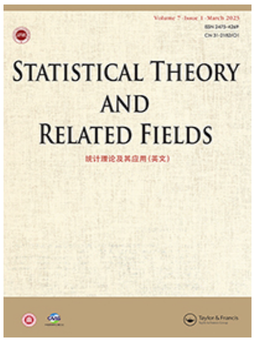

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Rates of convergence of powered order statistics from general error distribution

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ABSTRACT
Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with common general error distribution GED(v) with shape parameter $v > 0$, and let $M_{n,r}$ denote the r-th largest order statistics of $X_1, X_2, \ldots, X_n$. With different normalizing constants the distributional expansions and the uniform convergence rates of normalized powered order statistics $|M_{n,r}|^p$ are established. An alternative method is presented to estimate the probability of the r-th extremes. Numerical analyses are provided to support the main results.

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Distributional expansion; uniform convergence rate; general error distribution; powered order statistic

1. Introduction

The general error distribution is an extension of the normal distribution. The density of general error distribution, say $g_v(x)$, is given by

$$g_v(x) = \frac{v \exp(-\frac{1}{2}|\frac{x}{\lambda}|^v)}{\lambda 2^{1+\frac{1}{v}}\Gamma(\frac{1}{v})}, \quad x \in \mathbb{R}, \tag{1}$$

where $v > 0$ is the shape parameter, and $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$ with $\Gamma(\cdot)$ denoting the Gamma function. It is known that $g_2(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$ is the standard normal density and $g_1(x) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\sqrt{2}|x|}$ is the density of Laplace distribution.

For the general error distribution with parameter v (denoted by GED(v)), Nelson (1991) used it in volatility models since GED(v) can capture the heavy tailedness of high frequency financial time series. Peng et al. (2009) considered the tail behaviour of GED(v). Peng et al. (2010) established the uniform convergence rates of normalized extremes under optimal normalizing constants. Jia and Li (2014) established the higher-order distributional expansions of normalized maximum. Aforementioned studies show that the optimal convergence rate is proportional to $1/\log n$, similar to the results of Hall (1979) and Nair (1981) on normal extremes. In order to improve the convergence rate of normal extremes, Hall (1980) studied the asymptotics of $|M_{n,r}|^p$, the powered r-th largest order statistics, and showed that the distribution of normalized $M_{n,r}^2$ converges to its limit at the rate of $1/(\log n)^2$ under optimal normalizing constants, while the convergence rates are still the order of $1/\log n$ in the case of $p \neq 2$. For more details, see Hall (1980). For more work on higher-order expansions of powered-extremes from normal samples, see Li and Peng (2018) and Zhou and Ling (2016). For other related work on distributional expansions of extremes, see Liao and Peng (2012) for lognormal distribution, Liao et al. (2014a, 2014b) for logarithmic general error distribution and skew-normal distribution, and Hashorva et al. (2016), Liao and Peng (2014, 2015), Liao et al. (2016) and Lu and Peng (2017) for bivariate Hüsler-Reiss models.

In extreme value theory and its applications, it is important to know the convergence rate of distribution of normalized maximum to its ultimate distribution, cf. Cao and Zhang (2021), Hall (1979, 1980), Leadbetter et al. (1983) and Nair (1981). For more advanced topics related to extreme value theory and its applications, see Zhang (2021) for an excellent review. Motivation of this paper is to consider the higher-order expansions and the uniform

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convergence rate of powered order statistics from GED(v) sample, and an alternative method to estimate the probability related to powered order statistics, which are the complete extensions the results given by Hall (1980) for GED(2) case. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common distribution function G_v following GED(v) with density given by (1). For positive integer r, let $M_{n,r}$ denote the r-th largest order statistics of $X_1, X_2, \ldots, X_n$ and $M_n = M_{n,1} = \max_{1\le k\le n}\{X_k\}$ for later use. It is known that the distributional convergence rate of normalized maximum may depend heavily on the normalizing constants, see Hall (1979, 1980), Leadbetter et al. (1983), Nair (1981) and Resnick (1987) for normal samples. For the GED(v) distribution and positive power index p, it is necessary to discuss how to find the normalizing constants $c_n > 0$ and d_n such that

$$\lim_{n\to\infty}\Big(\mathrm{P}\{|M_{n,r}|^p \le c_n x + d_n\} - \Lambda_r(x)\Big) = 0, \tag{2}$$

where $\Lambda_r(x) = \Lambda(x)\sum_{j=0}^{r-1} \mathrm{e}^{-jx}/j!$ with $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, and further work on distributional expansions and uniform convergence rates of $|M_{n,r}|^p$ with different normalizing constants.

For $v > 0$ with normalizing constants α_n and β_n given by

$$\alpha_n = \frac{2^{1/v}\lambda}{v(\log n)^{1-1/v}}, \quad \beta_n = 2^{1/v}\lambda\left(\log n\right)^{1/v} - \frac{2^{1/v}\lambda\left[((v-1)/v)\log\log n + \log\{2\Gamma(1/v)\}\right]}{v\left(\log n\right)^{1-1/v}}, \tag{3}$$

Peng et al. (2009) showed that

$$\frac{M_n - \beta_n}{\alpha_n} \stackrel{d}{\to} M, \tag{4}$$

where M follows the Gumbel extreme value distribution $\Lambda(x)$. With normalizing constants α_n^* and β_n^* given by

$$\alpha_n^* = p\alpha_n\beta_n^{p-1}, \quad \beta_n^* = \beta_n^p, \tag{5}$$

we will show that (2) holds by replacing c_n and d_n by α_n^* and β_n^*, respectively, and investigate further its higher-order expansions and the uniform convergence rates. Similarly, with $p \ne v$ the optimal convergence rates of $|M_{n,r}|^p$ are derived under the following normalizing constants

$$c_n = 2pv^{-1}\lambda^v b_n^{p-v}, \quad d_n = b_n^p, \tag{6}$$

where constant b_n is the solution of the equation

$$2^{1/v}\lambda^{1-v}\Gamma(1/v)b_n^{v-1}\exp\left(\frac{b_n^v}{2\lambda^v}\right) = n. \tag{7}$$

Note that for the normal case, it follows from (6) that $c_n = pb_n^{p-2}$ and $d_n = b_n^p$ since $v = 2$ and $\lambda = 1$, which are just the normalizing constants given by Hall (1980).

For the normal case, Hall (1980) showed that the optimal convergence rate of $M_{n,r}^2$ is the order of $1/(\log n)^2$ if we choose the normalizing constants $c_n^* = 2(1 - b_n^{-2})$ and $d_n^* = b_n^2 - 2b_n^{-2}$. For the powered r-th largest order statistics $|M_{n,r}|^p$ from the GED(v) sample, it follows from (6) that the convergence rate can be improved if $p = v$. By Equation (3.1) of Lemma 1 in Jia and Li (2014), for $v \in (0,1)\cup(1,+\infty)$ and large x we have

$$\begin{aligned}1 - G_v(x) = \frac{2\lambda^v}{v}\Big\{&1 + 2(v^{-1}-1)\lambda^v x^{-v} + 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}x^{-2v}\\ &+ 8(v^{-1}-1)(v^{-1}-2)(v^{-1}-3)\lambda^{3v}x^{-3v} + O(x^{-4v})\Big\}x^{1-v}g_v(x).\end{aligned} \tag{8}$$

Similar to Hall (1980), as $p = v$ we choose the optimal normalizing constants c_n^* and d_n^* as follows.

$$\begin{cases} d_n^* = b_n^v + 4(v^{-1}-1)\lambda^{2v}b_n^{-v}, \\ c_n^* = f(b_n^v) = 2\lambda^v + 4(v^{-1}-1)\lambda^{2v}b_n^{-v}, \end{cases} \tag{9}$$

where b_n is given by (7). Note that if $v = 2$ and $\lambda = 1$, for the normal case, $c_n^* = 2 - 2b_n^{-2}$ and $d_n^* = b_n^2 - 2b_n^{-2}$ are just the normalizing constants given by Hall (1980).

The rest of this paper is organized as follows. Section 2 provides the main results and Section 3 presents some numerical analyses. Auxiliary lemmas are deferred to Section 4. Section 5 gives the proofs of the main results.

2. Main results

In this section, we provide the higher-order distributional expansions and the uniform convergence rates of powered extremes under different normalizing constants. Furthermore, we showed a method to estimate the probabilities of the extremes. Throughout this paper, let $\Lambda_r(x) = \Lambda(x)\sum_{j=0}^{r-1} \mathrm{e}^{-jx}/j!$ for positive integer r and $\Lambda_r(x) = 0$ for $r \le 0$. Recall that the power index p is positive.

Theorem 2.1: *Let $\{X_n : n \ge 1\}$ be a sequence of independent random variables with common distribution $G_v(x)$, $v > 0$, and $M_{n,r}$ denotes the r-th maximal term of $\{X_1, X_2, \ldots, X_n\}$. Then,*

(i) *if $v = 1$ and $p = 1$, with normalizing constants α_n^* and β_n^* given by $\alpha_n^* = 2^{-\frac{1}{2}}$ and $\beta_n^* = 2^{-\frac{1}{2}}\log\frac{n}{2}$, we have*

$$\lim_{n\to\infty} n\left\{n\left[\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x)\right] - \frac{\mathrm{e}^{-(r+1)x}\left[(r-1)\mathrm{e}^x - 1\right]}{2(r-1)!}\Lambda(x)\right\}$$
$$= \frac{\mathrm{e}^{-(r+2)x}}{24(r-1)!}\left[(-3r^3 + 10r^2 - 9r + 2)\mathrm{e}^{2x} + (9r^2 - 11r + 2)\mathrm{e}^x + 3\mathrm{e}^{-x} - 9r + 1\right]\Lambda(x);$$

(ii) *if $v = 1$ and $p \neq 1$, with normalizing constants α_n^* and β_n^* given by $\alpha_n^* = p2^{-\frac{p}{2}}(\log\frac{n}{2})^{p-1}$ and $\beta_n^* = (2^{-\frac{1}{2}}\log\frac{n}{2})^p$, we have*

$$\lim_{n\to\infty}(\log n)\left\{\left(\log\frac{n}{2}\right)\left[\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x)\right] - \frac{(1-p)x^2\mathrm{e}^{-rx}}{2(r-1)!}\Lambda(x)\right\}$$
$$= \frac{(1-p)x^3\mathrm{e}^{-rx}}{24(r-1)!}\left[4(1-2p) - 3(1-p)rx + 3(1-p)x\mathrm{e}^{-x}\right]\Lambda(x);$$

(iii) *if $v \in (0,1) \cup (1,+\infty)$, with normalizing constants α_n^* and β_n^* given by (5), we have*

$$\lim_{n\to\infty}(\log\log n)\left\{\frac{\log n}{(\log\log n)^2}\left[\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x)\right] - \frac{(1-v^{-1})^3\mathrm{e}^{-rx}}{2(r-1)!}\Lambda(x)\right\}$$
$$= -(1-v^{-1})^2\left(1 + x - \log 2\Gamma(1/v)\right)\frac{\mathrm{e}^{-rx}}{(r-1)!}\Lambda(x);$$

(iv) *if $v \in (0,1) \cup (1,+\infty)$ and $p \neq v$, with normalizing constants c_n and d_n given by (6), we have*

$$\lim_{n\to\infty} b_n^v\left\{b_n^v\left[\mathrm{P}\left(|M_{n,r}|^p \le c_n x + d_n\right) - \Lambda_r(x)\right] - \Lambda(x)h_v(x)\frac{\mathrm{e}^{-(r-1)x}}{(r-1)!}\right\}$$
$$= \left[q_v(x) + \left(1 - (r-1)\mathrm{e}^x\right)\frac{h_v^2(x)}{2}\right]\frac{\mathrm{e}^{-(r-1)x}}{(r-1)!}\Lambda(x),$$

where

$$h_v(x) = [v^{-1}(v-p)\lambda^v x^2 - 2v^{-1}(1-v)\lambda^v x - 2(v^{-1}-1)\lambda^v]\mathrm{e}^{-x}, \tag{10}$$

and

$$q_v(x) = \left[-\frac{1}{2}\lambda^{2v}v^{-2}(v-p)^2x^4 + v^{-2}(v-p)\lambda^{2v}\left(2 - \frac{4}{3}v - \frac{4}{3}p\right)x^3 - 2v^{-2}(1-v)\lambda^{2v}x^2 - 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}x - 4(v^{-1}-1)(v^{-1}-1)\lambda^{2v}\right]\mathrm{e}^{-x}; \tag{11}$$

(v) *if* $v \in (0,1) \cup (1,+\infty)$ *and* $p = v$, *with normalizing constants* c_n^* *and* d_n^* *given by* (9), *we have*

$$\lim_{n\to\infty} b_n^v \left\{ b_n^{2v} \Big[\mathrm{P}\left(|M_{n,r}|^v \le c_n^* x + d_n^*\right) - \Lambda_r(x) \Big] - \Lambda(x) S_v(x) \frac{\mathrm{e}^{-(r-1)x}}{(r-1)!} \right\} = \Lambda(x) B(x) \frac{\mathrm{e}^{-(r-1)x}}{(r-1)!},$$

where

$$S_v(x) = 2(v^{-1}-1)\lambda^{2v}\big[x^2 - 2(v^{-1}-2)x - (3v^{-1}-5)\big]\mathrm{e}^{-x}, \tag{12}$$

and

$$\begin{aligned} B(x) = -\frac{4}{3}(v^{-1}-1)\lambda^{3v}\Big[&(4-v^{-1})(v^{-1}-1)x^3 - 6(v^{-1}-2)x^2 \\ &- 6(3v^{-1}-5)x + (2v^{-2}-22v^{-1}+32)\Big]\mathrm{e}^{-x}. \end{aligned} \tag{13}$$

Remark 2.1: Theorem 2.1 (i)–(ii) show the difference of the convergence rates for the powered-extremes of the Laplace distribution as $p = 1$ and $p \neq 1$, respectively. Meanwhile, it follows from (3) and (5)–(7) that $c_n = \alpha_n^*$ and $d_n = \beta_n^*$ as $v = 1$ since $\lambda = 2^{-3/2}$, so it is not necessary to consider the case of $v = 1$ in Theorem 2.1 (iv)–(v).

Remark 2.2: For $p \neq v$, with normalizing constants c_n and d_n given by (6), Theorem 2.1 (iv) shows that the convergence rate of $\mathrm{P}(|M_n|^p \le c_n x + d_n)$ to the extreme value distribution $\Lambda(x)$ is proportional to $1/\log n$ since $b_n^v \sim 2\lambda^v \log n$ by (7), while it can be improved to the order of $1/(\log n)^2$ with optimal choice of normalizing constants c_n^* and d_n^* given by (9) as $p = 2$, which coincides with the normal case studied by Hall (1980).

Theorem 2.2: *Let* $\{X_n : n \ge 1\}$ *be a sequence of independent random variables with common distribution* $G_v(x)$, $v > 0$, *and* $M_{n,r}$ *denotes the r-th maximal term of* $\{X_1, X_2, \ldots, X_n\}$. *The following results hold.*

(i) *If* $v = 1$ *and* $p = 1$, *with normalizing constants* α_n^* *and* β_n^* *given by* $\alpha_n^* = 2^{-\frac{1}{2}}$ *and* $\beta_n^* = 2^{-\frac{1}{2}} \log \frac{n}{2}$, *then* $\sup_{x\in\mathbb{R}} |\mathrm{P}(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*) - \Lambda_r(x)|$ *is the order of* $1/n$.
(ii) *If* $v = 1$ *and* $p \neq 1$, *with normalizing constants* α_n^* *and* β_n^* *given by* $\alpha_n^* = p2^{-\frac{p}{2}}(\log \frac{n}{2})^{p-1}$ *and* $\beta_n^* = (2^{-\frac{1}{2}} \log \frac{n}{2})^p$, *then* $\sup_{x\in\mathbb{R}} |\mathrm{P}(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*) - \Lambda_r(x)|$ *is the order of* $1/\log n$.
(iii) *If* $v \in (0,1) \cup (1,+\infty)$ *and* $p > 0$, *with normalizing constants* α_n^* *and* β_n^* *given by* (5), *then* $\sup_{x\in\mathbb{R}} |\mathrm{P}(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*) - \Lambda_r(x)|$ *is the order of* $(\log\log n)^2/\log n$.
(iv) *If* $v \in (0,1) \cup (1,+\infty)$ *and* $p \neq v$, *with normalizing constants* c_n *and* d_n *given by* (6), *then* $\sup_{x\in\mathbb{R}} |\mathrm{P}(|M_{n,r}|^p \le c_n x + d_n) - \Lambda_r(x)|$ *is the order of* $1/\log n$.
(v) *If* $v \in (0,1) \cup (1,+\infty)$ *and* $p = v$, *with normalizing constants* c_n^* *and* d_n^* *given by* (9), *then* $\sup_{x\in\mathbb{R}} |\mathrm{P}(|M_{n,r}|^p \le c_n^* x + d_n^*) - \Lambda_r(x)|$ *is the order of* $1/(\log n)^2$.

Theorem 2.3: *Let* $b_n > 0$ *be defined by* (7) *and*

$$\lambda_n(x) = \frac{\lambda^{v-1}}{2^{\frac{1}{v}}\Gamma\left(\frac{1}{v}\right)} n x^{1-v} \exp\left\{-\frac{x^v}{2\lambda^v}\right\}.$$

If $v > 1$ *and* $x \ge b_n$, *then*

$$\begin{aligned} Q_{1n}(x) &= \exp\left\{-\lambda_n(x)\left(1 - \frac{2(v-1)}{v}\lambda^v x^{-v} + \frac{4(v-1)(2v-1)}{v^2}\lambda^{2v}x^{-2v} + \frac{\lambda_n(x)}{2(n-1)}\right)\right\} \\ &< \mathrm{P}(M_n \le x) < Q_{2n}(x) = \exp\left\{-\lambda_n(x)\left(1 - \frac{2(v-1)}{v}\lambda^v x^{-v}\right)\right\}. \end{aligned} \tag{14}$$

If $0 < v < 1$, $x > \lambda[\frac{2(1-v)}{v}]^{\frac{1}{v}}$ *and* $x \ge b_n$, *then*

$$Q_{1n}(x) = \exp\left\{-z - \frac{z^2}{2n}\left(1 - \frac{z}{n}\right)^{-1}\right\} < \mathrm{P}(M_n \le x) < Q_{2n}(x) = \exp\big\{-\lambda_n(x)\big\}, \tag{15}$$

where $z = \lambda_n(x)\left(1 + \frac{2(v-1)}{v}\lambda^v x^{-v}\right)^{-1}$.

Similarly, for the bounds of the r-th order statistics with $r > 1$, we have

$$Q_{1n,r}(x) \le \mathrm{P}\left(M_{n,r} \le x\right) < Q_{2n,r}(x), \tag{16}$$

where

$$\begin{cases} Q_{1n,r} = \sum_{j=0}^{r-1} \frac{\lambda_n^j(x)\left(1+\frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)^{-j}}{j!} \exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x) - \frac{j(j-1)}{n}\right\}, \\ Q_{2n,r} = \sum_{j=0}^{r-1} \frac{\lambda_n^j(x)}{j!} \exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x)\left(1-\frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)\right\}, \end{cases} \tag{17}$$

for $\nu > 1$, and for $\nu \in (0,1)$,

$$\begin{cases} Q_{1n,r}(x) = \sum_{j=0}^{r-1} \frac{\lambda_n^j(x)}{j!} \exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x)\left(1+\frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)^{-1} - \frac{j(j-1)}{n}\right\}, \\ Q_{2n,r}(x) = \sum_{j=0}^{r-1} \frac{\lambda_n^j(x)\left(1+\frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)^{-j}}{j!} \exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x)\right\}. \end{cases} \tag{18}$$

3. Numerical analyses

In this section, small numerical analyses are provided to support the main results. We compare the actual values of probability of powered order statistics with its higher order expansions provided by Theorem 2.1, and with two bounds based on Theorem 2.3.

First we compare the actual values of $\mathrm{P}(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*)$ with the following asymptotics.

(i) If $\nu = 1$ and $p = 1$, the second-order and the third-order asymptotics are respectively given by $\Lambda_r(x) + \frac{e^{-(r+1)x}[(r-1)e^x-1]}{2n(r-1)!}\Lambda(x)$ and $\Lambda_r(x) + \frac{e^{-(r+1)x}[(r-1)e^x-1]}{2n(r-1)!}\Lambda(x) + \frac{e^{-(r+2)x}}{24n^2(r-1)!}\Lambda(x)[(-3r^3+10r^2-9r+2)e^{2x} + (9r^2-11r+2)e^x + 3e^{-x} - 9r + 1]$.

(ii) If $\nu = 1$ and $p \ne 1$, the second-order and the third-order asymptotics are respectively given by $\Lambda_r(x) + \frac{(1-p)x^2e^{-rx}}{2\log\frac{n}{2}(r-1)!}\Lambda(x)$ and $\Lambda_r(x) + \frac{(1-p)x^2e^{-rx}}{2\log\frac{n}{2}(r-1)!}\Lambda(x) + \frac{(1-p)x^3e^{-rx}}{24(r-1)!\log n\log\frac{n}{2}}\Lambda(x)[4(1-2p) - 3(1-p)rx + 3(1-p)xe^{-x}]$.

(iii) If $\nu \in (0,1)\cup(1,+\infty)$, the second-order and the third-order asymptotics are respectively given by $\Lambda_r(x) + \frac{(\log\log n)^2(1-\nu^{-1})^3e^{-rx}}{2(r-1)!\log n}\Lambda(x)$ and $\Lambda_r(x) + \frac{(\log\log n)^2(1-\nu^{-1})^3e^{-rx}}{2(r-1)!\log n}\Lambda(x) - \frac{(\log\log n)}{\log n}(1-\nu^{-1})^2(1+x-\log 2\Gamma(\frac{1}{\nu})) \times \frac{e^{-rx}}{(r-1)!}\Lambda(x)$.

(iv) If $\nu \in (0,1)\cup(1,+\infty)$ and $p \ne \nu$, the second-order and the third-order asymptotics are respectively given by $\Lambda_r(x) + \Lambda(x)h_\nu(x)\frac{e^{-(r-1)x}}{b_n^\nu(r-1)!}$ and $\Lambda_r(x) + \Lambda(x)h_\nu(x)\frac{e^{-(r-1)x}}{b_n^\nu(r-1)!} + \frac{e^{-(r-1)x}}{b_n^{2\nu}(r-1)!}\Lambda(x)[q_\nu(x) + (1-(r-1)e^x)\frac{h_\nu^2(x)}{2}]$.

(v) If $\nu \in (0,1)\cup(1,+\infty)$ and $p = \nu$, the second-order and the third-order asymptotics are respectively given by $\Lambda_r(x) + \Lambda(x)S_\nu(x)\frac{e^{-(r-1)x}}{b_n^{2\nu}(r-1)!}$ and $\Lambda_r(x) + \Lambda(x)S_\nu(x)\frac{e^{-(r-1)x}}{b_n^{2\nu}(r-1)!} + \Lambda(x)B(x)\frac{e^{-(r-1)x}}{b_n^{3\nu}(r-1)!}$.

For different parameters ν and p, with sample size $n = 1\,000$, Figure 1 shows the relationship between the actual values and the three asymptotics mentioned above with parameter $r = 1$ and given interval $x \in [-5, 10]$, which supports our findings. For the case of $r > 1$, it is difficult to calculate the actual value directly, so we estimate the actual values by calculating the empirical distribution function. For given $x \in [-6, 6]$, Figure 2 shows the difference between the estimated actual values with the three asymptotics as $r = 2$, which may be acceptable.

To end this section, we compare the two bounds given by Theorem 2.3 with the actural values of extremes.

For M_n, calculate $Q_{1n}(x)$, $Q_{2n}(x)$, $\mathrm{P}(M_n \le x)$ and relative error $e_n(x) = [Q_{2n}(x) - Q_{1n}(x)]/[1 - Q_{2n}(x)]$, where $Q_{1n}(x)$ and $Q_{2n}(x)$ are given by (14) and (15), respectively. Tables 1–4 show the results. For the case of $0 < \nu < 1$, Table 1 shows that for given $\nu = 0.2, 0.5$, with increasing sample size n, $e_n(x)$ decreases; and for given n, with increasing ν, $e_n(x)$ decreases. For given n and ν, $e_n(x)$ is a decreasing function of x. Table 2 shows the fact that for given $x = 5$, $e_n(x)$ is a decreasing function of n and ν. For similar facts for the case of $\nu > 1$, see Tables 3 and 4.

For the r-th order statistics $M_{n,r}$, calculate $Q_{1n,r}(x)$, $Q_{2n,r}(x)$, $\mathrm{P}(M_{n,r} \le x)$ and $e_{n,r}(x) = [Q_{2n,r}(x) - Q_{1n,r}(x)]/[1 - Q_{2n,r}(x)]$ with $Q_{1n,r}(x)$ and $Q_{2n,r}(x)$ given by (17) for $\nu > 1$, and (18) for $0 < \nu < 1$. Table 5 shows that $e_{n,r}(x)$ is an increasing function of n and x for given $\nu = 0.5$ and $r = 2$, and it is irregular for the case of $\nu = 5$;

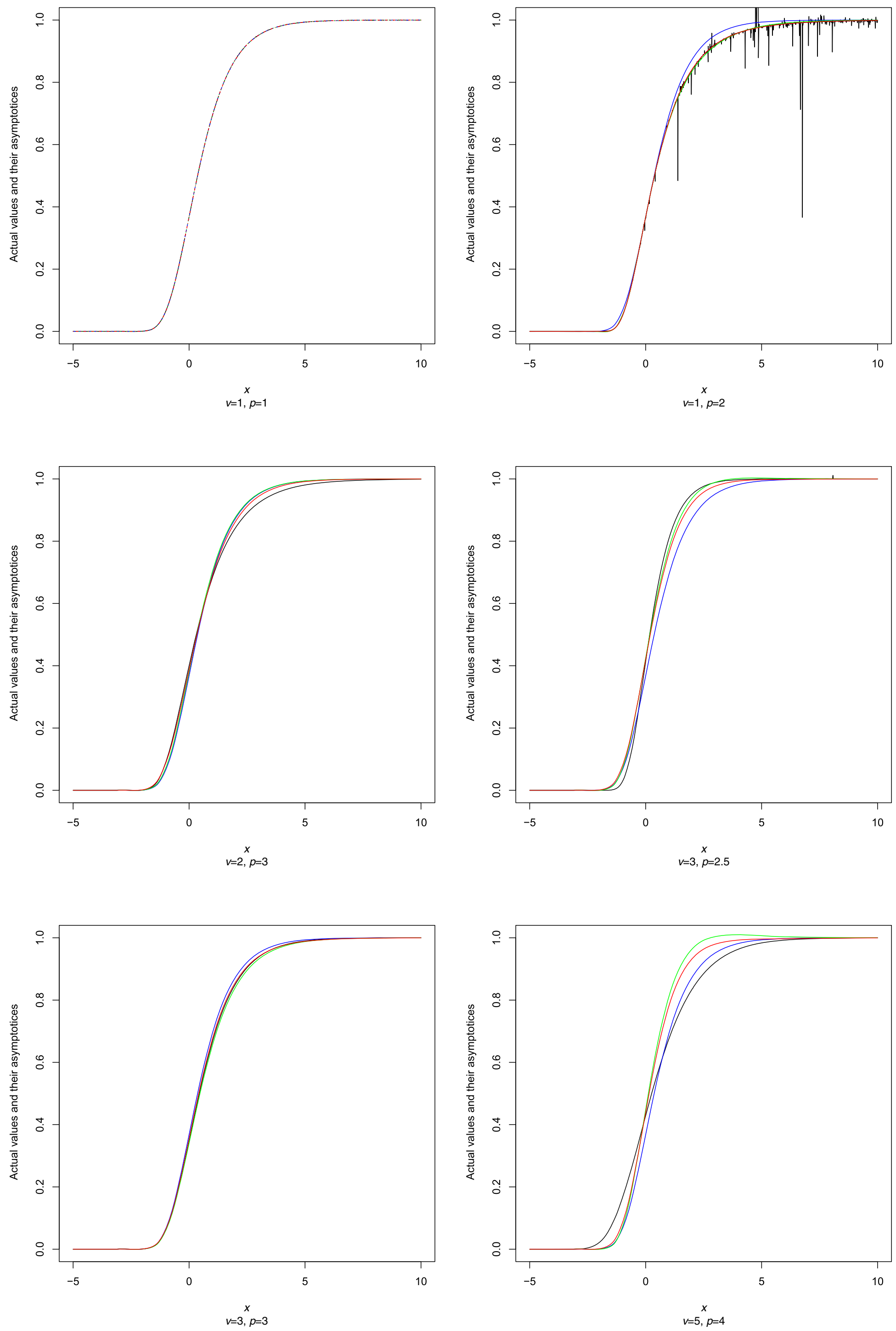

Figure 1. Actual values and its approximations with $n = 1000, r = 1, x \in [-5, 10]$. The actual values compared with the first-order asymptotics, the second-order asymptotics and the third-order asymptotics.

the two bounds are closer to the actual values. For given $x = 2.8$ and $n = 30$, Table 6 shows that $e_{n,r}(x)$ is an increasing function of r for given v. Tables 2–5 also show the fact that two relative accurate bounds control the range of the probability of the extremes, and provide an alternative to linear interpolation. It is an effective method to estimate the probability of the extremes.

4. Auxiliary lemmas

Let C be a positive constant with values varying from place to place. In order to prove the main results, we need some auxiliary lemmas.

Lemma 4.1: *Let $G_v(x)$ denote the* GED(v) *distribution function with parameter $v > 0$. We have the following results.*

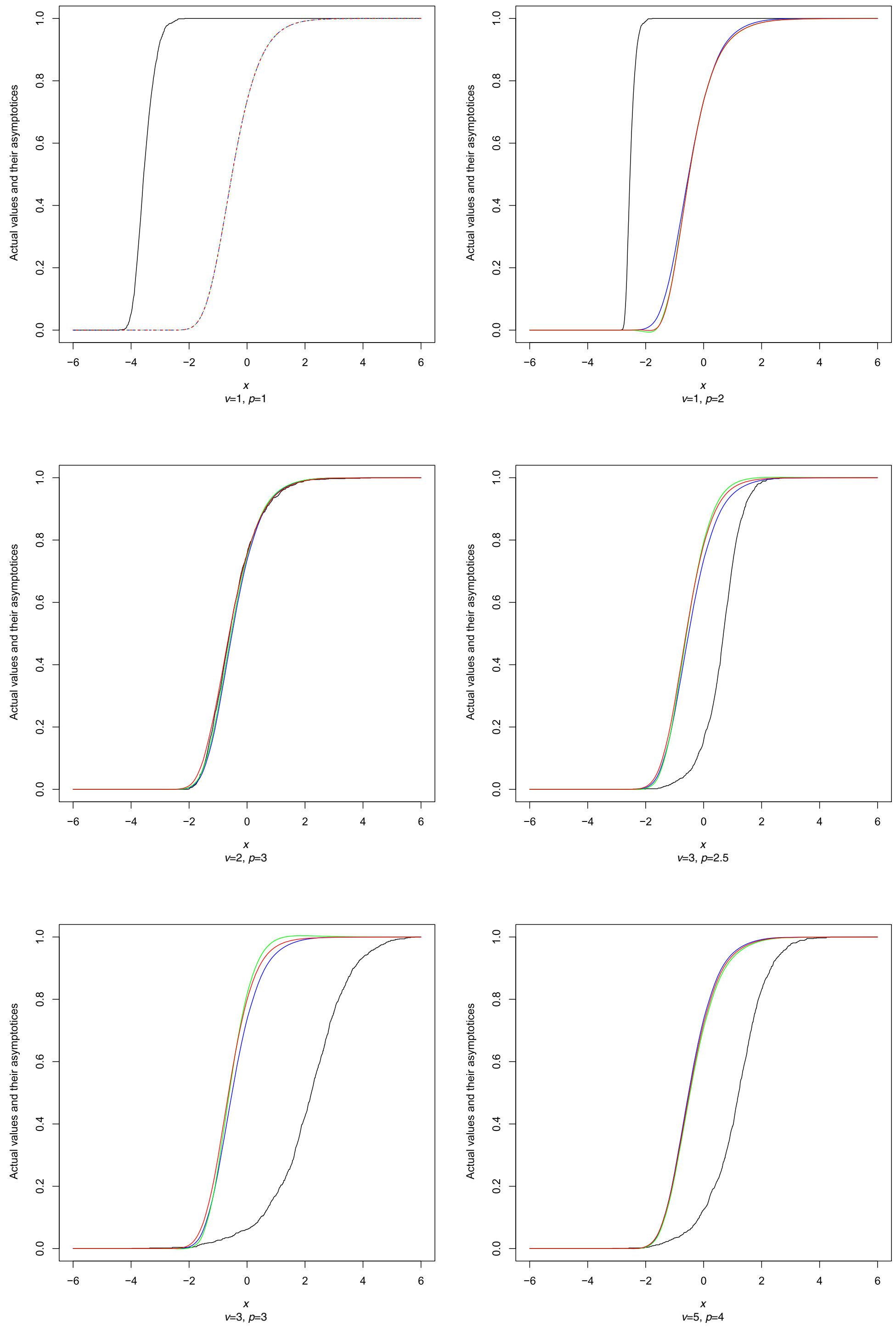

Figure 2. Actual values and its approximations with $r = 2$, $x \in [-6, 6]$. The actual values (estimated by empirical distribution) compared with the first-order asymptotics, the second-order asymptotics and the third-order asymptotics.

(i) *If $v = 1$ and $p = 1$, with normalizing constants α_n^* and β_n^* given by $\alpha_n^* = 2^{-\frac{1}{2}}$ and $\beta_n^* = 2^{-\frac{1}{2}} \log \frac{n}{2}$,*

$$1 - G_1\left(\alpha_n^* x + \beta_n^*\right) = n^{-1} e^{-x}. \tag{19}$$

(ii) *If $v = 1$ and $p \neq 1$, with normalizing constants α_n^* and β_n^* given by $\alpha_n^* = p2^{-\frac{p}{2}}(\log \frac{n}{2})^{p-1}$ and $\beta_n^* = (2^{-\frac{1}{2}} \log \frac{n}{2})^p$,*

$$1 - G_1\left((\alpha_n^* x + \beta_n^*)^{\frac{1}{p}}\right) = n^{-1} e^{-x} \left\{1 - \frac{(1-p)x^2}{2 \log \frac{n}{2}} + \frac{(1-p)[3(1-p)x - 4(1-2p)]x^3}{24(\log \frac{n}{2})^2} + o\left(\frac{1}{(\log \frac{n}{2})^2}\right)\right\}. \tag{20}$$

Table 1. Comparison of $Q_{1n}(x)$, $Q_{2n}(x)$ and $P(M_n \le x)$ along with n and x for given $v = 0.2$ and $v = 0.5$, respectively.

n	x	$Q_{1n}(x)$	$P(M_n \le x)$	$Q_{2n}(x)$	$e_n(x)$
			$v = 0.2$		
10	2.2	0.8983167	0.9043888	0.9358172	0.5842761
	2.4	0.9098966	0.9149719	0.9426480	0.5710592
	2.6	0.9196319	0.9240890	0.9484505	0.5590450
	2.8	0.9278988	0.9317987	0.9534248	0.5480593
	3.0	0.9349809	0.9383530	0.9577238	0.5379611
50	2.2	0.5849886	0.6050284	0.7177201	0.4702123
	2.4	0.6236778	0.6412669	0.7442996	0.4717309
	2.6	0.6577642	0.6738590	0.7674909	0.4719245
	2.8	0.6878648	0.7024419	0.7878294	0.4711521
	3.0	0.7145187	0.7274972	0.8057521	0.4696747
1000	2.2	0.0000220	0.0000432	0.0013155	0.0012952
	2.4	0.0000793	0.0001383	0.0027224	0.0026504
	2.6	0.0002298	0.0003727	0.0050288	0.0048232
	2.8	0.0005624	0.0008555	0.0084848	0.0079902
	3.0	0.0012030	0.0017244	0.0133054	0.0122656
			$v = 0.5$		
10	2.2	0.7944583	0.8021804	0.8343381	0.2407301
	2.4	0.8263512	0.8324911	0.8589261	0.2309067
	2.6	0.8527809	0.8576013	0.8795175	0.2219126
	2.8	0.8747526	0.8786060	0.8968032	0.2136752
	3.0	0.8930810	0.8961783	0.9113530	0.2061209
50	2.2	0.3164867	0.3321699	0.4043062	0.1474238
	2.4	0.3853216	0.3998509	0.4674973	0.1543197
	2.6	0.4510112	0.4639029	0.5262867	0.1589052
	2.8	0.5121842	0.5235652	0.5800771	0.1616794
	3.0	0.5681386	0.5780590	0.6286851	0.1630597
1000	2.2	0.0000000	0.0000000	0.0000000	0.0000000
	2.4	0.0000000	0.0000000	0.0000002	0.0000002
	2.6	0.0000001	0.0000002	0.0000027	0.0000025
	2.8	0.0000015	0.0000024	0.0000186	0.0000171
	3.0	0.0000123	0.0000174	0.0000930	0.0000808

Table 2. Comparison of $Q_{1n}(x)$, $Q_{2n}(x)$ and $P(M_n \le x)$ along with n and $0 < v < 1$ for given $x = 5$.

n	v	$Q_{1n}(x)$	$P(M_n \le x)$	$Q_{2n}(x)$	$e_n(x)$
10	0.1	0.9913561	0.9917302	0.9949114	0.6986797
	0.2	0.9718511	0.9730040	0.9808212	0.4677135
	0.4	0.9685581	0.9693446	0.9743350	0.2250904
	0.6	0.9802534	0.9804977	0.9821276	0.1048652
	0.9	0.9933760	0.9933885	0.9934844	0.0166275
50	0.1	0.9575212	0.9593293	0.9748146	0.6866442
	0.2	0.8669590	0.8721136	0.9077145	0.4416245
	0.4	0.8523703	0.8558368	0.8780950	0.2110231
	0.6	0.9050900	0.9062187	0.9137756	0.1007322
	0.9	0.9673159	0.9673766	0.9678435	0.0164072
1000	0.1	0.4197284	0.4358670	0.6004002	0.4521316
	0.2	0.0575402	0.0647830	0.1442068	0.1012705
	0.4	0.0409794	0.0444446	0.0742729	0.0359647
	0.6	0.1360928	0.1395276	0.1647368	0.0342934
	0.9	0.5144780	0.5151241	0.5201194	0.0117558

(iii) *If $v \in (0,1) \cup (1,+\infty)$, with normalizing constants α_n^* and β_n^* given by* (5),

$$1 - G_v\big((\alpha_n^* x + \beta_n^*)^{\frac{1}{p}}\big) = n^{-1}e^{-x}\left\{1 - \frac{(1-v^{-1})^3(\log\log n)^2}{2\log n} + \frac{(1-v^{-1})^2\left(1 + x - \log\left\{2\Gamma\left(\frac{1}{v}\right)\right\}\right)\log\log n}{\log n} + o\left(\frac{\log\log n}{\log n}\right)\right\}. \quad (21)$$

Table 3. Comparison of $Q_{1n}(x)$, $Q_{2n}(x)$ and $P(M_n \le x)$ along with n and x for given $v = 2$ and $v = 5$, respectively.

n	x	$Q_{1n}(x)$	$P(M_n \le x)$	$Q_{2n}(x)$	$e_n(x)$
			$v = 2$		
10	2.2	0.8606846	0.8693495	0.8799128	0.1601191
	2.4	0.9175650	0.9209835	0.9257873	0.1107936
	2.6	0.9530429	0.9543538	0.9564622	0.0785345
	2.8	0.9742495	0.9747405	0.9756380	0.0569938
	3.0	0.9864026	0.9865827	0.9869544	0.0422991
50	2.2	0.4725810	0.4965603	0.5274706	0.1161614
	2.4	0.6505338	0.6626118	0.6800738	0.0923336
	2.6	0.7863017	0.7916750	0.8004588	0.0709481
	2.8	0.8777258	0.8799238	0.8839822	0.0539265
	3.0	0.9338416	0.9346899	0.9364520	0.0410768
1000	2.2	0.0000003	0.0000008	0.0000028	0.0000025
	2.4	0.0001844	0.0002662	0.0004478	0.0002636
	2.6	0.0081639	0.0093527	0.0116622	0.0035395
	2.8	0.0736564	0.0774285	0.0848927	0.0122787
	3.0	0.2543747	0.2590304	0.2689741	0.0199710
			$v = 5$		
10	2.2	0.9821805	0.9834947	0.9851923	0.2033925
	2.4	0.9976378	0.9976947	0.9978047	0.0760575
	2.6	0.9998323	0.9998337	0.9998375	0.0320048
	2.8	0.9999948	0.9999948	0.9999948	0.0146745
	3.0	0.9999999	0.9999999	0.9999999	0.0071845
50	2.2	0.9140302	0.9201532	0.9281218	0.1960497
	2.4	0.9882446	0.9885267	0.9890717	0.0756840
	2.6	0.9991618	0.9991689	0.9991877	0.0319932
	2.8	0.9999739	0.9999741	0.9999742	0.0146743
	3.0	0.9999997	0.9999997	0.9999997	0.0071845
1000	2.2	0.1656638	0.1893227	0.2249572	0.0765034
	2.4	0.7893864	0.7939048	0.8027048	0.0675045
	2.6	0.9833679	0.9835085	0.9838796	0.0317388
	2.8	0.9994775	0.9994824	0.9994851	0.0146707
	3.0	0.9999942	0.9999942	0.9999943	0.0071845

Table 4. Comparison of $Q_{1n}(x)$, $Q_{2n}(x)$ and $P(M_n \le x)$ along with n and $v > 1$ for given $x = 3.5$.

n	v	$Q_{1n}(x)$	$P(M_n \le x)$	$Q_{2n}(x)$	$e_n(x)$
10	1.5	0.9890469	0.9891272	0.9892410	0.0180354
	2	0.9976627	0.9976761	0.9977128	0.0218939
	3	0.9999782	0.9999817	0.9999784	0.0124723
	3.3	0.9999970	0.9999982	0.9999971	0.0095311
50	1.5	0.9464242	0.9468053	0.9473501	0.0175860
	2	0.9883682	0.9884346	0.9886161	0.0217793
	3	0.9998909	0.9999086	0.9998922	0.0124717
	3.3	0.9999851	0.9999909	0.9999853	0.0095310
1000	1.5	0.3324471	0.3351304	0.3390082	0.0099262
	2	0.7913621	0.7924260	0.7953414	0.0194435
	3	0.9978200	0.9981739	0.9978469	0.0124587
	3.3	0.9997023	0.9998177	0.9997051	0.0095297

(iv) *If $v \in (0,1) \cup (1,+\infty)$ and $p \neq v$, with normalizing constants c_n and d_n given by* (6),

$$\begin{aligned}
&1 - G_v\left((c_n x + d_n)^{\frac{1}{p}}\right) \\
&= n^{-1}e^{-x}\left\{1 - \left[v^{-1}(v-p)\lambda^v x^2 - 2v^{-1}(1-v)\lambda^v x - 2(v^{-1}-1)\lambda^v\right]b_n^{-v}\right. \\
&+ \left[\frac{1}{2}\lambda^{2v}v^{-2}(v-p)^2x^4 - v^{-2}(v-p)\lambda^{2v}\left(2 - \frac{4}{3}v - \frac{4}{3}p\right)x^3 + 2v^{-2}(1-v)\lambda^{2v}x^2\right. \\
&\left.\left. + \ 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}x + 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}\right]b_n^{-2v} + o(b_n^{-2v})\right\}. \qquad (22)
\end{aligned}$$

Table 5. Comparison of $Q_{1n,r}(x)$, $Q_{2n,r}(x)$ and $P(M_{n,r} \le x)$ along with n and x for given $r = 2$, and $v = 0.5$ and $v = 5$, respectively.

n	x	$Q_{1n,r}(x)$	$P(M_{n,r} \le x)$	$Q_{2n,r}(x)$	$e_{n,r}(x)$
			$v = 0.5$		
10	2.2	0.9441538	0.9809615	1.0275749	−3.0252524
	2.4	0.9561505	0.9865216	1.0236780	−2.8519126
	2.6	0.9652290	0.9903596	1.0202631	−2.7159771
	2.8	0.9721745	0.9930530	1.0173107	−2.6074151
	3.0	0.9775434	0.9949541	1.0147799	−2.5193993
50	2.2	0.6178074	0.7023217	0.8725027	1.9976531
	2.4	0.6901408	0.7697601	0.9158538	2.6823909
	2.6	0.7498686	0.8229683	0.9473843	3.7539350
	2.8	0.7984150	0.8645627	0.9698151	5.6783455
	3.0	0.8374802	0.8966240	0.9854238	10.1496531
1000	2.2	0.0000000	0.0000000	0.0000003	0.0000003
	2.4	0.0000001	0.0000002	0.0000050	0.0000049
	2.6	0.0000019	0.0000035	0.0000452	0.0000433
	2.8	0.0000203	0.0000336	0.0002687	0.0002484
	3.0	0.0001359	0.0002086	0.0011489	0.0010141
			$v = 5$		
10	2.2	0.9957721	0.9998767	1.0050571	−1.8360237
	2.4	0.9996190	0.9999976	1.0004398	−1.8661697
	2.6	0.9999815	1.0000000	1.0000206	−1.9000942
	2.8	0.9999996	1.0000000	1.0000004	−1.9280712
	3.0	1.0000000	1.0000000	1.0000000	−1.9479278
50	2.2	0.9766794	0.9967876	1.0216924	−2.0750555
	2.4	0.9980460	0.9999352	1.0021319	−1.9165386
	2.6	0.9999072	0.9999997	1.0001025	−1.9056094
	2.8	0.9999980	1.0000000	1.0000022	−1.9283273
	3.0	1.0000000	1.0000000	1.0000000	−1.9479318
1000	2.2	0.3475245	0.5046753	0.6785480	1.0297758
	2.4	0.9416464	0.9771526	1.0146908	−4.9721211
	2.6	0.9980190	0.9998634	1.0018935	−2.0461942
	2.8	0.9999597	0.9999999	1.0000432	−1.9344294
	3.0	0.9999997	1.0000000	1.0000003	−1.9480282

(v) *If* $v \in (0, 1) \cup (1, +\infty)$ *and* $p = v$, *with normalizing constants* c_n^* *and* d_n^* *given by* (9),

$$\begin{aligned} 1 - G_v\left((c_n^* x + d_n^*)^{\frac{1}{p}}\right) &= n^{-1}e^{-x}\Bigg\{1 - 2(v^{-1}-1)\lambda^{2v}[x^2 - 2(v^{-1}-2)x - 3v^{-1} + 5]b_n^{-2v} \\ &+ (v^{-1}-1)\lambda^{3v}\left[\frac{4}{3}(v^{-1}-1)(4-v^{-1})x^3 - 8(v^{-1}-2)x^2 - 8(3v^{-1}-5)x\right]b_n^{-3v} \\ &+ \frac{8}{3}(v^{-1}-1)(v^{-2} - 11v^{-1} + 16)\lambda^{3v}b_n^{-3v} + o(b_n^{-3v})\Bigg\}. \end{aligned} \tag{23}$$

Proof: (i) If $v = 1$ and $p = 1$, by (1) the Laplace density $g_1(x)$ is given by

$$g_1(x) = \frac{1}{\sqrt{2}}\exp(-\sqrt{2}|x|). \tag{24}$$

By (24) and the values of α_n^* and β_n^* given by Theorem 2.1(i), we get

$$\begin{aligned} 1 - G_1\left(\alpha_n^* x + \beta_n^*\right) &= \frac{1}{\sqrt{2}}\int_{2^{-\frac{1}{2}}x + 2^{-\frac{1}{2}}\log\frac{n}{2}}^{\infty} \exp\left(-\sqrt{2}x\right)\mathrm{d}x \\ &= n^{-1}e^{-x}. \end{aligned}$$

(ii) If $v = 1$ and $p \neq 1$, note that

$$(\alpha_n^* x + \beta_n^*)^{\frac{1}{p}} = \frac{\log\frac{n}{2}}{\sqrt{2}}\left(1 + \frac{x}{\log\frac{n}{2}} + \frac{(1-p)x^2}{2(\log\frac{n}{2})^2} + \frac{(1-p)(1-2p)x^3}{6(\log\frac{n}{2})^3} + o\left(\frac{1}{(\log\frac{n}{2})^3}\right)\right). \tag{25}$$

The claimed result (20) follows from (24) and (25).

Table 6. Comparison of $Q_{1n,r}(x)$, $Q_{2n,r}(x)$ and $\mathrm{P}(M_{n,r} \le x)$ along with v and $r \ge 2$ for given $x = 2.8$ and $n = 30$.

v	r	$Q_{1n,r}(x)$	$\mathrm{P}(M_{n,r} \le x)$	$Q_{2n,r}(x)$	$e_{n,r}(x)$
0.1	2	0.9757843	0.9987123	1.0229409	−2.0555686
	3	0.9762135	0.9999790	1.0244196	−1.9740741
	4	0.9762174	0.9999998	1.0244468	−1.9728299
0.2	2	0.9148533	0.9810867	1.0614478	−2.3856747
	3	0.9226255	0.9987718	1.0833325	−1.9285032
	4	0.9229524	0.9999419	1.0849718	−1.9067431
0.5	2	0.8933923	0.9432802	1.0120160	−9.8721219
	3	0.9278140	0.9933405	1.0706215	−2.0221525
	4	0.9311391	0.9994266	1.0784966	−1.8772465
0.7	2	0.9250118	0.9487153	0.9825676	3.3016461
	3	0.9623665	0.9943041	1.0327901	−2.1477084
	4	0.9660732	0.9995364	1.0390663	−1.8684416
0.9	2	0.9556584	0.9607125	0.9701595	0.4859510
	3	0.9885106	0.9962277	1.0075575	−2.5202665
	4	0.9914847	0.9997351	1.0115326	−1.7383721
1.5	2	0.9759006	0.9889270	1.0028975	−9.3174327
	3	0.9857182	0.9994570	1.0155287	−1.9197014
	4	0.9861701	0.9999806	1.0162521	−1.8509573
2	2	0.9879828	0.9972920	1.0076344	−2.5740894
	3	0.9904276	0.9999357	1.0109907	−1.8709591
	4	0.9904814	0.9999989	1.0110858	−1.8586337
3	2	0.9984374	0.9999367	1.0016303	−1.9584774
	3	0.9984965	0.9999998	1.0017115	−1.8785071
	4	0.9984967	1.0000000	1.0017118	−1.8782153
5	2	0.9999988	1.0000000	1.0000013	−1.9281993
	3	0.9999988	1.0000000	1.0000013	−1.9280120
	4	0.9999988	1.0000000	1.0000013	−1.9280120

(iii) If $v \in (0,1) \cup (1,+\infty)$, with normalizing constants α_n^* and β_n^* given by (5), $z_{n,p}(x) = (\alpha_n^* x + \beta_n^*)^{\frac{1}{p}}$ and $z_{v,n} = \frac{v-1}{v} \log\log n + \log 2\Gamma(1/v)$, we have

$$z_{n,p}^{1-v}(x) = 2^{\frac{1-v}{v}} \lambda^{1-v} (\log n)^{\frac{1-v}{v}} \left\{ 1 + \frac{(v^{-1}-1)[x - \log 2\Gamma(\frac{1}{v})] + (v^{-1}-1)^2 (\log\log n)}{\log n} - \frac{(v^{-1}-1)^3 (\log\log n)^2}{2(\log n)^2} + o\left(\left(\frac{\log\log n}{\log n} \right)^2 \right) \right\}, \tag{26}$$

and

$$\begin{aligned} & g_v\big(z_{n,p}(x)\big) \\ &= \frac{v}{\lambda 2^{1+\frac{1}{v}} \Gamma(\frac{1}{v})} \exp\left[-(\log n) \left(1 - \frac{z_{v,n}}{v \log n} \right)^v \left(1 + \frac{px}{v\log n - z_{v,n}} \right)^{\frac{v}{p}} \right] \\ &= \frac{v e^{-x}}{\lambda 2^{\frac{1}{v}} n} (\log n)^{\frac{v-1}{v}} \exp\left[-\frac{\frac{(1-v^{-1})^3}{2} (\log\log n)^2 + (1-v^{-1})^2 \left(\log 2\Gamma\left(\frac{1}{v}\right) - x \right) \log\log n}{\log n} + o\left(\frac{\log\log n}{\log n} \right) \right] \\ &= \frac{v e^{-x}}{\lambda 2^{\frac{1}{v}} n} (\log n)^{\frac{v-1}{v}} \left[1 - \frac{\frac{(1-v^{-1})^3}{2} (\log\log n)^2 + (1-v^{-1})^2 \left(\log 2\Gamma\left(\frac{1}{v}\right) - x \right) \log\log n}{\log n} + o\left(\frac{\log\log n}{\log n} \right) \right]. \end{aligned}$$

Further,

$$\begin{aligned} & 1 + 2(v^{-1}-1)\lambda^v \big(z_{n,p}(x)\big)^{-v} + 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v} \big(z_{n,p}(x)\big)^{-2v} \\ &= 1 + \frac{v^{-1}-1}{\log n} - \frac{(v^{-1}-1)^2 \log\log n}{\log^2 n} + o\left(\frac{\log\log n}{\log^2 n} \right). \end{aligned} \tag{27}$$

Combining (8) and (26)–(27), we derive (21).

(iv) If $v \in (0,1) \cup (1,+\infty)$ and $p \neq v$, with normalizing constants c_n and d_n, we have

$$z_{n,p}(x) = (c_n x + d_n)^{\frac{1}{p}} = b_n \left(1 + \frac{2pv^{-1}\lambda^v x}{b_n^v}\right)^{\frac{1}{p}}.$$

By using arguments similar to (26)–(27), we have

$$z_{n,p}^{1-v}(x) = b_n^{1-v}\left(1 + \frac{2v^{-1}(1-v)\lambda^v x}{b_n^v} + \frac{2(1-v)(1-v-p)v^{-2}\lambda^{2v}x^2}{b_n^{2v}} + o\left(\frac{1}{b_n^{2v}}\right)\right), \tag{28}$$

and by (7),

$$g_v\big(z_{n,p}(x)\big) = n^{-1}\mathrm{e}^{-x}\frac{v}{2\lambda^v}b_n^{v-1}\left\{1 - \frac{v^{-1}(v-p)\lambda^v x^2}{b_n^v} - \frac{4\lambda^{2v}v^{-2}(v-p)(v-2p)x^3 - 3\lambda^{2v}v^{-2}(v-p)^2x^4}{6b_n^{2v}} + o\left(\frac{1}{b_n^{2v}}\right)\right\}. \tag{29}$$

Further,

$$1 + 2(v^{-1}-1)\lambda^v\big(z_{n,p}(x)\big)^{-v} + 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}\big(z_{n,p}(x)\big)^{-2v} = 1 + \frac{2(v^{-1}-1)\lambda^v}{b_n^v} + \frac{4(v^{-1}-1)(v^{-1}-2)\lambda^{2v} - 4(v^{-1}-1)\lambda^{2v}x}{b_n^{2v}} + o\left(\frac{1}{b_n^{2v}}\right). \tag{30}$$

Combining (8) and (28)–(30), we derive the desired result (22).

(v) If $v \in (0,1) \cup (1,+\infty)$ and $p = v$, with normalizing constants c_n^* and d_n^* given by (9), let

$$z_{n,v}(x) = (c_n^* x + d_n^*)^{\frac{1}{v}} = b_n\left(1 + \frac{2\lambda^v}{b_n^v}x + \frac{4(v^{-1}-1)\lambda^{2v}(x+1)}{b_n^{2v}}\right)^{\frac{1}{v}}.$$

By arguments similar to (26)–(27), we can get

$$\big(z_{n,v}(x)\big)^{1-v} = b_n^{1-v}\left(1 + \frac{2(v^{-1}-1)\lambda^v x}{b_n^v} + \frac{4(v^{-1}-1)^2\lambda^{2v}(x+1) + 2(v^{-1}-1)(v^{-1}-2)\lambda^{2v}x^2}{b_n^{2v}} + \frac{8(v^{-1}-1)^2(v^{-1}-2)\lambda^{3v}x(x+1)}{b_n^{3v}} + o\left(\frac{1}{b_n^{3v}}\right)\right), \tag{31}$$

and by (7),

$$g_v\big(z_{n,v}(x)\big) = \frac{v\mathrm{e}^{-x}}{2\lambda^v}n^{-1}b_n^{v-1}\left\{1 - \frac{2(v^{-1}-1)\lambda^v(x+1)}{b_n^v} + \frac{2(v^{-1}-1)^2\lambda^{2v}(x+1)^2}{b_n^{2v}} - \frac{4(v^{-1}-1)^3\lambda^{3v}(x+1)^3}{3b_n^{3v}} + o\left(\frac{1}{b_n^{3v}}\right)\right\}. \tag{32}$$

Further,

$$\begin{aligned}
&1 + 2(v^{-1}-1)\lambda^v\big(z_{n,v}(x)\big)^{-v} + 4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}\big(z_{n,v}(x)\big)^{-2v} \\
&\quad + 8(v^{-1}-1)(v^{-1}-2)(v^{-1}-3)\lambda^{3v}\big(z_{n,v}(x)\big)^{-3v} \\
&= 1 + \frac{2(v^{-1}-1)\lambda^v}{b_n^v} + \frac{4(v^{-1}-1)(v^{-1}-2)\lambda^{2v} - 4(v^{-1}-1)\lambda^{2v}x}{b_n^{2v}} \\
&\quad + \frac{8(v^{-1}-1)\lambda^{3v}x^2 - 8(v^{-1}-1)(3v^{-1}-5)\lambda^{3v}x}{b_n^{3v}} \\
&\quad + \frac{8(v^{-1}-1)\lambda^{3v}(v^{-2}-6v^{-1}+7)}{b_n^{3v}} + o\left(\frac{1}{b_n^{3v}}\right).
\end{aligned} \tag{33}$$

Combining (8) and (31)–(33), we derive (23). ■

Lemma 4.2: *Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution $G_v(x)$ with parameter $v > 0$ and $M_{n,r}$ denotes the r-th maximal term of $\{X_1, X_2, \ldots, X_n\}$ for $1 \leq r \leq n$. Assume that there exists positive constant $z_n(x)$ such that when $n(1 - G_v(z_n(x)) \to \mathrm{e}^{-x}$,*

$$\mathrm{P}\left(|M_{n,r}|^p \leq z_n^p(x)\right) - \Lambda_r(x) = \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,v}(x))(r-1-\mathrm{e}^{-x})\right](1-\theta_{n,v}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1}), \quad (34)$$

where $\theta_{n,v}(x) = n\mathrm{e}^x(1 - G_v(z_n(x)))$.

Proof: By arguments similar to Hall (1980), we have

$$\begin{aligned}
&\mathrm{P}\left(|M_{n,r}|^p \leq z_n^p(x)\right) - \Lambda_r(x) \\
&= \sum_{j=0}^{r-1}\binom{n}{j}\left[1 - n^{-1}\mathrm{e}^{-x}\theta_{n,v}(x)\right]^{n-j}\left[n^{-1}\mathrm{e}^{-x}\theta_{n,v}(x)\right]^j - \Lambda_r(x) + O(n^{r-1}2^{-n}) \\
&= \sum_{j=0}^{r-1}\left[1 - n^{-1}\mathrm{e}^{-x}\theta_{n,v}(x)\right]^n\theta_{n,v}^j(x)\frac{\mathrm{e}^{-jx}}{j!} - \Lambda_r(x) + O(n^{-1}) \\
&= \Lambda(x)\sum_{j=0}^{r-1}\left\{\exp\left[(1-\theta_{n,v}(x))\mathrm{e}^{-x}\right]\right\}\left[1 - (1-\theta_{n,v}(x))\right]^j\frac{\mathrm{e}^{-jx}}{j!} - \Lambda_r(x) + O(n^{-1}) \\
&= \Lambda(x)\sum_{j=0}^{r-1}\left[1 + (1-\theta_{n,v}(x))\mathrm{e}^{-x} + \frac{(1-\theta_{n,v}(x))^2}{2}\mathrm{e}^{-2x}\right] \\
&\quad \times\left[1 - j(1-\theta_{n,v}(x)) + \frac{j(j-1)}{2}(1-\theta_{n,v}(x))^2\right]\frac{\mathrm{e}^{-jx}}{j!} - \Lambda_r(x) + O(n^{-1}) \\
&= \Lambda(x)(1-\theta_{n,v}(x))\left(\Lambda_r(x) - \Lambda_{r-1}(x)\right)\mathrm{e}^{-x} \\
&\quad - \frac{1}{2}\Lambda(x)(1-\theta_{n,v}(x))^2\left[\Lambda_{r-1}(x) - \Lambda_{r-2}(x) - \left(\Lambda_r(x) - \Lambda_{r-1}(x)\right)\right]\mathrm{e}^{-2x} + O(n^{-1}) \\
&= \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,v}(x))(r-1-\mathrm{e}^{-x})\right](1-\theta_{n,v}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1})
\end{aligned}$$

since

$$\sum_{j=0}^{r-1}\frac{j\mathrm{e}^{-jx}}{j!} = \mathrm{e}^{-x}\Lambda_{r-1}(x), \quad \sum_{j=0}^{r-1}\frac{j^2\mathrm{e}^{-jx}}{j!} = \mathrm{e}^{-2x}\Lambda_{r-2}(x) + \mathrm{e}^{-x}\Lambda_{r-1}(x). \quad (35)$$

The proof is completed. ■

Lemma 4.3: *Let $\{X_n : n \geq 1\}$ satisfy the assumptions of Lemma 4.2 and $M_{n,r}$ denotes the r-th largest order statistics of $X_1, X_2, \ldots, X_n$. The following results hold.*

(i) *If $v = 1$ and $p \neq 1$, with normalizing constants α_n^* and β_n^* given by $\alpha_n^* = p2^{-\frac{p}{2}}(\log\frac{n}{2})^{p-1}$ and $\beta_n^* = (2^{-\frac{1}{2}}\log\frac{n}{2})^p$, let $\beta_n' = 2\log\log\frac{n}{2}$, and then*

$$\sup_{x\leq-\beta_n'}|\mathrm{P}(|M_{n,r}|^p \leq \alpha_n^*x + \beta_n^*) - \Lambda_r(x)| \leq \frac{C}{\log n}. \quad (36)$$

(ii) *If $v \in (0,1) \cup (1,+\infty)$, with normalizing constants α_n^* and β_n^* given by (5), let $\beta_n' = 2\log\left(\frac{\log n}{(\log\log n)^2}\right)$, and then*

$$\sup_{x\leq-\beta_n'}|\mathrm{P}(|M_{n,r}|^p \leq \alpha_n^*x + \beta_n^*) - \Lambda_r(x)| \leq C\frac{(\log\log n)^2}{\log n}. \quad (37)$$

(iii) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p \neq \nu$, with normalizing c_n and d_n given by (6), let $\beta_n' = 2\log(a_n^{-1} b_n \nu^{-1})$, and then*

$$\sup_{x \le -\beta_n'} |\,\mathrm{P}(|M_{n,r}|^p \le c_n x + d_n) - \Lambda_r(x)| \le \frac{C}{\log n}. \tag{38}$$

(iv) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p = \nu$, with normalizing c_n^* and d_n^* given by (9), let $\beta_n' = 4\log(a_n^{-1} b_n \nu^{-1})$, and then*

$$\sup_{x \le -\beta_n'} |\,\mathrm{P}(|M_{n,r}|^p \le c_n^* + d_n^*) - \Lambda_r(x)| \le \frac{C}{(\log n)^2}. \tag{39}$$

Proof: Note that

$$\begin{aligned}\Delta_n(x) &= |\,\mathrm{P}(|M_{n,r}|^p \le c_n x + d_n) - \Lambda_r(x)| \\ &= \left|\sum_{j=1}^{r-1} \binom{n}{j} G^{n-j}(z_n)[1-G(z_n)]^j - \Lambda_r(x) + O(n^{r-1}2^{-n})\right| \\ &\le r n^{r-1} \left(\frac{1}{2}\right)^{n-r} + \Lambda_r(-\beta_n'),\end{aligned} \tag{40}$$

and

$$\begin{aligned}\Lambda_r(-\beta_n') &= \Lambda(-\beta_n') \sum_{j=0}^{r-1} \frac{e^{j\beta_n'}}{j!} = C\Lambda(-\beta_n') \frac{1-e^{r\beta_n'}}{1-r^{\beta_n'}} \\ &\le C\Lambda(-\beta_n') e^{(r-1)\beta_n'} \le C e^{-\beta_n'}.\end{aligned} \tag{41}$$

If $\nu = 1$ and $p \neq \nu$, note that $\beta_n' = 2\log\log\frac{n}{2}$ and $rn^{r-1}(\frac{1}{2})^{n-r} = o(n^{-\alpha})$ for any $\alpha > 0$. It follows from (40) and (41) that (36) holds. If $\nu \in (0,1) \cup (1,+\infty)$, and $\beta_n' = 2\log\left(\frac{\log n}{(\log\log n)^2}\right)$, from (40) and (41) we can get (37). If $\nu \in (0,1) \cup (1,+\infty)$ and $p \neq \nu$, recall that $a_n b_n^{-1} \nu \sim \frac{1}{\log n}$ and $\beta_n' = 2\log(a_n^{-1} b_n \nu^{-1})$, and then (38) follows. If $\nu \in (0,1) \cup (1,+\infty)$, $p = \nu$, and $\beta_n' = 4\log(a_n^{-1} b_n \nu^{-1})$, from (40) and (41) we can get (39). ■

The following is about the Mills' type inequalities of GED.

Lemma 4.4: (i) *For $0 < \nu < 1$, as $x > [\frac{2(1-\nu)}{\nu}]^{\frac{1}{\nu}}$ we have*

$$\frac{2\lambda^\nu}{\nu} x^{1-\nu} < \frac{1-G(x)}{g_\nu(x)} < \frac{2\lambda^\nu}{\nu} x^{1-\nu} \left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu} x^{-\nu}\right)^{-1}.$$

(ii) *For $\nu > 1$, for all $x > 0$ we have*

$$\frac{2\lambda^\nu}{\nu} x^{1-\nu} \left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu} x^{-\nu}\right)^{-1} < \frac{1-G_\nu(x)}{g_\nu(x)} < \frac{2\lambda^\nu}{\nu} x^{1-\nu}.$$

Proof: Note that assertion (ii) is just Lemma 2.2 of Peng et al. (2009). We only show that assertion (i) holds. For $x > [\frac{2(1-\nu)}{\nu}]^{\frac{1}{\nu}}$ as $0 < \nu < 1$,

$$\frac{1}{x^\nu} \int_x^\infty e^{-\frac{t^\nu}{2}}\, dt > \int_x^\infty \frac{1}{t^\nu} e^{-\frac{t^\nu}{2}}\, dt = \frac{\nu}{2(1-\nu)} \int_x^\infty e^{-\frac{t^\nu}{2}}\, dt - \frac{x^{1-\nu}}{1-\nu} e^{-\frac{x^\nu}{2}}$$

implies that

$$\int_x^\infty e^{-\frac{t^\nu}{2}} dt < \left(\frac{\nu-1}{x} + \frac{\nu x^{\nu-1}}{2}\right)^{-1} \exp\left(-\frac{x^\nu}{2}\right) \tag{42}$$

and

$$\int_x^\infty e^{-\frac{t^\nu}{2}} dt > \left(\frac{\nu x^{\nu-1}}{2}\right)^{-1} \exp\left(-\frac{x^\nu}{2}\right). \tag{43}$$

It follows from (42) that

$$1 - G_\nu(x) = \frac{\nu}{2^{1+\frac{1}{\nu}}\Gamma(\frac{1}{\nu})} \int_{x/\lambda}^{\infty} e^{-\frac{t^\nu}{2}} dt$$
$$> \frac{\nu}{2^{1+\frac{1}{\nu}}\Gamma(\frac{1}{\nu})} \left(\frac{\nu}{2}\left(\frac{x}{\lambda}\right)^{\nu-1}\right)^{-1} \exp\left(-\frac{x^\nu}{2\lambda^\nu}\right) = g_\nu(x)\frac{2\lambda^\nu}{\nu}x^{1-\nu}.$$

Similarly, by (43),

$$1 - G_\nu(x) < g_\nu(x)\frac{2\lambda^\nu}{\nu}x^{1-\nu}\left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)^{-1}.$$

Hence, the assertion follows. ■

Recall that $z_{n,p}(x) = (c_n x + d_n)^{\frac{1}{p}}$ and for large enough n, $z_{n,p}(x) > 0$ and $z_{n,p}(x) > \lambda[\frac{2(1-\nu)}{\nu}]^{\frac{1}{\nu}}$.

Lemma 4.5: (i) *If $\nu = 1$ and $p \neq 1$, let $\alpha'_n = \frac{\log \frac{n}{2}}{4(p+1)}$ and $|x| \leq \alpha'_n$, and then*

$$\left|e^{-x} - n\big(1 - G_\nu(z_{n,p}(x))\big)\right| \leq \frac{C}{\log n}.$$

(ii) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p > 0$, let $\alpha'_n = \frac{a_n^{-1}b_n}{4(\nu+2)}\frac{(1+\delta_n a_n b_n^{-1})}{r_n}$ and $|x| \leq \alpha'_n$, and then*

$$\left|e^{-x} - n\big(1 - G_\nu(z_{n,p}(x))\big)\right| \leq C(x^2 + |x| + 1)\delta_n \exp\left(\frac{1}{4}|x| - x\right),$$

where $\delta_n = \frac{\beta_n - b_n}{a_n} \sim \frac{(\nu-1)^3}{\nu^2}\frac{(\log\log n)^2}{\log n}$, $r_n = \alpha_n/a_n$, and $r_n - 1 \sim \frac{(\nu-1)^3}{\nu^2}\frac{\log\log n}{\log n}$.

(iii) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p \neq \nu$, let $\alpha'_n = \frac{a_n^{-1}b_n}{4(p+2\nu)}$ and $|x| \leq \alpha'_n$, and then*

$$\left|e^{-x} - n\big(1 - G_\nu(z_{n,p}(x))\big)\right| \leq C(x^2 + |x| + 1)a_n b_n^{-1}\nu \exp\left(\frac{1}{4}|x| - x\right).$$

(iv) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p = \nu$, let $\alpha'_n = \frac{a_n^{-1}b_n}{4(\nu+2)}$ and $|x| \leq \alpha'_n$, and then*

$$\left|e^{-x} - n\big(1 - G_\nu(z_{n,p}(x))\big)\right| \leq C(|x| + 1)(a_n b_n^{-1}\nu)^2 e^{-x}.$$

Proof: (i) If $\nu = 1$ and $p \neq 1$, then $z_{n,p}(x) = \frac{\log \frac{n}{2}}{\sqrt{2}}\left(1 + \frac{px}{\log \frac{n}{2}}\right)^{\frac{1}{p}}$. It follows from Lemma 1 of Hall (1980) and (20) that

$$e^{-x} - n[1 - G(z_{n,p}(x))] \leq e^{-x} - \frac{n}{2}\exp\left\{-\log\frac{n}{2}\left(1 + \frac{1}{\log \frac{n}{2}}x + \frac{2}{(\log \frac{n}{2})^2}x^2\right)\right\}$$
$$\leq \frac{2x^2 e^{-x}}{\log \frac{n}{2}} \leq \frac{C}{\log n},$$

and

$$n[1 - G(z_{n,p}(x))] - e^{-x} \leq \frac{n}{2}\exp\left\{-\log\frac{n}{2}\left(1 + \frac{x}{\log \frac{n}{2}} - \frac{px^2}{(\log \frac{n}{2})^2}\right)\right\} - e^{-x}$$
$$\leq e^{-x}\cdot\frac{px^2}{\log \frac{n}{2}}\exp\left\{\frac{px^2}{\log \frac{n}{2}}\right\} \leq e^{-\frac{3}{4}x}\cdot\frac{px^2}{\log \frac{n}{2}} \leq \frac{C}{\log n}.$$

Combining the above results, we can get (i).

(ii) In case of $\nu > 1$, $z_{n,p}(x) = b_n\left(1 + pa_nb_n^{-1} \cdot \frac{r_n}{1+\delta_n a_n b_n^{-1}}\right)^{\frac{1}{p}}(1 + \delta_n a_n b_n^{-1})$. By Lemma 1 of Hall (1980), Lemma 4.4 and some tedious calculation, we have

$$e^{-x} - n[1 - G(z_{n,p}(x))] \le 2e^{-x}a_nb_n^{-1}\nu(|x| + 1 + x^2) + e^{-x}\delta_n$$

and

$$\begin{aligned}
n[1 - G(z_{n,p}(x))] - e^{-x} &\le n \cdot \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right) - e^{-x} \\
&\le \left[(\nu - 1)a_nb_n^{-1}|x| + pa_nb_n^{-1}x^2\right]\exp\left\{(\nu - 1)a_nb_n^{-1}|x| + pa_nb_n^{-1}x^2 - x\right\} \\
&\le \nu a_nb_n^{-1}\left(\nu|x| + px^2\right)\exp\left(\frac{1}{4} + \frac{1}{4}|x| - x\right),
\end{aligned}$$

which implies (ii) for the case $\nu > 1$.

If $0 < \nu < 1$, it follows from Lemma 1 of Hall (1980) and Lemma 4.4 that

$$\begin{aligned}
e^{-x} - n[1 - G(z_{n,p}(x))] &\le e^{-x} - n \cdot \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right) \\
&\le e^{-x}\left\{1 - \exp\left\{-2\nu a_nb_n^{-1}x^2\right\}(1 + (1 - \nu)\delta_n a_nb_n^{-1})\right\} \\
&\le e^{-x}\left\{2\nu a_nb_n^{-1}x^2 + 2\nu(1 - \nu)\delta_n\left(a_nb_n^{-1}x\right)^2\right\} \\
&\le 2x^2a_nb_n^{-1}\nu e^{-x} + \frac{2\nu(1 - \nu)}{16}\delta_n e^{-x},
\end{aligned}$$

and

$$\begin{aligned}
&n[1 - G(z_{n,p}(x))] - e^{-x} \\
&\le n \cdot \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right)\left(1 + \frac{2(\nu - 1)\lambda^{\nu}}{\nu} z_{n,p}^{-\nu}(x)\right)^{-1} - e^{-x} \\
&\le e^{-x}\left\{\exp\left[(1 - \nu)a_nb_n^{-1}x - a_nb_n^{-1}(\nu - 1)(1 - \nu a_nb_n^{-1}x) + pa_nb_n^{-1}x^2 - \delta_n\right] - 1\right\} \\
&\le Ca_nb_n^{-1}\nu\left(|x| + 1 + x^2\right)\exp\left(\frac{1}{4} + \frac{1}{4}|x| - x\right) + |\delta_n|\exp\left(\frac{1}{4} + \frac{1}{4}|x| - x\right),
\end{aligned}$$

which implies (ii) for the case $0 < \nu < 1$.

(iii) If $p \ne \nu$, in case of $\nu > 1$, $z_{n,p}(x) = b_n(1 + a_nb_n^{-1}px)^{\frac{1}{p}}$. It follows from Lemma 1 of Hall (1980) and Lemma 4.4 that

$$\begin{aligned}
e^{-x} - n[1 - G(z_{n,p}(x))] &\le e^{-x} - n \cdot \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} z_{n,p}^{1-\nu}(x)\exp\left\{-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right\}\left(1 - \frac{2(\nu - 1)\lambda^{\nu}}{\nu} z_{n,p}^{-\nu}(x)\right) \\
&\le e^{-x} - \left(1 - (\nu - 1)a_nb_n^{-1}x\right)\exp\left\{-x - 2a_nb_n^{-1}x^2\right\} \\
&\quad \times\left[1 - \frac{2(\nu - 1)\lambda^{\nu}}{\nu} b_n^{-\nu}(1 - a_nb_n^{-1}\nu x + x^2(a_nb_n^{-1})^2(8\nu^2 + p\nu))\right] \\
&\le e^{-x}\left\{1 - (1 - (\nu - 1)a_nb_n^{-1}x)\exp\left\{-2a_nb_n^{-1}x^2\right\}\right. \\
&\quad \left.\times\left[1 - (\nu - 1)a_nb_n^{-1}\left(1 - a_nb_n^{-1}\nu x + (a_nb_n^{-1}\nu)^2x^2\left(8 + \frac{p}{\nu}\right)\right)\right]\right\} \\
&\le e^{-x}\{1 - [1 - a_nb_n^{-1}\nu(|x| + 2)]\exp(-2a_nb_n^{-1}x^2)\} \\
&\le e^{-x}\{1 - [1 - a_nb_n^{-1}\nu(|x| + 2)](1 - 2a_nb_n^{-1}x^2)\} \\
&\le 2e^{-x}a_nb_n^{-1}\nu(|x| + 1 + x^2)
\end{aligned}$$

since

$$\begin{aligned}
&\left(1 - (\nu - 1)a_nb_n^{-1}x\right)\left[1 - (\nu - 1)a_nb_n^{-1}\left(1 - a_nb_n^{-1}\nu x + (a_nb_n^{-1}\nu)^2x^2\left(8 + \frac{p}{\nu}\right)\right)\right] \\
&\ge 1 - a_nb_n^{-1}\nu(|x| + 2)
\end{aligned}$$

as $|x| \le \alpha_n'$. Next,

$$\begin{aligned} n[1-G(z_{n,p}(x))]-\mathrm{e}^{-x} &\le n\cdot\frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right)-\mathrm{e}^{-x} \\ &\le (1+pa_nb_n^{-1}x)^{\frac{1-\nu}{p}}\exp\big(-x+pa_nb_n^{-1}x^2\big)-\mathrm{e}^{-x} \\ &\le \big[(\nu-1)a_nb_n^{-1}|x|+pa_nb_n^{-1}x^2\big]\exp\big\{(\nu-1)a_nb_n^{-1}|x|+pa_nb_n^{-1}x^2-x\big\} \\ &\le \nu a_nb_n^{-1}\big(\nu|x|+px^2\big)\exp\left(\frac{1}{4}+\frac{1}{4}|x|-x\right), \end{aligned}$$

which implies (iii) for the case $\nu>1$.

If $p\neq\nu$, in case of $0<\nu<1$, it follows from Lemma 1 of Hall (1980) and Lemma 4.4 that

$$\begin{aligned} \mathrm{e}^{-x}-n[1-G(z_{n,p}(x))] &\le \mathrm{e}^{-x}-n\cdot\frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right) \\ &\le \mathrm{e}^{-x}-\big[1+(1-\nu)a_nb_n^{-1}x-(1-\nu)(a_nb_n^{-1})^2px^2\big]\exp\big(-x-2\nu a_nb_n^{-1}x^2\big) \\ &\le \mathrm{e}^{-x}\big[1-\exp(-2\nu a_nb_n^{-1}x^2)\big]\le 2x^2a_nb_n^{-1}\nu\mathrm{e}^{-x}, \end{aligned}$$

and

$$\begin{aligned} &n[1-G(z_{n,p}(x))]-\mathrm{e}^{-x} \\ &\le n\cdot\frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}z_{n,p}^{1-\nu}(x)\exp\left(-\frac{z_{n,p}^{\nu}(x)}{2\lambda^{\nu}}\right)\left(1+\frac{2(\nu-1)\lambda^{\nu}}{\nu}z_{n,p}^{-\nu}(x)\right)^{-1}-\mathrm{e}^{-x} \\ &\le \mathrm{e}^{-x}\big\{\exp\big[(1-\nu)a_nb_n^{-1}-a_nb_n^{-1}(\nu-1)(1-\nu a_nb_n^{-1}x)+pa_nb_n^{-1}x^2\big]-1\big\} \\ &\le \big[(1-\nu)a_nb_n^{-1}|x|+(1-\nu)a_nb_n^{-1}-(1-\nu)\nu(a_nb_n^{-1})^2|x|+pa_nb_n^{-1}x^2\big] \\ &\quad\times\exp\big\{(1-\nu)a_nb_n^{-1}|x|+(1-\nu)a_nb_n^{-1}-(1-\nu)\nu(a_nb_n^{-1})^2|x|+pa_nb_n^{-1}x^2-x\big\} \\ &\le \big[a_nb_n^{-1}|x|+a_nb_n^{-1}-\nu(a_nb_n^{-1})^2|x|+pa_nb_n^{-1}x^2\big] \\ &\quad\times\exp\big\{a_nb_n^{-1}|x|+a_nb_n^{-1}-\nu(a_nb_n^{-1})^2|x|+pa_nb_n^{-1}x^2-x\big\} \\ &\le a_nb_n^{-1}\nu\left(\frac{1}{\nu}|x|+\frac{1}{\nu}-a_nb_n^{-1}|x|+\frac{p}{\nu}x^2\right)\exp\left(\frac{1}{4}+\frac{1}{4}|x|-x\right) \\ &\le Ca_nb_n^{-1}\nu\big(|x|+1+x^2\big)\exp\left(\frac{1}{4}+\frac{1}{4}|x|-x\right), \end{aligned}$$

which implies (iii) for the case of $0<\nu<1$.

(iv) If $p=\nu$, the proof is similar to the case of (iii). If $\nu>1$ and $z_{n,p}(x)=b_n[1+\nu a_nb_n^{-1}x+\nu(1-\nu)(a_nb_n^{-1})^2(1+x)]^{\frac{1}{\nu}}$, we have

$$\begin{aligned} &\mathrm{e}^{-x}-n[1-G(z_{n,p}(x))] \\ &\le \mathrm{e}^{-x}-\big[1-(\nu-1)a_nb_n^{-1}x+(\nu-1)^2(a_nb_n^{-1})^2(x+1)\big] \\ &\quad\times\exp\big\{-x+(\nu-1)(a_nb_n^{-1})(1+x)\big\}\big[1-(\nu-1)a_nb_n^{-1}\big] \\ &\le \mathrm{e}^{-x}\big\{1-[1-\nu a_nb_n^{-1}x-\nu a_nb_n^{-1}-(\nu a_nb_n^{-1})^3(x+1)][1+\nu a_nb_n^{-1}(1+x)]\big\} \\ &\le 2(a_nb_n^{-1}\nu)^2\mathrm{e}^{-x}(x^2+|x|+1), \end{aligned}$$

and

$$\begin{aligned} &n[1-G(z_{n,p}(x))]-\mathrm{e}^{-x} \\ &\le \mathrm{e}^{-x}\big\{\exp\{(1-\nu)a_nb_n^{-1}x+(1-\nu)^2(a_nb_n^{-1})^2(1+x)+(\nu-1)a_nb_n^{-1}(1+x)\}-1\big\} \\ &\le \big[(\nu-1)a_nb_n^{-1}+(\nu-1)^2(a_nb_n^{-1})^2(1+|x|)\big]\exp\big\{(\nu-1)a_nb_n^{-1}+(\nu-1)^2(a_nb_n^{-1})^2(1+|x|)-x\big\} \\ &\le C(|x|+1)(a_nb_n^{-1}\nu)^2\mathrm{e}^{-x}. \end{aligned}$$

Therefore, the result in (iv) for the case of $\nu>1$ can be proved.

If $p = \nu$, in case of $0 < \nu < 1$,

$$\begin{aligned}
\mathrm{e}^{-x} - n[1 - G(z_{n,p}(x))] &\le \mathrm{e}^{-x}\big\{1 - \exp\big[-(1-\nu)a_n b_n^{-1}(1+x)\big]\big\} \\
&\le C\mathrm{e}^{-x} a_n b_n^{-1}(1+|x|),
\end{aligned}$$

and

$$\begin{aligned}
&n[1 - G(z_{n,p}(x))] - \mathrm{e}^{-x} \\
&\le \mathrm{e}^{-x}\Big\{\exp\Big[(1-\nu)a_n b_n^{-1}|x| + (1-\nu)^2(a_n b_n^{-1})^2(1+x) - (1-\nu)a_n b_n^{-1}(1+x) \\
&\quad + a_n b_n^{-1}(1-\nu)(1-\nu a_n b_n^{-1}x - \nu(1-\nu)(a_n b_n^{-1})^2(1+x))\Big] - 1\Big\} \\
&\le \mathrm{e}^{-x}\big\{\exp\big[(1-\nu)^2(a_n b_n^{-1})^2(1+x) - \nu(1-\nu)(a_n b_n^{-1})^2 x + \nu(1-\nu)^2(a_n b_n^{-1})^3(1+x)\big] - 1\big\} \\
&\le \big[(1-\nu)^2(a_n b_n^{-1})^2(1+|x|) - \nu(1-\nu)(a_n b_n^{-1})^2|x| + \nu(1-\nu)^2(a_n b_n^{-1})^3(1+|x|)\big] \\
&\quad \times \exp\big\{(1-\nu)^2(a_n b_n^{-1})^2(1+|x|) - \nu(1-\nu)(a_n b_n^{-1})^2|x| + \nu(1-\nu)^2(a_n b_n^{-1})^3(1+|x|)\big\} \\
&\le C(|x|+1)(a_n b_n^{-1}\nu)^2\mathrm{e}^{-x}.
\end{aligned}$$

Hence the result of (iv) can be proved for $0 < \nu < 1$. ■

Lemma 4.6: (i) *If $\nu = 1$ and $p \ne 1$, let $\alpha_n' = \frac{\log \frac{n}{2}}{4(p+1)}$ and $x \ge \alpha_n'$, and then*

$$\left|\mathrm{e}^{-x} - n[1 - G_\nu(z_{n,p}(x))]\right| \le \exp\left\{-C\log\frac{n}{2}\right\}.$$

(ii) *If $\nu \in (0,1) \cup (1,+\infty)$, let $\alpha_n' = \frac{a_n^{-1}b_n}{4(p+2\nu)} \cdot \frac{1+\delta_n a_n b_n^{-1}}{r_n}$ and $x \ge \alpha_n'$, and then*

$$\left|\mathrm{e}^{-x} - n[1 - G_\nu(z_{n,p}(x))]\right| \le C\exp\big\{-Ca_n^{-1}b_n\nu^{-1}\big\}.$$

(iii) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p \ne \nu$, let $\alpha_n' = \frac{a_n^{-1}b_n}{4(p+2\nu)}$ and $x \ge \alpha_n'$, and then*

$$\left|\mathrm{e}^{-x} - n[1 - G_\nu(z_{n,p}(x))]\right| \le C\exp\big\{-Ca_n^{-1}b_n\nu^{-1}\big\}.$$

(iv) *If $\nu \in (0,1) \cup (1,+\infty)$ and $p = \nu$, let $\alpha_n' = \frac{a_n^{-1}b_n}{4(\nu+2)}$ and $x \ge \alpha_n'$, and then*

$$\left|\mathrm{e}^{-x} - n[1 - G_\nu(z_{n,p}(x))]\right| \le C\exp\big\{-C(a_n^{-1}b_n\nu^{-1})^2\big\}.$$

Proof: (i) We can get the assertion of (i) since

$$\begin{aligned}
n\big[1 - G_\nu(z_{n,p}(\alpha_n'))\big] &\le \frac{n}{2}\exp\left\{-\log\frac{n}{2}\left(1 + \frac{1}{\log\frac{n}{2}}\alpha_n' - \frac{p}{(\log\frac{n}{2})^2}(\alpha_n')^2\right)\right\} \\
&\le \exp\left\{-\frac{3}{4}\alpha_n'\right\} \le \exp\left\{-C\log\frac{n}{2}\right\}.
\end{aligned}$$

(ii) In case of $\nu > 1$,

$$\begin{aligned}
n\big[1 - G_\nu(z_{n,p}(\alpha_n'))\big] &\le \left(1 + pa_n b_n^{-1}\alpha_n' \frac{r_n}{1+\delta_n a_n b_n^{-1}}\right)^{\frac{1-\nu}{p}} \\
&\quad \times \exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1 - \left(1 + a_n b_n^{-1}p\alpha_n' \cdot \frac{r_n}{1+\delta_n a_n b_n^{-1}}\right)^{\frac{\nu}{p}}(1+\delta_n a_n b_n^{-1})^\nu\right]\right\} \\
&\le \exp\big\{-Ca_n^{-1}b_n\nu^{-1}\big\},
\end{aligned}$$

and for the case $0 < \nu < 1$,

$$\begin{aligned} n\big[1-G_\nu(z_{n,p}(\alpha_n'))\big] &\le \left(1+pa_nb_n^{-1}\alpha_n'\cdot\frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{1-\nu}{p}} \\ &\quad\times\exp\left\{-\frac{b_n^\nu}{2\lambda^\nu}\left[1-\left(1+pa_nb_n^{-1}\alpha_n'\frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{\nu}{p}}\left(1+\delta_na_nb_n^{-1}\right)^\nu\right]\right\} \\ &\quad\times\left(1+\frac{2(\nu-1)\lambda^\nu}{\nu}b_n^{-\nu}\left(1+pa_nb_n^{-1}\alpha_n'\cdot\frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{-\nu}{p}}\left(1+\delta_na_nb_n^{-1}\right)^\nu\right)^{-1} \\ &\le C\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-\left(\frac{5p+8\nu}{4p+8\nu}\right)^{\frac{\nu}{p}}\right]\right\}\le C\exp\big\{-Ca_n^{-1}b_n\nu^{-1}\big\}. \end{aligned}$$

Hence, we can get (ii).

(iii) If $p \neq \nu$, in case of $\nu > 1$,

$$n\left[1-G_\nu(z_{n,p}(\alpha_n'))\right]\le\left(1+pa_nb_n^{-1}\alpha_n'\right)^{\frac{1-\nu}{p}}\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-(1+a_nb_n^{-1}p\alpha_n')^{\frac{\nu}{p}}\right]\right\}\le\exp\left\{-Ca_n^{-1}b_n\nu^{-1}\right\},$$

and for the case $0 < \nu < 1$,

$$\begin{aligned} n\left[1-G_\nu(z_{n,p}(\alpha_n'))\right] &\le \left(1+a_nb_n^{-1}p\alpha_n'\right)^{\frac{1-\nu}{p}}\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-(1+a_nb_n^{-1}p\alpha_n')^{\frac{\nu}{p}}\right]\right\} \\ &\quad\times\left(1+\frac{2(\nu-1)\lambda^\nu}{\nu}b_n^{-\nu}(1+pa_nb_n^{-1}\alpha_n')^{-\frac{\nu}{p}}\right)^{-1} \\ &\le C\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-\left(\frac{5p+8\nu}{4p+8\nu}\right)^{\frac{\nu}{p}}\right]\right\}\le C\exp\left\{-Ca_n^{-1}b_n\nu^{-1}\right\}. \end{aligned}$$

Therefore, we can get (iii).

(iv) If $p = \nu$, in case of $\nu > 1$

$$\begin{aligned} n\left[1-G_\nu(z_{n,p}(\alpha_n'))\right] &\le \left[1+\nu a_nb_n^{-1}\alpha_n'-\nu(\nu-1)(a_nb_n^{-1})^2(1+\alpha_n')\right]^{\frac{1-\nu}{\nu}} \\ &\quad\times\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-(1+\nu a_nb_n^{-1}\alpha_n'+\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n'))\right]\right\} \\ &\le \exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-(1+\nu a_nb_n^{-1}\alpha_n'+\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n'))\right]\right\} \\ &\le \exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[-\nu a_nb_n^{-1}\alpha_n'+\nu(\nu-1)(a_nb_n^{-1})^2(1+\alpha_n')\right]\right\} \\ &\le C\exp\left\{-\frac{b_n^\nu}{2\lambda^\nu}\alpha_n'+(\nu-1)a_nb_n^{-1}(1+\alpha_n')\right\} \\ &\le C\exp\left\{-C(a_n^{-1}b_n\nu^{-1})^2\right\}, \end{aligned}$$

and for the case of $0 < \nu < 1$,

$$\begin{aligned} n\left[1-G_\nu(z_{n,p}(\alpha_n'))\right] &\le \left[1+\nu a_nb_n^{-1}\alpha_n'+\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n')\right]^{\frac{1-\nu}{\nu}} \\ &\quad\times\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[1-(1+\nu a_nb_n^{-1}\alpha_n'+\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n'))\right]\right\} \\ &\quad\times\left\{1+\frac{2(\nu-1)\lambda^\nu}{\nu}b_n^{-\nu}\left[1+\nu a_nb_n^{-1}\alpha_n'+\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n')\right]^{-1}\right\}^{-1} \\ &\le C\exp\left\{\frac{b_n^\nu}{2\lambda^\nu}\left[-\nu a_nb_n^{-1}\alpha_n'-\nu(1-\nu)(a_nb_n^{-1})^2(1+\alpha_n'))\right]\right\} \\ &\le C\exp\left\{-C(a_n^{-1}b_n\nu^{-1})^2\right\}. \end{aligned}$$

Then, we complete the proof of (iv). ■

Let

$$A_n = \exp\left\{-n\left(1 - G_\nu(z_{n,p}(x))\right) + e^{-x}\right\}, \quad B_n(x) = \exp\{-R_n(x)\},$$

where $R_n(x) = -n[1 - G_\nu(z_{n,p}(x))] - n\log G(z_n)$ and $0 < R_n(x) \le \frac{n(1-G_\nu(z_{n,p}(x)))^2}{2G_\nu(z_{n,p}(x))}$.

Lemma 4.7: (i) *If* $\nu = 1$ *and* $p \ne 1$, *let* $\alpha_n' = \frac{\log\frac{n}{2}}{2(p+1)}$ *and* $x \ge -\alpha_n'$, *and then*

$$\left(1 - G_\nu(z_{n,p}(x))\right) \le Cn^{-\frac{11}{16}}, \quad |B_n(x) - 1| \le Cn^{-\frac{1}{4}}.$$

(ii) *If* $\nu \in (0,1)\cup(1,+\infty)$ *and* $p > 0$, *let* $\alpha_n' = \frac{a_n^{-1}b_n}{4(p+2\nu)} \cdot \frac{1+\delta_n a_n b_n^{-1}}{r_n}$ *and* $x \ge -\alpha_n'$, *and then*

$$\left(1 - G_\nu(z_{n,p}(x))\right) \le Cn^{-\frac{15}{16}}, \quad |B_n(x) - 1| \le Cn^{-\frac{3}{4}}.$$

(iii) *If* $\nu \in (0,1)\cup(1,+\infty)$ *and* $p \ne \nu$, *let* $\alpha_n' = \frac{a_n^{-1}b_n}{4(p+2\nu)}$ *and* $x \ge -\alpha_n'$, *and then*

$$\left(1 - G_\nu(z_{n,p}(x))\right) \le Cn^{-\frac{15}{16}}, \quad |B_n(x) - 1| \le Cn^{-\frac{3}{4}}.$$

(iv) *If* $\nu \in (0,1)\cup(1,+\infty)$ *and* $p = \nu$, *let* $\alpha_n' = \frac{a_n^{-1}b_n}{4(\nu+2)}$ *and* $x \ge -\alpha_n'$, *and then*

$$\left(1 - G_\nu(z_{n,p}(x))\right) \le Cn^{-\frac{3}{4}}, \quad |B_n(x) - 1| \le Cn^{-\frac{1}{4}}.$$

Proof: (i) If $\nu = 1$ and $p \ne 1$, since $x \ge -\alpha_n'$, then

$$\begin{aligned}1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha_n'))\\ &\le \frac{1}{2}\exp\left\{-\log\frac{n}{2}\left(1 - \frac{1}{\log\frac{n}{2}}\alpha_n' - \frac{p}{(\log\frac{n}{2})^2}(\alpha_n')^2\right)\right\} \le \exp\left\{-\frac{5}{16}\log\frac{n}{2}\right\} \le Cn^{-\frac{11}{16}},\end{aligned}$$

and

$$|B_n(x) - 1| = \left|\exp\{-R_n(x)\} - 1\right| \le R_n(x) \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2G_\nu(z_{n,p}(x))} \le Cn^{-\frac{3}{8}}.$$

Therefore, we can get (i).

(ii) If $\nu > 1$, since $a_n^{-1}b_n \sim \nu\log n$,

$$\begin{aligned}1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha_n'))\\ &\le n^{-1}\frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}b_n^{1-\nu}\left(1 - pa_nb_n^{-1}\alpha_n' \cdot \frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{1-\nu}{p}}\\ &\quad\times\exp\left\{-\frac{b_n^\nu}{2\lambda^\nu}\left(1 - pa_nb_n^{-1}\alpha_n' \cdot \frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{\nu}{p}}\right\}\\ &\le n^{-1}\left(1 - pa_nb_n^{-1}\alpha_n' \cdot \frac{r_n}{1+\delta_na_nb_n^{-1}}\right)^{\frac{1-\nu}{p}}\exp\left(\frac{r_n}{1+\delta_na_nb_n^{-1}}\alpha_n' + pa_nb_n^{-1}(\alpha_n')^2\right)\\ &\le Cn^{-1}\exp\left(\frac{r_n}{1+\delta_na_nb_n^{-1}}\alpha_n' + pa_nb_n^{-1}(\alpha_n')^2\right)\\ &\le Cn^{-1}\exp\left\{\frac{a_n^{-1}b_n}{16\nu}\right\} \le Cn^{-\frac{15}{16}},\end{aligned}$$

and

$$|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2G_\nu(z_{n,p}(x))} < Cn^{-\frac{3}{4}}.$$

In case of $0 < \nu < 1$, noting that $a_n^{-1} b_n \sim \nu \log n$, we have

$$
\begin{aligned}
1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha_n')) \\
&\le n^{-1} \frac{2^{-\frac{1}{\nu}} \lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} b_n^{1-\nu} \left(1 - p a_n b_n^{-1} \alpha_n' \frac{r_n}{1 + \delta_n a_n b_n^{-1}}\right)^{\frac{1-\nu}{p}} \\
&\quad \times \exp\left[- \frac{b_n^\nu \left(1 - p a_n b_n^{-1} \alpha_n' \frac{r_n}{1+\delta_n a_n b_n^{-1}}\right)^{\frac{\nu}{p}}}{2\lambda^\nu} \left(1 + \delta_n a_n b_n^{-1}\right)^\nu \right] \\
&\quad \times \left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu} b_n^{-\nu} \left(1 - p a_n b_n^{-1} \alpha_n' \frac{r_n}{1 + \delta_n a_n b_n^{-1}}\right)^{-\frac{\nu}{p}} \left(1 + \delta_n a_n b_n^{-1}\right)^\nu \right)^{-1} \\
&\le C n^{-1} \exp\left(\alpha_n' + p a_n b_n^{-1} (\alpha_n')^2\right) \le C n^{-1} \exp\left\{ \frac{a_n^{-1} b_n}{16\nu} \right\} \le C n^{-\frac{15}{16}},
\end{aligned}
$$

and

$$
|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2 G_\nu(z_{n,p}(x))} < C n^{-\frac{3}{4}}.
$$

Therefore, we can get (ii).

(iii) If $p \ne \nu$, in case of $\nu > 1$ and noting that $a_n^{-1} b_n \sim \nu \log n$, we have

$$
\begin{aligned}
1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha_n')) \\
&\le n^{-1} \frac{2^{-\frac{1}{\nu}} \lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} b_n^{1-\nu} \left(1 - p a_n b_n^{-1} \alpha_n'\right)^{\frac{1-\nu}{p}} \exp\left[- \frac{b_n^\nu (1 - p a_n b_n^{-1} \alpha_n')^{\frac{\nu}{p}}}{2\lambda^\nu} \right] \\
&\le n^{-1} \left(1 - p a_n b_n^{-1} \alpha_n'\right)^{\frac{1-\nu}{p}} \exp\left(\alpha_n' + p a_n b_n^{-1} (\alpha_n')^2\right) \\
&\le C n^{-1} \exp\left(\alpha_n' + p a_n b_n^{-1} (\alpha_n')^2\right) \le C n^{-1} \exp\left\{ \frac{a_n^{-1} b_n}{16\nu} \right\} \le C n^{-\frac{15}{16}},
\end{aligned}
$$

and

$$
|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2 G_\nu(z_{n,p}(x))} < C n^{-\frac{3}{4}}.
$$

If $p \ne \nu$, in case of $0 < \nu < 1$, noting that $a_n^{-1} b_n \sim \nu \log n$, we have

$$
\begin{aligned}
1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha_n')) \\
&\le n \frac{2^{-\frac{1}{\nu}} \lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})} b_n^{1-\nu} \left(1 - p a_n b_n^{-1} \alpha_n'\right)^{\frac{1-\nu}{p}} \exp\left[- \frac{b_n^\nu (1 - p a_n b_n^{-1} \alpha_n')^{\frac{\nu}{p}}}{2\lambda^\nu} \right] \\
&\quad \times \left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu} b_n^{-\nu} \left(1 - p a_n b_n^{-1} \alpha_n'\right)^{-\frac{\nu}{p}} \right)^{-1} \\
&\le C n^{-1} \exp\left(\alpha_n' + p a_n b_n^{-1} (\alpha_n')^2\right) \le C n^{-1} \exp\left\{ \frac{a_n^{-1} b_n}{16\nu} \right\} \le C n^{-\frac{15}{16}},
\end{aligned}
$$

and

$$
|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2 G_\nu(z_{n,p}(x))} < C n^{-\frac{3}{4}}.
$$

Therefore, we can get (iii).

(iv) If $p = \nu$, in case of $\nu > 1$, noting that $a_n^{-1} b_n \sim \nu \log n$, we have

$$\begin{aligned} &1 - G_\nu(z_{n,p}(x)) \\ &\quad \le 1 - G_\nu(z_{n,p}(-\alpha'_n)) \\ &\quad \le n^{-1}\left[1 - \nu a_n b_n^{-1}\alpha'_n - \nu(\nu-1)(a_n b_n^{-1})^2(1-\alpha'_n)\right]^{\frac{1-\nu}{\nu}} \exp\left\{\alpha'_n + (\nu-1)a_n b_n^{-1}(1-\alpha'_n)\right\} \\ &\quad \le Cn^{-1}\exp\{\alpha'_n + (\nu-1)a_n b_n^{-1}(1-\alpha'_n)\} \le Cn^{-1}\exp\left\{\frac{a_n^{-1}b_n}{4\nu}\right\} \le Cn^{-\frac{3}{4}}, \end{aligned}$$

and

$$|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2G_\nu(z_{n,p}(x))} < Cn^{-\frac{1}{4}}.$$

If $p = \nu$, in case of $0 < \nu < 1$, noting that $a_n^{-1} b_n \sim \nu \log n$, we have

$$\begin{aligned} 1 - G_\nu(z_{n,p}(x)) &\le 1 - G_\nu(z_{n,p}(-\alpha'_n)) \\ &\le n^{-1}\left[1 - \nu a_n b_n^{-1}\alpha'_n + \nu(1-\nu)(a_n b_n^{-1})^2(1-\alpha'_n)\right]^{\frac{1-\nu}{\nu}} \\ &\quad \times \exp\left\{\alpha'_n + (1-\nu)a_n b_n^{-1}(1-\alpha'_n))\right\} \\ &\quad \times \left[1 + \frac{2(\nu-1)\lambda^\nu}{\nu} b_n^{-\nu}\left[1 - \nu a_n b_n^{-1}\alpha'_n + \nu(1-\nu)(a_n b_n^{-1})^2(1-\alpha'_n)\right]^{-1}\right]^{-1} \\ &\le Cn^{-1}\exp\left\{\alpha'_n - (1-\nu)a_n b_n^{-1}(1-\alpha'_n)\right\} \le Cn^{-1}\exp\left\{\frac{a_n^{-1}b_n}{4\nu}\right\} \le Cn^{-\frac{3}{4}}, \end{aligned}$$

and

$$|B_n(x) - 1| \le \frac{n[1 - G_\nu(z_{n,p}(x))]^2}{2G_\nu(z_{n,p}(x))} < Cn^{-\frac{1}{4}}.$$

The desired result (iv) follows. ■

5. Proofs

Proof of Theorem 2.1: (i) Let $z_{n,1}(x) = (\alpha_n^* x + \beta_n^*)$ with α_n^* and β_n^* given by Theorem 2.1 (i). By using (19) and (35), and some tedious calculation we have

$$\begin{aligned} &\mathrm{P}\left(|M_{n,r}| \le z_{n,1}(x)\right) - \Lambda_r(x) \\ &\quad = \Lambda(x)\frac{(r-1)\mathrm{e}^x - 1}{2n\{(r-1)!\}}\mathrm{e}^{-(r+1)x} + \frac{\mathrm{e}^{-(r+2)x}}{24n^2\{(r-1)!\}}\Big[(-3r^3 + 10r^2 - 9r + 2)\mathrm{e}^{2x} \\ &\qquad + (9r^2 - 11r + 2)\mathrm{e}^x + 3\mathrm{e}^{-x} - 9r + 1\Big]\Lambda(x) + o(n^{-2}). \end{aligned} \tag{44}$$

Hence, the desired results follow from (44).

(ii) Let $z_{n,p}(x) = (\alpha_n^* x + \beta_n^*)^{1/p}$ with α_n^* and β_n^* given by Theorem 2.1(ii). By using (20) we have

$$1 - \theta_{n,1}(x) = \frac{(1-p)x^2}{2\log\frac{n}{2}} - \frac{(1-p)[3(1-p)x - 4(1-2p)]x^3}{24(\log\frac{n}{2})^2} + o\big((\log n)^{-2}\big). \tag{45}$$

It follows from Lemma 4.3 and (45) that

$$\begin{aligned} &\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x) \\ &\quad = \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,1}(x))(r - 1 - \mathrm{e}^{-x})\right](1-\theta_{n,1}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1}) \\ &\quad = \Lambda(x)\left\{\frac{(1-p)x^2}{2\log\frac{n}{2}} + \frac{(1-p)x^3}{24(\log\frac{n}{2})^2}\Big[4(1-2p) - 3(1-p)rx + 3(1-p)x\mathrm{e}^{-x}\Big]\right\}\frac{\mathrm{e}^{-rx}}{(r-1)!} + o\big((\log n)^{-2}\big). \end{aligned} \tag{46}$$

Hence, following (60) we get the desired results.

(iii) Let $z_{n,p}(x) = (\alpha_n^* x + \beta_n^*)^{1/p}$ with α_n^* and β_n^* given by (iii) of Theorem 2.1. By using (21) we have

$$1 - \theta_{n,\nu}(x) = \frac{(1-\nu^{-1})^3(\log\log n)^2}{2\log n} - \frac{(1-\nu^{-1})^2\left(1 + x - \log 2\Gamma\left(\frac{1}{\nu}\right)\right)\log\log n}{\log n} + o\left(\frac{\log\log n}{\log n}\right). \tag{47}$$

It follows from Lemma 4.3 and (47) that

$$\begin{aligned} &\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x) \\ &= \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,\nu}(x))(r - 1 - \mathrm{e}^{-x})\right](1-\theta_{n,\nu}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1}) \\ &= \Lambda(x)\left[\frac{(1-\nu^{-1})^3(\log\log n)^2}{2\log n} - \frac{(1-\nu^{-1})^2(1 + x - \log 2\Gamma(\frac{1}{\nu}))\log\log n}{\log n}\right]\frac{\mathrm{e}^{-rx}}{(r-1)!} + o\left(\frac{\log\log n}{\log n}\right). \end{aligned} \tag{48}$$

Therefore, following (61), we get the desired results.

(iv) Let $z_{n,p}(x) = (c_n x + b_n)^{1/p}$ with c_n and d_n given by (iv) of Theorem 2.1 and $h_\nu(x)$, $q_\nu(x)$ are given by (10) and (11). By using (22) we have

$$1 - \theta_{n,\nu}(x) = h_\nu(x)\mathrm{e}^x b_n^{-\nu} + q_\nu(x)\mathrm{e}^x b_n^{-2\nu} + o(b_n^{-2\nu}). \tag{49}$$

It follows from Lemma 4.3 and (49) that

$$\begin{aligned} &\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x) \\ &= \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,\nu}(x))(r - 1 - \mathrm{e}^{-x})\right](1-\theta_{n,\nu}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1}) \\ &= \Lambda(x)\left[h_\nu(x)b_n^{-\nu} + \left(q_\nu(x) + \frac{1-(r-1)\mathrm{e}^x}{2}h_\nu^2\right)b_n^{-2\nu}\right]\frac{\mathrm{e}^{-(r-1)x}}{(r-1)!} + o(b_n^{-2\nu}), \end{aligned} \tag{50}$$

implying the desired results.

(v) Let $z_{n,p}(x) = (c_n^* x + d_n^*)^{1/\nu}$ with c_n^* and d_n^* given by Theorem 2.1(v) and $S_\nu(x)$ and $B(x)$ be given by (12) and (13). By using (23) we have

$$1 - \theta_{n,\nu}(x) = S_\nu(x)\mathrm{e}^x b_n^{-2\nu} + B(x)\mathrm{e}^x b_n^{-3\nu} + o(b_n^{-3\nu}). \tag{51}$$

It follows from Lemma 4.3 and (51) that

$$\begin{aligned} &\mathrm{P}\left(|M_{n,r}|^p \le \alpha_n^* x + \beta_n^*\right) - \Lambda_r(x) \\ &= \Lambda(x)\left[1 - \frac{1}{2}(1-\theta_{n,\nu}(x))(r - 1 - \mathrm{e}^{-x})\right](1-\theta_{n,\nu}(x))\frac{\mathrm{e}^{-rx}}{(r-1)!} + O(n^{-1}) \\ &= \Lambda(x)\left[S_\nu(x)b_n^{-2\nu} + B(x)b_n^{-3\nu}\right]\frac{\mathrm{e}^{-(r-1)x}}{(r-1)!} + o(b_n^{-3\nu}). \end{aligned} \tag{52}$$

Hence, by using (52), we derive the desired results. ∎

Proof of Theorem 2.2: The lower bounds are from Theorem 2.1. The rest is to derive the upper bounds. By the arguments similar to Hall (1980) and some tedious calculations, we have

$$\begin{aligned}\Delta_n(x) &= |P(|M_{n,r}| \le z_{n,p}(x)) - \Lambda_r(x)| \\ &= \left|\sum_{j=0}^{r-1} \frac{G_\nu^{n-j}(z_{n,p}(x))[n(1-G_\nu(z_{n,p}(x)))]^j}{j!} - \Lambda_r(x) + o(n^{-1})\right| \\ &\le \left|\sum_{j=0}^{r-1} G_\nu^{n-j}(z_{n,p}(x))\left[n\left(1-G_\nu^{n-j}(z_{n,p}(x))\right)\right]^j - \Lambda(x)\mathrm{e}^{-jx}\right| + C(n^{-1}) \\ &\le \sum_{j=0}^{r-1} |G_\nu^{n-j}(z_{n,p}(x)) - \Lambda(x)|\left[n\left(1-G_\nu^{n-j}(z_{n,p}(x))\right)\right]^j \\ &\quad + \Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1-G_\nu^{n-j}(z_{n,p}(x))\right)\right]^j - \mathrm{e}^{-jx}\right| + C(n^{-1}). \end{aligned} \tag{53}$$

For $0 \le j < r$,

$$\begin{aligned}\left|G_\nu^{n-j}(z_{n,p}(x)) - \Lambda(x)\right| &\le \left|G_\nu^{n-j}(z_{n,p}(x)) - G_\nu^{n}(z_{n'p}(x))\right| + \left|G_\nu^{n}(z_{n,p}(x)) - \Lambda(x)\right| \\ &\le |G_\nu^{n}(z_{n,p}(x)) - \Lambda(x)| + o\left(\frac{1}{n}\right). \end{aligned} \tag{54}$$

Note that

$$\begin{aligned}\left|G_\nu^{n}(z_{n,p}(x)) - \Lambda(x)\right| &< \Lambda(x)|A_n(x)-1| + \Lambda(x)|B_n(x)-1| \\ &< \left|\mathrm{e}^{-x} - n\left(1-G_\nu(z_{n,p}(x))\right)\right| \exp\left\{-\mathrm{e}^{-x} + \left|\mathrm{e}^{-x} - n\left(1-G_\nu(z_{n,p}(x))\right)\right|\right\} \\ &\quad + \Lambda(x)\,|B_n(x)-1|. \end{aligned} \tag{55}$$

(i) If $\nu = 1$ and $p = 1$, by (19) and (53)–(55), we have

$$\Delta_n(x) \le \sum_{j=0}^{r-1} \mathrm{e}^{-jx}\Lambda(x)|B_n(x)-1| + C(n^{-1})$$

and $|B_n(x) - 1| \le \mathrm{e}^{-2x}/n$. Therefore, combining Theorem 2.1 (i), the assertion of (i) can be proved.

(ii) If $\nu = 1$ and $p \ne 1$, by Lemma 4.5 (i) for $|x| \le \alpha'_n$ we have

$$\begin{aligned}&\Lambda(x)\sum_{j=0}^{r-1}\left|[n\left(1-G_\nu(z_{n,p}(x))\right)]^j - \mathrm{e}^{-jx}\right| \\ &< \Lambda(x)\sum_{j=1}^{r-1}\left|n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x}\right| \times |\mathrm{e}^{-x} + (n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x})\theta|^{j-1} \qquad (0<\theta<1) \\ &\le C\Lambda(x)\left|n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x}\right|\left[\mathrm{e}^{-x} + \left|n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x}\right| + 1\right]^{r-2} \\ &\le C\Lambda(x)\left|n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x}\right|\left[\mathrm{e}^{-(r-2)x} + \left|n\left(1-G_\nu(z_{n,p}(x))\right) - \mathrm{e}^{-x}\right|^{r-2} + 1\right] \\ &\le \frac{C}{\log n}. \end{aligned} \tag{56}$$

For $x \ge \alpha'_n$, by the arguments similar to those used in (56), it follows from Lemma 4.6(i) that

$$\Lambda(x)\sum_{j=0}^{r-1}\left|[n\Psi_n(x)]^j - \mathrm{e}^{-jx}\right| \le \frac{C}{\log n}.$$

It follows from (53)–(55) that $\Delta_n(x) \le \frac{C}{\log n}$ as $x \ge 0$.

Suppose that $-\beta_n \le x \le 0$, in view of (54) and Lemma 4.7(i),

$$\left|G_\nu^n(z_{n,p}(x)) - \Lambda(x)\right| \le C\left\{\frac{1}{\log\frac{n}{2}}\exp\left(-\frac{1}{8}\mathrm{e}^{|x|}\right) + \exp(-\mathrm{e}^{|x|})n^{-\frac{3}{4}}\right\}.$$

It follows from (3.19) that

$$\sum_{j=0}^{r-1}\left|G_\nu^{n-j}(z_{n,p}(x)) - \Lambda(x)\right|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j \le \frac{C}{\log n}. \tag{57}$$

Combining with Lemma 4.3(i) and Theorem 2.1(ii), we complete the proof of (ii).

(iii) If $\nu \in (0,1) \cup (1,+\infty)$, from Lemma 4.1 (iii), Lemma 4.5 (ii) and the similar arguments used in (56), for $|x| \le \alpha_n'$ recall that $\delta_n \sim \frac{(\nu-1)^3}{\nu^2}\frac{(\log\log n)^2}{\log n}$, and we have

$$\Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j - \mathrm{e}^{-jx}\right| \le C\delta_n.$$

Next, from Lemma 4.6(ii), for $x \ge \alpha_n'$,

$$\Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j - \mathrm{e}^{-jx}\right| \le Ca_n b_n^{-1}\nu.$$

Therefore for $x \ge 0$, it follows from (53)–(55) that $\Delta_n(x) \le C\delta_n$.

Note that for the case $-\beta_n' \le x \le 0$, Lemma 4.5 (ii) also implies that

$$\begin{aligned}\left|\mathrm{e}^{-x} - n\big(1 - G_\nu(z_{n,p}(x))\big)\right| &\le C(x^2 + |x| + 1)\exp\left(\frac{5}{4}|x|\right)\delta_n\\ &\le C(x^2 + |x| + 1)\exp\left(\frac{3}{4}|x|\right) \le C\exp\left(\frac{7}{8}|x|\right),\end{aligned}$$

and in view of (55) and Lemma 4.7(ii), we can get

$$\begin{aligned}&\left|G_\nu^n(z_{n,p}(x)) - \Lambda(x)\right|\\ &\le C\left\{(x^2 + |x| + 1)\delta_n\exp\left[\frac{5|x|}{4} - \mathrm{e}^{|x|} + C_1\exp\left(\frac{7}{8}|x|\right)\right] + \Lambda(x)n^{-\frac{3}{4}}\right\}\\ &\le C\left\{\delta_n\exp\left(-\frac{1}{8}\mathrm{e}^{|x|}\right) + \exp(-\mathrm{e}^{|x|})n^{-\frac{3}{4}}\right\}.\end{aligned}$$

Now, from (54) and by the similar arguments used in (57), we have

$$\sum_{j=0}^{r-1}|G^{n-j}(z_n) - \Lambda(x)|[n\Psi_n(x)]^j \le C\delta_n.$$

Hence, combining with Lemma 4.3(ii), Theorem 2.1(iii) and (53)–(55), (iii) can be proved.

(iv) If $\nu \in (0,1) \cup (1,+\infty)$ and $p \ne \nu$, by Lemma 4.5 (iii) and the similar arguments used in (56), for $|x| \le \alpha_n'$, we have

$$\Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j - \mathrm{e}^{-jx}\right| \le Ca_n b_n^{-1}\nu.$$

Next, from Lemma 4.6 (iii), for $x \ge \alpha_n'$,

$$\Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j - \mathrm{e}^{-jx}\right| \le Ca_n b_n^{-1}\nu.$$

Therefore for $x \ge 0$, it follows from (53)–(55) that $\Delta_n(x) \le C \cdot a_n b_n^{-1}\nu$.

Note that for the case $-\beta_n' \le x \le 0$, Lemma 4.5 (iii) also implies that

$$\left|e^{-x} - n\left(1 - G_\nu(z_{n,p}(x))\right)\right| \le C(x^2 + |x| + 1)\exp\left(\frac{5}{4}|x|\right) a_n b_n^{-1} \nu$$
$$\le C(x^2 + |x| + 1)\exp\left(\frac{3}{4}|x|\right) \le C\exp\left(\frac{7}{8}|x|\right),$$

and in view of (55) and Lemma 4.7 (iii), we can get

$$\left|G_\nu^n(z_{n,p}(x)) - \Lambda(x)\right|$$
$$\le C\left\{(x^2 + |x| + 1)a_n b_n^{-1}\nu \exp\left[\frac{5|x|}{4} - e^{|x|} + C_1\exp\left(\frac{7}{8}|x|\right)\right] + \Lambda(x)n^{-\frac{3}{4}}\right\}$$
$$\le C\left\{a_n b_n^{-1}\nu \exp\left(-\frac{1}{8}e^{|x|}\right) + \exp(-e^{|x|})n^{-\frac{3}{4}}\right\}.$$

Now, from (54) and by the similar arguments used in (57), we have

$$\sum_{j=0}^{r-1} |G^{n-j}(z_n) - \Lambda(x)|[n\Psi_n(x)]^j \le C a_n b_n^{-1}\nu.$$

Hence, combining with Lemma 4.3 (iii), Theorem 2.1 (iv) and (53)–(55), (iv) can be proved.

(v) If $\nu \in (0,1) \cup (1,+\infty)$ and $p = \nu$, for $x \ge 0$, it follows from Lemmas 4.5-4.6 (iv) and (53) that

$$\Delta_n(x) \le C\left[\sum_{j=0}^{r-1} |G_\nu^{n-j}(z_{n,p}(x)) - \Lambda(x)| + (a_n b_n^{-1}\nu)^2\right],$$

and combining (54)–(55), we can get $\Delta_n(x) \le C(a_n b_n^{-1}\nu)^2$.

Note that for $-\beta_n' \le x \le 0$, by Lemma 4.5 (iv) and the similar arguments used in (56),

$$\Lambda(x)\sum_{j=0}^{r-1}\left|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j - e^{-jx}\right| \le C(a_n b_n^{-1}\nu)^2,$$

and in view of (55) and Lemma 4.7 (iv),

$$\left|G_\nu^n(z_{n,p}(x)) - \Lambda(x)\right|$$
$$\le C\{(|x| + 1)(a_n b_n^{-1}\nu)^2 \exp[|x| - e^{|x|} + C] + \Lambda(x)n^{-\frac{1}{4}}\}$$
$$\le C\{(a_n b_n^{-1}\nu)^2 \exp\left(-e^{|x|}\right) + \exp(-e^{|x|})n^{-\frac{1}{4}}\}.$$

Finally, by (54) and the similar arguments used in (57), we have

$$\sum_{j=0}^{r-1} |G_\nu^{n-j}(z_{n,p}(x)) - \Lambda(x)|\left[n\left(1 - G_\nu(z_{n,p}(x))\right)\right]^j \le C(a_n b_n^{-1}\nu)^2.$$

Combining with Lemma 4.3(iv) and Theorem 2.1(v), we finish the proof of (v). ■

Proof of Theorem 2.3: For the r-th largest order statistics, we have

$$\mathrm{P}\left(M_{n,r} \le x\right) = \sum_{j=0}^{r-1} \frac{n!}{j!(n-j)!} G_\nu^{(n-j)}(x)\left(1 - G_\nu(x)\right)^j, \tag{58}$$

and for any $0 < z < n$,

$$\exp\left[-z - \frac{z^2}{2n}\left(1 - \frac{z}{n}\right)^{-1}\right] < \left(1 - \frac{z}{n}\right)^n < e^{-z}. \tag{59}$$

We first consider the bounds of $\mathrm{P}(M_n \leq x)$ as $\nu > 1$. It follows from (59) and Lemma 4.4(ii) that

$$\mathrm{P}\left(M_n \leq x\right) < \left[1 - \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}x^{1-\nu}\exp\left(-\frac{x^\nu}{2\lambda^\nu}\right)\left(1 - \frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)\right]^n$$
$$< \exp\left\{-\lambda_n(x)\left(1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu}\right)\right\}.$$

For the lower bound, note that

$$1 - G_\nu(x) = \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}x^{1-\nu}\exp\left(-\frac{x^\nu}{2\lambda^\nu}\right)\left[1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu} + \frac{4(\nu-1)(2\nu-1)}{\nu^2}\lambda^{2\nu}x^{-2\nu}g(x)\right] \tag{60}$$

with

$$1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu} + \frac{4(\nu-1)(2\nu-1)}{\nu^2}\lambda^{2\nu}x^{-2\nu}g(x) < 1.$$

By (59) and (60) we have

$$\mathrm{P}\left(M_n \leq x\right) = \left\{1 - \frac{\lambda_n(x)\left[1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu} + \frac{4(\nu-1)(2\nu-1)}{\nu^2}\lambda^{2\nu}x^{-2\nu}g(x)\right]}{n}\right\}^n$$
$$> \exp\left\{-z - \frac{z^2}{2n}\left(1 - \frac{z}{n}\right)^{-1}\right\}, \tag{61}$$

where $z = \lambda_n(x)\left[1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu} + \frac{4(\nu-1)(2\nu-1)}{\nu^2}\lambda^{2\nu}x^{-2\nu}g(x)\right]$. Noting that for $x \geq b_n$ and $0 < z < \lambda_n(x) \leq 1$, we have

$$\frac{z^2}{2n}\left(1 - \frac{z}{n}\right)^{-1} \leq \frac{z^2}{2n}\left(1 - \frac{1}{n}\right)^{-1} = \frac{z^2}{2(n-1)}. \tag{62}$$

It follows from (61) and (62) that

$$\mathrm{P}\left(M_n \leq x\right) > \exp\left\{-\lambda_n(x)\left(1 - \frac{2(\nu-1)}{\nu}\lambda^\nu x^{-\nu} + \frac{4(\nu-1)(2\nu-1)}{\nu^2}\lambda^{2\nu}x^{-2\nu} + \frac{\lambda_n(x)}{2(n-1)}\right)\right\}, \tag{63}$$

which is the desired lower bound.

For the bounds of the r-th order statistics as $\nu > 1$, by (58), (59) and Lemma 4.4(ii), we have

$$\mathrm{P}\left(M_{n,r} \leq x\right) < \sum_{j=0}^{r-1}\frac{n!}{j!(n-j)!}\left[1 - \frac{\lambda_n(x)}{n}\left(1 - \frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)\right]^{n-j}\left(\frac{\lambda_n(x)}{n}\right)^j$$
$$< \sum_{j=0}^{r-1}\frac{\lambda_n^j(x)}{j!}\exp\left\{-\left(1 - \frac{j}{n}\right)\lambda_n(x)\left(1 - \frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)\right\}.$$

For the lower bound, by (63), (59) and Lemma 4.4(ii), we have

$$\mathrm{P}\left(M_{n,r} \leq x\right) > \sum_{j=0}^{r-1}\frac{n!}{j!(n-j)!}\exp\left\{-\left(1 - \frac{j}{n}\right)\lambda_n(x)\right\}\left[\frac{\lambda_n(x)}{n}\left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)^{-1}\right]^j$$
$$> \sum_{j=0}^{r-1}\frac{\lambda_n^j(x)\left(1 + \frac{2(\nu-1)\lambda^\nu x^{-\nu}}{\nu}\right)^{-j}}{j!}\exp\left\{-\left(1 - \frac{j}{n}\right)\lambda_n(x) - \frac{j(j-1)}{n}\right\}.$$

The remainder is to consider the bounds as $0 < \nu < 1$. For the bounds of $\mathrm{P}(M_n \leq x)$ as $0 < \nu < 1$, it follows from (59) and Lemma 4.4(i) that

$$\mathrm{P}\left(M_n \leq x\right) < \left[1 - \frac{2^{-\frac{1}{\nu}}\lambda^{\nu-1}}{\Gamma(\frac{1}{\nu})}x^{1-\nu}\exp\left(-\frac{x^\nu}{2\lambda^\nu}\right)\right]^n < \exp\{-\lambda_n(x)\}$$

and

$$\mathrm{P}\left(M_n \leq x\right) > \left[1 - \frac{2^{-\frac{1}{v}}\lambda^{v-1}}{\Gamma(\frac{1}{v})}x^{1-v}\exp\left(-\frac{x^v}{2\lambda^v}\right)\left(1+\frac{2(v-1)}{v}\lambda^v x^{-v}\right)^{-1}\right]^n$$
$$> \exp\left\{-z - \frac{z^2}{2n}\left(1-\frac{z}{n}\right)^{-1}\right\},$$

where $z = \lambda_n(x)(1 + \frac{2(v-1)}{v}\lambda^v x^{-v})^{-1}$.

For the bounds of the r-th order statistics, it follows from (59) and Lemma 4.4(i) that

$$\mathrm{P}\left(M_{n,r} \leq x\right) < \sum_{j=0}^{r-1}\frac{n!}{j!(n-j)!}\left[1-\frac{\lambda_n(x)}{n}\right]^{n-j}\left(\frac{\lambda_n(x)}{n}\left(1+\frac{2(v-1)}{v}\lambda^v x^{-v}\right)^{-1}\right)^j$$
$$< \sum_{j=0}^{r-1}\frac{\lambda_n^j(x)\left(1+\frac{2(v-1)\lambda^v x^{-v}}{v}\right)^{-j}}{j!}\exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x)\right\}.$$

By similar arguments used in (63), one can show that

$$\mathrm{P}\left(M_{n,r} \leq x\right) > \sum_{j=0}^{r-1}\frac{n!}{j!(n-j)!}\exp\left(-\left(1-\frac{j}{n}\right)\lambda_n(x)\left(1+\frac{2(v-1)}{v}\lambda^v x^{-v}\right)^{-1}\right)\left(\frac{\lambda_n(x)}{n}\right)^j$$
$$> \sum_{j=0}^{r-1}\frac{\lambda_n^j(x)}{j!}\exp\left\{-\left(1-\frac{j}{n}\right)\lambda_n(x)\left(1+\frac{2(v-1)\lambda^v x^{-v}}{v}\right)^{-1} - \frac{j(j-1)}{n}\right\}.$$

Thus, the proof is completed. ■

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