing convex ordering for a k-out-of-2 system composed of only two dependent components, whose lifetimes share a symmetric survival copula. To proceed, we make the following assumption on the survival copula.

Assumption 3.7: For a weighted k-out-of-2 system with two dependent components sharing a survival copula $\hat{C}$, assume that $\hat{C}$ is symmetric and $u\hat{C}(u, v)$ is AD in $(u, v) \in (0, 1)^2$.

Theorem 3.7: *Under Assumptions 3.1, 3.2, 3.4 and 3.7, if $\boldsymbol{\lambda} \in \mathcal{D}_2$ and $\boldsymbol{\lambda} \overset{\text{w}}{\succeq} \boldsymbol{\mu}$, then $\psi(t;\boldsymbol{w},\boldsymbol{X_\lambda}) \geq_{\text{icx}} \psi(t;\boldsymbol{w},\boldsymbol{X_\mu})$, for all $t \in \mathbb{R}_+$.*

Proof: According to the proof of Theorem 3.3, we can write

$$\begin{aligned}
\phi_4(\boldsymbol{\lambda}) &:= \mathbb{E}[u(\psi(t;\boldsymbol{w},\boldsymbol{X}_{\boldsymbol{\lambda}}))] \\
&= \hat{C}(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))u(w_1+w_2) \\
&\quad + [\bar{F}(t;\lambda_1) - \hat{C}(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))]u(w_1) \\
&\quad + [\bar{F}(t;\lambda_2) - \hat{C}(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))]u(w_2) \\
&\quad + [1 - \bar{F}(t;\lambda_1) - \bar{F}(t;\lambda_2) \\
&\quad + \hat{C}(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))]u(0).
\end{aligned}$$

It can be derived that

$$\begin{aligned}
\frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_1} &= \frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}\hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2)) \\
&\quad \times [u(w_1+w_2) - u(w_2)] \\
&\quad + \frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}[1 - \hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))] \\
&\quad \times [u(w_1) - u(0)], \\
\frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_2} &= \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2)) \\
&\quad \times [u(w_1+w_2) - u(w_2)] \\
&\quad + \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}[1 - \hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))] \\
&\quad \times [u(w_1) - u(0)],
\end{aligned}$$

where $\hat{C}_1(u,v)$ and $\hat{C}_2(u,v)$ stands for the partial derivative of $\hat{C}(u,v)$ with respective to u and v, respectively. Upon using Assumptions 3.1, 3.2, 3.4 and 3.7, we have $\frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_1} < 0$, $\frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_2} < 0$, $\lambda_1 \geq \lambda_2$, $\bar{F}(t;\lambda_1) \leq \bar{F}(t;\lambda_2)$,

$$\begin{aligned}
0 &\leq \bar{F}(t;\lambda_1)\hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2)) \\
&\leq \bar{F}(t;\lambda_2)\hat{C}_1(\bar{F}(t;\lambda_2),\bar{F}(t;\lambda_1)) \\
&= \bar{F}(t;\lambda_2)\hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}\hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2)) \\
&= -\left[\left(-\frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}\frac{1}{\bar{F}(t;\lambda_1)}\right)\bar{F}(t;\lambda_1)\right. \\
&\quad \left.\times\ \hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))\right] \\
&\geq -\left[\left(-\frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\frac{1}{\bar{F}(t;\lambda_2)}\right)\bar{F}(t;\lambda_1)\right. \\
&\quad \left.\times\ \hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))\right] \\
&\geq -\left[\left(-\frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\frac{1}{\bar{F}(t;\lambda_2)}\right)\bar{F}(t;\lambda_2)\right. \\
&\quad \left.\times\ \hat{C}_1(\bar{F}(t;\lambda_2),\bar{F}(t;\lambda_1))\right] \\
&= \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2)).
\end{aligned}$$

Then, by applying the convexity of u we have

$$\begin{aligned}
&\frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_1} - \frac{\partial\phi_4(\boldsymbol{\lambda})}{\partial\lambda_2} \\
&= [u(w_1+w_2) - u(w_2)] \\
&\quad \times \left[\frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}\hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))\right. \\
&\quad \left. - \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))\right] \\
&\quad + \left\{\frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1}[1 - \hat{C}_1(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))]\right. \\
&\quad \left. - \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}[1 - \hat{C}_2(\bar{F}(t;\lambda_1),\bar{F}(t;\lambda_2))]\right\} \\
&\quad \times [u(w_1) - u(0)] \\
&\geq [u(w_1) - u(0)]\left(\frac{\partial\bar{F}(t;\lambda_1)}{\partial\lambda_1} - \frac{\partial\bar{F}(t;\lambda_2)}{\partial\lambda_2}\right) \geq 0.
\end{aligned}$$

Hence, the proof is completed by applying Lemma 2.1. ■

In light of the proof of Theorem 3.7 and the setting of Theorem 3.5, the following result can be obtained immediately.

Theorem 3.8: *Under Assumptions* 3.1, 3.2, 3.4 *and* 3.7, *if* $\boldsymbol{\lambda} \in \mathcal{I}_2$ *and* $\boldsymbol{\lambda} \succeq_{\mathrm{w}} \boldsymbol{\mu}$, *then* $\psi(t;\boldsymbol{w},\boldsymbol{X}_{\boldsymbol{\lambda}}) \leq_{\mathrm{icv}} \psi(t;\boldsymbol{w},\boldsymbol{X}_{\boldsymbol{\mu}})$, *for all* $t \in \mathbb{R}_+$.

As an illustration of Assumption 3.7, let us consider the Farlie–Gumble–Morgenstern (FGM) survival copula and the family of Archimedean survival copulas.

Example 3.2: (a) The FGM survival copula (cf. Nelsen, 2007) is given by

$$\hat{C}(u,v) = uv[1 + \theta(1-u)(1-v)], \quad -1 \leq \theta \leq 1.$$

Note that

$$\begin{aligned}
u\hat{C}_1(u,v) &= uv + \theta uv(1-u)(1-v) + u^2v \\
&\quad - \theta u^2v(1-v), \\
v\hat{C}_1(v,u) &= uv + \theta uv(1-u)(1-v) + uv^2 \\
&\quad - \theta v^2u(1-u).
\end{aligned}$$

Then, for $0 \leq u \leq v \leq 1$, we have

$$\begin{aligned}
&v\hat{C}_1(v,u) - u\hat{C}_1(u,v) \\
&\quad = uv^2 - \theta v^2u(1-u) - u^2v + \theta u^2v(1-v)
\end{aligned}$$

$$= uv[v - \theta v(1-u) - u + \theta u(1-v)]$$
$$= uv(v-u)(1-\theta) \geq 0,$$

which implies the AD property of $u\hat{C}_1(u, v)$ in $(u, v) \in [0, 1]^2$, irrespective of the dependence structure.

(b) The Archimedean survival copula (cf. Nelsen, 2007) has its explicit expression

$$\hat{C}(u, v) = \psi(\psi^{-1}(u) + \psi^{-1}(v)),$$

where the generator $\psi : \mathbb{R}_+ \mapsto [0, 1]$ is a decreasing and continuous function such that $\psi(0) = 1$ and $\psi(+\infty) = 0$. Observe that

$$u\hat{C}_1(u, v) = u(\psi^{-1})'(u)\psi'(\psi^{-1}(u) + \psi^{-1}(v)).$$

Thus, for $0 \leq u \leq v \leq 1$, the AD property of $u\hat{C}_1(u, v)$ in (u, v) is equivalent to $u\hat{C}_1(u, v) \leq v\hat{C}_1(v, u)$, i.e., $u(\psi^{-1})'(u) \geq v(\psi^{-1})'(v)$. This boils down to showing that $u(\psi^{-1})'(u)$ is decreasing in $u \in [0, 1]$. Since $(\psi^{-1})'(u) = 1/\psi'(\psi^{-1}(u))$, by letting $t = \psi^{-1}(u)$ we need to show that $\psi(t)/\psi'(t)$ is increasing in $t \in \mathbb{R}_+$. It equals to saying that $\log \psi$ is concave. For example, the Gumbel–Hougaard survival copula (see (4.2.9) in Table 4.1 of Nelsen, 2007) has log-concave generator meaning that the two components lifetimes are negative lower orthant dependent (NLOD). To this regard, we cannot obtain the similar ordering result with that of Theorem 3.4 since the component lifetimes are required to be positively dependent according to Theorem 3.7 of Li et al. (2016) and Proposition 5.4 of Cai Wei (2014) under the framework of Archimedean copulas. We leave it as an open problem for further investigation.

Next, let us present a result on the usual stochastic ordering for the weighted k-out-of-2 system. We leave it as an open problem extending the result to the case when the system has more than three components.

Theorem 3.9: *Consider two weighted k-out-of-2 systems with common weights (w_1, w_2). Under Assumption 3.3, if $X_{\lambda_2} \geq_{\rm st} X_{\mu_2}$ and $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm st} \min\{X_{\mu_1}, X_{\mu_2}\}$, then $T(k; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq_{\rm st} T(k; \boldsymbol{w}, \boldsymbol{X_\mu})$, for all $k > 0$.*

Proof: By using (1), it is equivalent to showing that $\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k) \geq \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k)$, for all $k > 0$ and $t \in \mathbb{R}_+$. The proof can be completed by distinguishing the values of k in accordance with the following four cases:

Case 1: $k \leq w_1$. For this case, it is clear that $\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k) = \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k) = 1$, for all $t \in \mathbb{R}_+$.

Case 2: $w_1 < k \leq w_2$. For this case, by using $X_{\lambda_2} \geq_{\rm st} X_{\mu_2}$ we have

$$\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k)$$
$$= F(t; \lambda_1)\bar{F}(t; \lambda_2) + \bar{F}(t; \lambda_1)\bar{F}(t; \lambda_2)$$
$$= \bar{F}(t; \lambda_2) \geq \bar{F}(t; \mu_2) = \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k).$$

Case 3: $w_2 < k \leq w_1 + w_2$. By applying $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm st} \min\{X_{\mu_1}, X_{\mu_2}\}$, we have

$$\begin{aligned}\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k) &= \bar{F}(t; \lambda_1)\bar{F}(t; \lambda_2) \\ &\geq \bar{F}(t; \mu_1)\bar{F}(t; \mu_2) \\ &= \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k).\end{aligned}$$

Case 4: $k > w_1 + w_2$. For this case, it is easy to see that $\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k) = \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k) = 0$, for all $t \in \mathbb{R}_+$.

Therefore, we have $\mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq k) \geq \mathbb{P}(\psi(t; \boldsymbol{w}, \boldsymbol{X_\mu}) \geq k)$, for all $k > 0$ and $t \in \mathbb{R}_+$, which proves the desired result. ■

Remark 3.3: According to Theorem 3.9, the condition $X_{\lambda_2} \geq_{\rm st} X_{\mu_2}$ means that the reliability of component with lifetime X_{λ_2} is higher than that of X_{μ_2}. Further, $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm st} \min\{X_{\mu_1}, X_{\mu_2}\}$ indicates that the series system with components having lifetimes X_{λ_1} and X_{λ_2} is better than the system with components having lifetimes X_{μ_1} and X_{μ_2}. These two conditions can be simplified as follows for the scale and PHR models.

(a) Scale model: In this case, we have $\bar{F}(t; \lambda) = \bar{F}(\lambda t)$, $t \in \mathbb{R}_+$. Thus $X_{\lambda_2} \geq_{\rm st} X_{\mu_2}$ is equivalent to saying that $\lambda_2 \leq \mu_2$. On the other hand, $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm st} \min\{X_{\mu_1}, X_{\mu_2}\}$ boils down to $\bar{F}(\lambda_1 t)\bar{F}(\lambda_2 t) \geq \bar{F}(\mu_1 t)\bar{F}(\mu_2 t)$, for all $t \in \mathbb{R}_+$. Furthermore, suppose that $\lambda_1 \geq \lambda_2$, $\mu_1 \geq \mu_2$, and $(\lambda_1, \lambda_2) \overset{\rm w}{\succeq} (\mu_1, \mu_2)$. Let $\phi_5(\lambda_1, \lambda_2) = \bar{F}(\lambda_1 t)\bar{F}(\lambda_2 t)$. We need to ensure that $\phi_5(\lambda_1, \lambda_2) \geq \phi_5(\mu_1, \mu_2)$. If the baseline hazard rate function h_F is decreasing, then

$$\begin{aligned}&\frac{\partial \phi_5(\lambda_1, \lambda_2)}{\partial \lambda_1} - \frac{\partial \phi_5(\lambda_1, \lambda_2)}{\partial \lambda_2} \\ &\quad = t\phi_5(\lambda_1, \lambda_2)(h_F(\lambda_2 t) - h_F(\lambda_1 t)) \geq 0,\end{aligned}$$

which implies the desired result by Lemma 2.1.

(b) PHR model: In this case, we have $\bar{F}(t; \lambda) = \bar{F}^\lambda(t)$, $t \in \mathbb{R}_+$. Then, the conditions $X_{\lambda_2} \geq_{\rm st} X_{\mu_2}$ and $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm st} \min\{X_{\mu_1}, X_{\mu_2}\}$ reduce to $\lambda_2 \leq \mu_2$ and $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, respectively.

To conclude, we present an ordering result for the hazard rate ordering in the next theorem. The proof is easy to be conducted by following the discussions of Theorem 3.9, and thus omitted for brevity.

Theorem 3.10: *Consider two weighted k-out-of-2 systems with common weights (w_1, w_2). Under Assumption 3.3, if $X_{\lambda_2} \geq_{\rm hr} X_{\mu_2}$ and $\min\{X_{\lambda_1}, X_{\lambda_2}\} \geq_{\rm hr} \min\{X_{\mu_1}, X_{\mu_2}\}$, then $T(k; \boldsymbol{w}, \boldsymbol{X_\lambda}) \geq_{\rm hr} T(k; \boldsymbol{w}, \boldsymbol{X_\mu})$, for all $0 < k \leq w_1 + w_2$.*

4. Conclusion

We have studied stochastic comparisons on total capacity of two weighted k-out-of-n systems in the sense of the expectation ordering, the increasing convex [concave] ordering, and the usual stochastic ordering. Some useful majorisation-type orders are employed to establish sufficient conditions for four kinds of stochastic orders. Some examples are also presented to illustrate the conditions and assumptions needed in the results.

As a future work, more studies are needed on the generalisations of Theorems 3.7, 3.8 and 3.9 to the case of weighted k-out-of-n systems with more than three dependent components. Besides, it is of great interest to extend the current studies to the setting of randomly weighted k-out-of-n systems (cf. Eryilmaz, 2013; Zhang et al., 2018). We are currently working on these problems and hope to report some valuable findings in a future paper.

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Notes on contributor

Yiying Zhang is an Assistant Professor at School of Statistics and Data Science, Nankai University. He obtained PhD degree with a focus on actuarial science in Sep. 2018 at Department of Statistics and Actuarial Science, The University of Hong Kong. His research interests contain several hot topics in reliability theory and actuarial science, including reliability analysis, redundancies allocation, systemic risks, credibility theory, optimal reinsurance design, and dependent risk models.

ORCID

Yiying Zhang http://orcid.org/0000-0001-9092-0000

References

Boland, P. J., & Proschan, F. (1988). Multivariate arrangement increasing functions with applications in probability and statistics. *Journal of Multivariate Analysis*, *25*(2), 286–298. https://doi.org/10.1016/0047-259X(88)90052-8

Cai, J., & Wei, W. (2014). Some new notions of dependence with applications in optimal allocation problems. *Insurance: Mathematics and Economics*, *55*, 200–209. https://doi.org/10.1016/j.insmatheco.2014.01.009

Cai, J., & Wei, W. (2015). Notions of multivariate dependence and their applications in optimal portfolio selections with dependent risks. *Journal of Multivariate Analysis*, *138*, 156–169. https://doi.org/10.1016/j.jmva.2014.12.011

Ding, W., Yang, J., & Ling, X. (2017). On the skewness of extreme order statistics from heterogeneous samples. *Communications in Statistics-Theory and Methods*, *46*(5), 2315–2331. https://doi.org/10.1080/03610926.2015.1041984

Eryilmaz, S. (2013). On reliability analysis of a k-out-of-n system with components having random weights. *Reliability Engineering & System Safety*, *109*, 41–44. https://doi.org/10.1016/j.ress.2012.07.010

Eryilmaz, S. (2015). Capacity loss and residual capacity in weighted k-out-of-n: G systems. *Reliability Engineering & System Safety*, *136*, 140–144. https://doi.org/10.1016/j.ress.2014.12.008

Eryilmaz, S., & Kan, C. (2020). Reliability based modeling and analysis for a wind power system integrated by two wind farms considering wind speed dependence. *Reliability Engineering & System Safety*, *203*, 107077. https://doi.org/10.1016/j.ress.2020.107077

Eryilmaz, S., & Sarikaya, K. (2014). Modeling and analysis of weighted-k-out-of-n: G system consisting of two different types of components. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, *228*(3), 265–271. https://doi.org/10.1177/1748006X13515647

Hollander, M., Proschan, F., & Sethuraman, J. (1977). Functions decreasing in transposition and their applications in ranking problems. *The Annals of Statistics*, *5*(4), 722–733. https://doi.org/10.1214/aos/1176343895

Li, X., You, Y., & Fang, R. (2016). On weighted k-out-of-n systems with statistically dependent component lifetimes. *Probability in the Engineering and Informational Sciences*, *30*(4), 533–546. https://doi.org/10.1017/S0269964816000231

Marshall, A. W., Olkin, I., & Arnold, B. C. (2011). *Inequalities: theory of majorization and its applications*. Springer.

Meshkat, R. S., & Mahmoudi, E. (2017). Joint reliability and weighted importance measures of a k-out-of-n system with random weights for components. *Journal of Computational and Applied Mathematics*, *326*, 273–283. https://doi.org/10.1016/j.cam.2017.05.042

Nelsen, R. B. (2007). *An introduction to copulas*. Springer Science & Business Media.

Rahmani, R.-A., Izadi, M., & Khaledi, B.-E. (2016a). Importance of components in k-out-of-n system with components having random weights. *Journal of Computational and Applied Mathematics*, *296*, 1–9. https://doi.org/10.1016/j.cam.2015.09.009

Rahmani, R.-A., Izadi, M., & Khaledi, B.-E. (2016b). Stochastic comparisons of total capacity of weighted-k-out-of-n systems. *Statistics & Probability Letters*, *117*, 216–220. https://doi.org/10.1016/j.spl.2016.06.005

Samaniego, F. J., & Shaked, M. (2008). Systems with weighted components. *Statistics & Probability Letters*, *78*(6), 815–823. https://doi.org/10.1016/j.spl.2007.09.049

Shaked, M., & Shanthikumar, J. G. (2007). *Stochastic orders*. Springer Science & Business Media.

Wu, J.-S., & Chen, R.-J. (1994). An algorithm for computing the reliability of weighted-k-out-of-n systems. *IEEE Transactions on Reliability*, *43*(2), 327–328. https://doi.org/10.1109/24.295016

Zhang, Y. (2018). Optimal allocation of active redundancies in weighted k-out-of-n systems. *Statistics & Probability Letters*, *135*, 110–117. https://doi.org/10.1016/j.spl.2017.12.002

Zhang, Y. (2021). Reliability analysis of randomly weighted k-out-of-n systems with heterogeneous components. *Reliability Engineering & System Safety*, *205*, 107184. https://doi.org/10.1016/j.ress.2020.107184

Zhang, Y., Ding, W., & Zhao, P. (2018). On total capacity of k-out-of-n systems with random weights. *Naval Research Logistics*, *65*(4), 347–359. https://doi.org/10.1002/nav.v65.4

Zhang, Y., Cai, X., Zhao, P., & Wang, H. (2018). Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components. *Statistics*, *53*(1), 126–147. https://doi.org/10.1080/02331888.2018.1546705