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# Time-dependent reliability analysis for repairable consecutive- $k$ -out-of- $n:F$ system

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## ABSTRACT

In a repairable consecutive  $C(k, n : F)$  system, after the system operates for a certain time, some components may fail, some failed components may be repaired and the state of the system may change. The models developed in the existing literature usually assume that the state of the system varies over time depending on the values of  $n$  and  $k$  and the state of the system is known. Since the system reliability will vary over time, it is of great interest to analyse the time-dependent system reliability. In this paper, we develop a novel and simple method that utilizes the eigenvalues of the transition rate matrix of the system for the computation of time-dependent system reliability when the system state is known. In addition, the transition performance probabilities of the system from a known state to the possible states are also analysed. Computational results are presented to illustrate the applicability and accuracy of the proposed method.

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## 1. Introduction

In many engineering applications, consecutive- $k$ -out-of- $n$  systems are encountered and the evaluation of the system reliability is of interest. For example, consecutive- $k$ -out-of- $n$  systems appear in telecommunications, microwave relay stations, oil pipeline systems, vacuum systems in accelerators, computer ring networks and spacecraft relay stations. These systems can be classified with respect to the logical or physical connections among components as either linear or circular and the functioning principle as either failed ( $F$ ) or good ( $G$ ). A consecutive- $k$ -out-of- $n:F$  system, or a  $C(k, n : F)$  system for short, consists of an ordered sequence of  $n$  components such that the system fails if and only if at least  $k$  consecutive components fail. Another special type of system related to the  $C(k, n : F)$  system is the consecutive- $k$ -out-of- $n:G$  system, or a  $C(k, n : G)$  system for short. A  $C(k, n : G)$  system consists of an ordered sequence of  $n$  components such that the system works if and only if at least  $k$  consecutive components work. Both  $C(k, n : F)$  system and  $C(k, n : G)$  system can be either linear or circular. The first report for these systems was presented by Kontoleon (1980). Then, much research work has been done on consecutive- $k$ -out-of- $n$  systems in the literature since Kontoleon (1980). For further reference, see, for example, Chiang and Niu (1981), Bollinger and Salvia (1982), Derman et al. (1982), Zuo and Kuo (1990), Chang et al. (2000), Kuo and Zuo (2003), Eryilmaz (2010) and Gökdereli and Gurcan (2016). However, in the aforementioned work,

it was assumed that the components in the system are non-repairable. If we consider that components are repairable, then such a system is called a repairable system.

Repairable  $C(k, n : F)$  and  $C(k, n : G)$  systems have also been studied extensively in the literature (see, for example, Cheng & Zhang, 2001; Eryilmaz, 2014; Guan & Wu, 2006; Hongda et al., 2019; Lam & Ng, 2001; Lam & Zhang, 1999, 2000; Wang et al., 2021; Yam et al., 2003; Yuan & Cui, 2013; Zhang & Lam, 1998; Zhang & Wang, 1996; Zhang et al., 1998, 2000). In most of these research papers, it is assumed that the working time and the repair time of the components are both exponentially distributed and the repair completely restores all the properties of failed components. Moreover, when the system is in a failed state, each failed component in the system is classified as either a key component or an ordinary component. A failed component is called a key component if repairing this component will return the system to a working state (Zhang et al., 2000). The key components have a higher priority in repair than ordinary components. Then, by using the definition of generalized transition probability, for both linear and circular systems, the state transition rate matrix of the system can be derived. Usually, the Laplace transform method or Runge-Kutta method is used to obtain the reliability of a consecutive- $k$ -out-of- $n$  system.

In most of the models for repairable  $C(k, n : F)$  and  $C(k, n : G)$  systems in the existing literatures, the system reliability is calculated by assuming that all

components are working in the system within a certain period of time. However, in a repairable system, the state of the system may change over time after the system starts operating. In this paper, we study the repairable linear and circular  $C(k, n : F)$  systems by assuming that the state of the system is known. We aim to provide a simple method to determine the time-dependent system reliability and analyse the transition performance probabilities of the system when the system state is known. We have first modified the transition rate matrix considered in Cheng and Zhang (2001) and Yam et al. (2003) and then proposed a novel method to compute the time-dependent system reliability using the eigenvalues of the transition rate matrix. The proposed method can be served as a simple alternative method to the Laplace transform and the Runge-Kutta method for the determination of time-dependent reliability and transition performance probabilities of the system. Also, it can be applied to both repairable and irreparable systems. In Section 4.1 and 4.2, to show the accuracy of our method we consider the case that the system is irreparable.

This paper is organized as follows. In Section 2, the preliminary assumptions, the state transition probabilities and the transition rate matrix are presented for the repairable  $C(k, n : F)$  system with  $n$  linearly and circularly arranged components. In Section 3, we present the proposed method for evaluation of the system reliability and transition performance probabilities when the state of the system is known. In Section 4, some numerical examples are provided to illustrate the usefulness of the proposed method for different linear and circular repairable  $C(k, n : F)$  systems. In Section 5, some concluding remarks are provided.

## 2. Repairable $C(k, n : F)$ systems and model assumptions

Here is the list of the model assumptions used throughout this paper:

**Assumption 2.1:** The system is a linear or circular repairable  $C(k, n : F)$  system with identical components.

**Assumption 2.2:** At time  $t = 0$ , all components are new and working.

**Assumption 2.3:** There is a single repairman in the system and the repair completely restores all properties of failed components.

**Assumption 2.4:** The working time and the repair time of the components are both exponentially distributed with the parameters  $\lambda > 0$  and  $\mu > 0$ , respectively.

**Assumption 2.5:** While the system is in the failed state, the components that have not failed will not fail.

**Assumption 2.6:** The probability that two or more than two components fail or complete their repairs simultaneously in a very small time interval is negligible.

Based on these model assumptions, the state of the system at time  $t$ , denoted by  $N(t)$ , varies depending on the values of  $n$  and  $k$ . For linear and circular repairable  $C(k, n : F)$  systems, the possible states can be expressed as follows:

$$N(t) = \begin{cases} 0, & \text{if at time } t, \text{ all components work the} \\ & \text{system works,} \\ -1, & \text{if at time } t, \text{ one component fails the} \\ & \text{system works,} \\ \vdots & \\ -k, & \text{if at time } t, k \text{ components fail the} \\ & \text{system works,} \\ \vdots & \\ -d, & \text{if at time } t, d \text{ components fail the} \\ & \text{system works,} \\ 1, & \text{if at time } t, l \text{ components fail the} \\ & \text{system fails,} \end{cases}$$

where  $l = k, k + 1, \dots, d + 1$  and  $d$  takes different values depending on whether the system is linear or circular. If the system is linear,  $d = n - \lfloor n/k \rfloor$  and  $\lfloor x \rfloor$  is the largest integer less than or equal to value  $x$ . If the system is circular,  $d = n - \lceil n/k \rceil$  and  $\lceil x \rceil$  is the smallest integer greater than or equal to value  $x$ .

In our proposed method, all states  $(k, k + 1, \dots, d + 1)$  that caused the system to fail were considered as a single state. Then,  $\{N(t), t \geq 0\}$  is a continuous-time homogeneous Markov process with state space  $\Omega = \{0, -1, \dots, -d, 1\}$ . The set of working state is  $W = \{0, -1, \dots, -d\}$  and the set of failed states is  $F = \{1\}$ .

According to  $N(t)$ , the generalized transition probability from state  $i$  to state  $j$  in time  $\Delta t$  is defined as

$$p_{ij}(\Delta t) = P\{N(t + \Delta t) = j \mid N(t) = i\}. \quad (1)$$

In order to obtain the transition rate matrix  $Q$ , we need to calculate the state transition probabilities of the system. These state transition probabilities are presented as follows (for the detailed derivations, one can refer to Cheng & Zhang, 2001; Yam et al., 2003):

For  $i = 0$  and  $j \in \Omega$  in (1), we have

$$p_{0j}(\Delta t) = \begin{cases} 1 - n\lambda\Delta t + o(\Delta t), & j = 0, \\ n\lambda\Delta t + o(\Delta t), & j = -1, \\ o(\Delta t), & \text{for all other } j \text{ values.} \end{cases} \quad (2)$$

For  $1 \leq i \leq d$  and  $j \in \Omega$  in (1), we have

$$p_{-ij}(\Delta t) = \begin{cases} \mu \Delta t + o(\Delta t), & j = -(i-1), \\ (i+1) \frac{M_{-(i+1)}}{M_{-i}} & j = -(i+1), \\ \lambda \Delta t + o(\Delta t), & j = -i, \\ 1 - ((n-i)\lambda + \mu) \Delta t + o(\Delta t), & \text{for all other } j \text{ values.} \\ o(\Delta t), & \end{cases} \quad (3)$$

For an explain (3) with  $j = -(i+1)$ , refer to Section 4 of Guan and Wu (2006). In Equation (3),  $M_{-i}$  is the number of different possible cases which the system under consideration in state  $-i$ . We need to find a formula of  $M_{-i}$  for  $-i \in W$  and  $i \neq 0$ .

In linear  $C(k, n : F)$  systems

$$M_{-i} = M_{-i}^L = \sum_{r=0}^m (-1)^r \binom{n-i+1}{r} \binom{n-kr}{n-i},$$

where  $m = \min(n-i+1, [i/k])$ .

In circular  $C(k, n : F)$  systems

$$M_{-i} = M_{-i}^C = M_{-i}^L - \sum_{l=k}^{\min(i, 2(k-1))} (2k-1-l) \times \sum_{r=0}^{m_l} (-1)^r \binom{n-i-1}{r} \binom{n-l-kr-2}{i-l-kr},$$

where  $m_l = \min(n-i+1, [(i-l)/k])$ .

For  $1 \leq i \leq d$  and  $j = 1$  in (1), we have

$$p_{-i1}(\Delta t) = \left( (n-i) - (i+1) \frac{M_{-(i+1)}}{M_{-i}} \right) \lambda \Delta t + o(\Delta t). \quad (4)$$

For  $i = 1$  and  $j \in \Omega$  in (1), we have

$$p_{1j}(\Delta t) = o(\Delta t). \quad (5)$$

Using the state transition probabilities in the system in Equations (2)–(5), we can obtain the transition rate matrix  $Q$  as

$$Q = (q_{ij}),$$

where

$$q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{p_{ij}(\Delta t)}{\Delta t},$$

and

$$q_j = \lim_{\Delta t \rightarrow 0} \frac{1 - p_{jj}(\Delta t)}{\Delta t},$$

for  $i, j \in \Omega$  and  $i \neq j$ . Also,  $q_{jj} = -q_j, j \in \Omega$ . In the following section, we present a novel and simple method to compute the time-dependent system reliability by using the eigenvalues of the matrix  $Q$ .

### 3. Proposed method

Let  $\Upsilon(t) = (\Psi_{ij}(t); i, j \in \Omega \text{ and } t \geq 0)$  be a transition matrix function defining a continuous-time homogeneous Markov process, where  $\Psi_{ij}(t)$  represents the time-dependent performance probability of the system between known state  $i$  and possible state  $j$  at time  $t$ . If  $Q$  was a finite matrix the solution of

$$\frac{d}{dt} \Upsilon(t) = \Upsilon(t)Q$$

could be written down at once in the form

$$\Upsilon(t) = e^{Qt} = \sum_{k=0}^{\infty} \frac{(Qt)^k}{k!}, \quad Q^0 = I,$$

where  $I = (I_{ij})$  is the unit matrix for  $i, j \in \Omega$ .

The abovementioned matrix  $Q$  is a  $\xi \times \xi$  real square matrix, where  $\xi$  is equal to the number of elements of the state space  $\Omega$ . Then, there exists a singular value decomposition of  $Q$  of the form

$$Q = U \Sigma V^T,$$

where  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\xi-1}, \sigma_{\xi})$  has diagonal elements arranged in ascending order of magnitude and  $U = (u_1, u_2, \dots, u_{\xi})$  and  $V = (v_1, v_2, \dots, v_{\xi})$  are orthogonal. Now, consider the linear transformation of  $\xi$ -dimensional vectors defined by matrix  $Q$ . In this case,  $\Psi_{ij}(t)$  is found to be

$$\Psi_{ij}(t) = A_0^{ij} + A_1^{ij} e^{\sigma_1 t} + A_2^{ij} e^{\sigma_2 t} + \dots + A_{\xi-1}^{ij} e^{\sigma_{\xi-1} t}, \quad (6)$$

where  $A_0^{ij}, A_1^{ij}, \dots, A_{\xi-1}^{ij}$  are constant coefficients for  $i, j \in \Omega$  and  $\sigma_1, \sigma_2, \dots, \sigma_{\xi-1}$  are the non-zero eigenvalues of  $Q$ . To obtain the above constant coefficients, the following system of differential equations is solved:

$$\frac{d}{dt} \Upsilon^{(k)}(t=0) = Q^k, \quad (7)$$

for  $k = 0, 1, 2, \dots, \xi - 1$ . We can express (7) as

$$\begin{aligned} A_0^{ij} + A_1^{ij} + A_2^{ij} + \dots + A_{\xi-1}^{ij} &= (I_{ij}), \\ (\sigma_1) A_1^{ij} + (\sigma_2) A_2^{ij} + \dots + (\sigma_{\xi-1}) A_{\xi-1}^{ij} &= (q_{ij}), \\ (\sigma_1)^2 A_1^{ij} + (\sigma_2)^2 A_2^{ij} + \dots + (\sigma_{\xi-1})^2 A_{\xi-1}^{ij} &= (q_{ij})^2, \\ &\vdots \\ (\sigma_1)^{\xi-1} A_1^{ij} + (\sigma_2)^{\xi-1} A_2^{ij} + \dots + (\sigma_{\xi-1})^{\xi-1} A_{\xi-1}^{ij} &= (q_{ij})^{\xi-1}. \end{aligned}$$

According to the Equation (6) for  $j = 1$ , we can obtain the time-dependent system reliability at time  $t$  when the state of the system is known as

$$R_i(t) = 1 - \Psi_{i1}(t), \quad (8)$$

where  $i \in \Omega$ .

In the proposed method, we need to obtain the eigenvalues of the matrix  $Q$ . When  $\mu = 0$ , the eigenvalues of the  $Q$  can be easily calculated based on the parameter  $\lambda$ . However, obtaining the eigenvalues of the matrix  $Q$  can be difficult and complex in general based on the parameters  $\lambda$  and  $\mu$ . In such a case, we overcome this difficulty by providing the numerical values of the parameters to simplify the computation. It is noteworthy that the existing methods in the literature deal with the assumption that all components in the system are working, whereas such an assumption is not needed in our proposed method. Hence, the proposed method can be applied to a wider range of situations in system engineering.

#### 4. Numerical illustrations

In the following, numerical examples in the existing literature are used to illustrate the proposed method developed in Section 3.

##### 4.1. A linear repairable $C(2, 5 : F)$ system

In this section, we analyse a linear repairable  $C(2, 5 : F)$  system. First, we focus on obtaining the time-dependent transition performance probabilities and the system reliability under the condition that there is a single failed component in the system at time  $t$ . Then, to show the accuracy of the proposed method, the situation where  $\mu = 0$  is analysed. Based on the results presented in Section 2, the state space  $\Omega = \{0, -1, -2, -3, 1\}$ , the set of working state  $W = \{0, -1, -2, -3\}$  and the set of failed states  $F = \{1\}$  of the system. The  $Q$ -matrix can be obtained as

$$Q = \begin{pmatrix} -5\lambda & 5\lambda & 0 & 0 & 0 \\ \mu & (-\mu + 4\lambda) & \frac{12}{5}\lambda & 0 & \frac{8}{5}\lambda \\ 0 & \mu & -(\mu + 3\lambda) & \frac{1}{2}\lambda & \frac{5}{2}\lambda \\ 0 & 0 & \mu & -(\mu + 2\lambda) & 2\lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now consider the  $Q$ -matrix for  $\lambda = 0.5$  and  $\mu = 1.5$ . For  $i = -1$  and  $j \in \Omega$  in (6) we have

$$\begin{aligned} \Psi_{-1j}(t) = & A_0^{-1j} + A_1^{-1j}e^{\sigma_1 t} + A_2^{-1j}e^{\sigma_2 t} \\ & + A_3^{-1j}e^{\sigma_3 t} + A_4^{-1j}e^{\sigma_4 t}, \end{aligned} \quad (9)$$

where  $A_0^{-1j}$ ,  $A_1^{-1j}$ ,  $A_2^{-1j}$ ,  $A_3^{-1j}$  and  $A_4^{-1j}$  are constant coefficients and  $\sigma_1 = -5.5045$ ,  $\sigma_2 = -3.2078$ ,  $\sigma_3 = -2.1437$  and  $\sigma_4 = -0.6439$  are the non-zero eigenvalues of  $Q$ . To determine the above constant coefficients, Equation (7) has to be solved for  $j \in \Omega$  and  $k = 0, 1, 2, 3, 4$ . Finally, we have the following results for Equation (9).

For  $j = 0$ ,

$$\begin{aligned} \Psi_{-10}(t) = & -0.2859e^{-5.5045t} - 0.0545e^{-3.2078t} \\ & + 0.0295e^{-2.1437t} + 0.3209e^{-0.6439t}. \end{aligned} \quad (10)$$

For  $j = -1$ ,

$$\begin{aligned} \Psi_{-1-1}(t) = & 0.5726e^{-5.5045t} + 0.0257e^{-3.2078t} \\ & + 0.0047e^{-2.1437t} + 0.397e^{-0.6439t}. \end{aligned} \quad (11)$$

For  $j = -2$ ,

$$\begin{aligned} \Psi_{-1-2}(t) = & -0.2887e^{-5.5045t} + 0.096e^{-3.2078t} \\ & + 0.0284e^{-2.1437t} + 0.2212e^{-0.6439t}. \end{aligned} \quad (12)$$

For  $j = -3$ ,

$$\begin{aligned} \Psi_{-1-3}(t) = & 0.024e^{-5.5045t} - 0.0339e^{-3.2078t} \\ & - 0.0199e^{-2.1437t} + 0.0298e^{-0.6439t}. \end{aligned} \quad (13)$$

For  $j = 1$ ,

$$\begin{aligned} \Psi_{-11}(t) = & 1 - 0.022e^{-5.5045t} - 0.0332e^{-3.2078t} \\ & + 0.0241e^{-2.1437t} - 0.9689e^{-0.6439t}. \end{aligned} \quad (14)$$

Using Equations (10)–(14), for  $t = 0.5, 1.0, \dots, 5$ , we can obtain the time-dependent transition performance probabilities for the linear repairable  $C(2, 5 : F)$  system (see, Table 1). Moreover, using Equation (14) for  $i = -1$  in Equation (8), we can obtain the time-dependent reliability of the system when a single component is in the failed state at time  $t$ .

In order to demonstrate the accuracy of our method, we consider the case that  $\mu = 0$  and the system is linear irreparable  $C(2, 5 : F)$  system. The  $Q$ -matrix in this situation becomes

$$\tilde{Q} = \begin{pmatrix} -5\lambda & 5\lambda & 0 & 0 & 0 \\ 0 & -4\lambda & \frac{12}{5}\lambda & 0 & \frac{8}{5}\lambda \\ 0 & 0 & -3\lambda & \frac{1}{2}\lambda & \frac{5}{2}\lambda \\ 0 & 0 & 0 & -2\lambda & 2\lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assume that at time  $t$  all the five components are working. To obtain the time-dependent system reliability, let

**Table 1.** Time-dependent transition performance probabilities for the linear repairable  $C(2, 5 : F)$  system when one component fails.

$t$	$\Psi_{-10}(t)$	$\Psi_{-1-1}(t)$	$\Psi_{-1-2}(t)$	$\Psi_{-1-3}(t)$	$\Psi_{-11}(t)$
0.5	0.2100	0.3310	0.1515	0.0095	0.2980
1.0	0.1675	0.2124	0.1156	0.0120	0.4925
1.5	0.1224	0.1517	0.0838	0.0103	0.6319
2.0	0.0887	0.1096	0.0608	0.0079	0.7330
2.5	0.0642	0.0794	0.0441	0.0059	0.8064
3.0	0.0465	0.0575	0.0320	0.0043	0.8596
3.5	0.0337	0.0417	0.0232	0.0031	0.8983
4.0	0.0244	0.0302	0.0168	0.0023	0.9263
4.5	0.0177	0.0219	0.0122	0.0016	0.9466
5.0	0.0128	0.0159	0.0088	0.0012	0.9613



$i = 0$  and  $j = 1$  in Equation (6), then

$$\Psi_{01}(t) = A_0^{01} + A_1^{01}e^{\sigma_1 t} + A_2^{01}e^{\sigma_2 t} + A_3^{01}e^{\sigma_3 t} + A_4^{01}e^{\sigma_4 t}, \quad (15)$$

where  $A_0^{01}, A_1^{01}, A_2^{01}, A_3^{01}$  and  $A_4^{01}$  are constant coefficients and  $\sigma_1 = -5\lambda, \sigma_2 = -4\lambda, \sigma_3 = -3\lambda$  and  $\sigma_4 = -2\lambda$  are the non-zero eigenvalues of  $\tilde{Q}$ . To determine the constant coefficients, we have to solve the following system of differential equations:

$$\begin{aligned} A_0^{01} + A_1^{01} + A_2^{01} + A_3^{01} + A_4^{01} &= 0, \\ (-5\lambda)A_1^{01} + (-4\lambda)A_2^{01} + (-3\lambda)A_3^{01} + (-2\lambda)A_4^{01} &= 0, \\ (-5\lambda)^2 A_1^{01} + (-4\lambda)^2 A_2^{01} + (-3\lambda)^2 A_3^{01} + (-2\lambda)^2 A_4^{01} &= 8\lambda^2, \\ (-5\lambda)^3 A_1^{01} + (-4\lambda)^3 A_2^{01} + (-3\lambda)^3 A_3^{01} + (-2\lambda)^3 A_4^{01} &= -42\lambda^3, \\ (-5\lambda)^4 A_1^{01} + (-4\lambda)^4 A_2^{01} + (-3\lambda)^4 A_3^{01} + (-2\lambda)^4 A_4^{01} &= 140\lambda^4. \end{aligned}$$

The solutions of the above equations are  $A_0^{01} = 1, A_1^{01} = -1, A_2^{01} = 4, A_3^{01} = -3, A_4^{01} = -1$ . Finally, Equation (15) can be obtained as

$$\Psi_{01}(t) = 1 - e^{-5\lambda t} + 4e^{-4\lambda t} - 3e^{-3\lambda t} - e^{-2\lambda t}. \quad (16)$$

Using Equation (16) for  $i = 0$  in Equation (8), we can obtain the time-dependent reliability of the system as

$$\begin{aligned} R_0(t) &= 1 - \Psi_{01}(t) \\ &= e^{-2\lambda t} + 3e^{-3\lambda t} - 4e^{-4\lambda t} + e^{-5\lambda t}. \end{aligned} \quad (17)$$

Note that Equation (17) is equivalent to Equation (12) presented in Zhang and Wang (1996).

#### 4.2. A circular repairable C(2, 6 : F) system

In this section the circular repairable C(2, 6 : F) system is considered. The state space is  $\Omega = \{0, -1, -2, -3, 1\}$ , the set of working state is  $W = \{0, -1, -2, -3\}$  and the set of failed states is  $F = \{1\}$ . Based on the results in Section 2, the Q-matrix is

$$Q = \begin{pmatrix} -6\lambda & 6\lambda & 0 & 0 & 0 \\ \mu & (-\mu + 5\lambda) & 3\lambda & 0 & 2\lambda \\ 0 & \mu & -(\mu + 4\lambda) & \frac{2}{3}\lambda & \frac{10}{3}\lambda \\ 0 & 0 & \mu & -(\mu + 3\lambda) & 3\lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assume that, at time  $t$ , two components fail. Let,  $\lambda = 0.5$  and  $\mu = 1.5$ , for  $i = -2$  and  $j \in \Omega$  in Equation (6), and then we have

$$\begin{aligned} \Psi_{-2j}(t) &= A_0^{-2j} + A_1^{-2j}e^{\sigma_1 t} + A_2^{-2j}e^{\sigma_2 t} + A_3^{-2j}e^{\sigma_3 t} \\ &\quad + A_4^{-2j}e^{\sigma_4 t}, \end{aligned} \quad (18)$$

where  $A_0^{-2j}, A_1^{-2j}, A_2^{-2j}, A_3^{-2j}$  and  $A_4^{-2j}$  are constant coefficients and  $\sigma_1 = -6.2507, \sigma_2 = -3.7738, \sigma_3 = -2.5737$  and  $\sigma_4 = -0.9018$  are the non-zero eigenvalues of  $Q$ . By applying the proposed method in Section 3 in determining the constant coefficients, we can obtain the following results for Equation (18):

For  $j = 0$ ,

$$\begin{aligned} \Psi_{-20}(t) &= 0.1501e^{-6.2507t} - 0.2039e^{-3.7738t} \\ &\quad - 0.13e^{-2.5737t} + 0.1838e^{-0.9018t}. \end{aligned} \quad (19)$$

For  $j = -1$ ,

$$\begin{aligned} \Psi_{-2-1}(t) &= -0.3254e^{-6.2507t} + 0.1052e^{-3.7738t} \\ &\quad - 0.037e^{-2.5737t} + 0.2572e^{-0.9018t}. \end{aligned} \quad (20)$$

For  $j = -2$ ,

$$\begin{aligned} \Psi_{-2-2}(t) &= 0.1879e^{-6.2507t} + 0.4237e^{-3.7738t} \\ &\quad + 0.225e^{-2.5737t} + 0.1634e^{-0.9018t}. \end{aligned} \quad (21)$$

For  $j = -3$ ,

$$\begin{aligned} \Psi_{-2-3}(t) &= -0.0193e^{-6.2507t} - 0.1825e^{-3.7738t} \\ &\quad + 0.1758e^{-2.5737t} + 0.026e^{-0.9018t}. \end{aligned} \quad (22)$$

For  $j = 1$ ,

$$\begin{aligned} \Psi_{-21}(t) &= 1 + 0.0066e^{-6.2507t} - 0.1425e^{-3.7738t} \\ &\quad - 0.2337e^{-2.5737t} - 0.6305e^{-0.9018t}. \end{aligned} \quad (23)$$

Using Equations (19)–(23), for  $t = 0.5, 1.0, \dots, 5$ , the time-dependent transition performance probabilities for the circular repairable C(2, 6 : F) system can be obtained (see, Table 2). In addition, using Equation (23) for  $i = -2$  in Equation (8), we can obtain the time-dependent reliability of the system when two components failed at time  $t$ .

**Table 2.** Time-dependent transition performance probabilities for the circular repairable C(2,6:F) system when two components fail.

$t$	$\Psi_{-20}(t)$	$\Psi_{-2-1}(t)$	$\Psi_{-2-2}(t)$	$\Psi_{-2-3}(t)$	$\Psi_{-21}(t)$
0.5	0.0569	0.1553	0.2387	0.0366	0.5125
1.0	0.0603	0.1033	0.0936	0.0197	0.7230
1.5	0.0441	0.0661	0.0485	0.0098	0.8316
2.0	0.0294	0.0422	0.0284	0.0052	0.8947
2.5	0.0191	0.0269	0.0175	0.0030	0.9335
3.0	0.0123	0.0172	0.0110	0.0018	0.9578
3.5	0.0078	0.0109	0.0070	0.0011	0.9731
4.0	0.0050	0.0069	0.0044	0.0007	0.9829
4.5	0.0032	0.0044	0.0028	0.0005	0.9891
5.0	0.0020	0.0028	0.0018	0.0003	0.9931

For  $\mu = 0$ , we have the circular irreparable  $C(2, 6 : F)$  system and the  $Q$ -matrix becomes

$$\check{Q} = \begin{pmatrix} -6\lambda & 6\lambda & 0 & 0 & 0 \\ 0 & -5\lambda & 3\lambda & 0 & 2\lambda \\ 0 & 0 & -4\lambda & \frac{2}{3}\lambda & \frac{10}{3}\lambda \\ 0 & 0 & 0 & -3\lambda & 3\lambda \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that all the five components are working at time  $t$ , for  $i = 0$  and  $j = 1$  in Equation (6), we have

$$\begin{aligned}\Psi_{01}(t) = & A_0^{01} + A_1^{01}e^{\sigma_1 t} + A_2^{01}e^{\sigma_2 t} + A_3^{01}e^{\sigma_3 t} \\ & + A_4^{01}e^{\sigma_4 t},\end{aligned}\quad (24)$$

where  $A_0^{01}$ ,  $A_1^{01}$ ,  $A_2^{01}$ ,  $A_3^{01}$  and  $A_4^{01}$  are constant coefficients and  $\sigma_1 = -6\lambda$ ,  $\sigma_2 = -5\lambda$ ,  $\sigma_3 = -4\lambda$  and  $\sigma_4 = -3\lambda$  are the non-zero eigenvalues of  $\tilde{Q}$ . To determine the constant coefficients, we have to solve the following system of differential equations:

$$\begin{aligned} A_0^{01} + A_1^{01} + A_2^{01} + A_3^{01} + A_4^{01} &= 0, \\ (-6\lambda)A_1^{01} + (-5\lambda)A_2^{01} + (-4\lambda)A_3^{01} + (-3\lambda)A_4^{01} &= 0, \\ (-6\lambda)^2 A_1^{01} + (-5\lambda)^2 A_2^{01} + (-4\lambda)^2 A_3^{01} + (-3\lambda)^2 A_4^{01} \\ &= 12\lambda^2, \\ (-6\lambda)^3 A_1^{01} + (-5\lambda)^3 A_2^{01} + (-4\lambda)^3 A_3^{01} + (-3\lambda)^3 A_4^{01} \\ &= -72\lambda^3, \\ (-6\lambda)^4 A_1^{01} + (-5\lambda)^4 A_2^{01} + (-4\lambda)^4 A_3^{01} + (-3\lambda)^4 A_4^{01} \\ &= 228\lambda^4. \end{aligned}$$

The solutions of the above equations are  $A_0^{01} = 1$ ,  $A_1^{01} = -2$ ,  $A_2^{01} = 6$ ,  $A_3^{01} = -3$ ,  $A_4^{01} = -2$ . Then, Equation (24) can be obtained as

$$\Psi_{01}(t) = 1 - 2e^{-6\lambda t} + 6e^{-5\lambda t} - 3e^{-4\lambda t} - 2e^{-3\lambda t}. \quad (25)$$

Finally, using Equation (25) for  $i = 0$  in Equation (8), the time-dependent reliability of the system can be obtained as

$$R_0(t) = 1 - \Psi_{01}(t) \\ = 2e^{-3\lambda t} + 3e^{-4\lambda t} - 6e^{-5\lambda t} + 2e^{-6\lambda t}. \quad (26)$$

Note that Equation (26) is equivalent to Equation (42) in Zhang et al. (2000).

### 4.3. A linear repairable $C(3, 8 : F)$ system

As another example, suppose we have a linear repairable  $C(3, 8 : F)$  system. In this case the state space is  $\Omega = \{0, -1, -2, -3, -4, -5, -6, 1\}$ , the set of working state is  $W = \{0, -1, -2, -3, -4, -5, -6\}$  and the set of

failed states is  $F = \{1\}$ . The Q-matrix is

$$Q = \begin{pmatrix} -8\lambda & 8\lambda & 0 & 0 \\ \mu & -(\mu + 7\lambda) & 7\lambda & 0 \\ 0 & \mu & -(\mu + 6\lambda) & \frac{75}{14}\lambda \\ 0 & 0 & \mu & -(\mu + 5\lambda) \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{14}\lambda \\ \frac{18}{5}\lambda & 0 & 0 & \frac{7}{5}\lambda \\ -(\mu + 4\lambda) & \frac{16}{9}\lambda & 0 & \frac{20}{9}\lambda \\ \mu & -(\mu + 3\lambda) & \frac{3}{8}\lambda & \frac{21}{8}\lambda \\ 0 & \mu & -(\mu + 2\lambda) & 2\lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that all the eight components are working at time  $t$ , if  $\lambda$  and  $\mu$  are given, we can obtain the time-dependent reliability of the system by using the method proposed in Section 2. Here, we consider three sets of parameters: (i)  $\lambda = 0.5, \mu = 1.5$ ; (ii)  $\lambda = 0.5, \mu = 3.0$ ; and (iii)  $\lambda = 0.25, \mu = 1.5$ .

Let,  $i = 0$  in Equation (8), we have

$$\begin{aligned} R_0(t) &= 1 - \Psi_{01}(t) \\ &= 1 - (A_0^{01} + A_1^{01} e^{\sigma_1 t} + A_2^{01} e^{\sigma_2 t} + \dots \\ &\quad + A_7^{01} e^{\sigma_7 t}), \end{aligned} \quad (27)$$

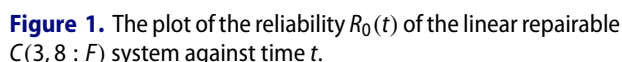
where  $A_0^{01}, A_1^{01}, \dots, A_7^{01}$  are constant coefficients and  $\sigma_1, \sigma_2, \dots, \sigma_7$  are the non-zero eigenvalues of  $Q$ . Using the above  $Q$ -matrix for each parameter sets respectively, the non-zero eigenvalues of  $Q$  are computed and they are presented in Table 3.

Using eigenvalues as seen in Table 3 and Equation (7) for  $k = 0, 1, 2, \dots, 7$ , we can obtain the constant coefficients (see, Table 4). Using the constant coefficients in Tables 3 and 4, we plotted the reliability  $R_0(t)$  against time  $t$  for the linear repairable  $C(3, 8 : F)$  system in Figure 1 for three different parameter sets.

**Table 3.** The non-zero eigenvalues of  $Q$  for three parameter sets.

$\sigma$	$\lambda = 0.5, \mu = 1.5$	$\lambda = 0.5, \mu = 3.0$	$\lambda = 0.25, \mu = 1.5$
$\sigma_1$	-8.3126	-11.1812	-5.5906
$\sigma_2$	-6.1101	-8.2729	-4.1365
$\sigma_3$	-4.4007	-5.9962	-2.9981
$\sigma_4$	-3.1176	-4.3836	-2.1918
$\sigma_5$	-2.3398	-3.3967	-1.6938
$\sigma_6$	-1.5893	-1.9523	-0.9761
$\sigma_7$	-0.6299	-0.3171	-0.1586

$A_k^{01}$	$\lambda = 0.5, \mu = 1.5$	$\lambda = 0.5, \mu = 3.0$	$\lambda = 0.25, \mu = 1.5$
$A_0^{01}$	1.0000	1.0000	1.0000
$A_1^{01}$	-0.0045	-0.0015	-0.0019
$A_2^{01}$	0.0026	-0.0015	-0.0015
$A_3^{01}$	-0.0251	-0.0069	-0.0069
$A_4^{01}$	-0.1525	-0.0162	-0.0162
$A_5^{01}$	0.0116	-0.0084	-0.0085
$A_6^{01}$	1.1515	0.2937	0.2938
$A_7^{01}$	-1.9836	-1.2589	-1.2587



In this section, we study a circular repairable  $C(3, 8 : F)$  system. The state space is  $\Omega = \{0, -1, -2, -3, -4, -5, 1\}$ , the set of working state is  $W = \{0, -1, -2, -3, -4, -5\}$  and the set of failed states is  $F = \{1\}$ . The  $Q$ -matrix is

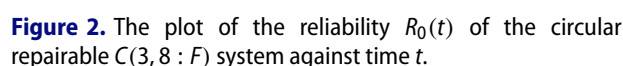
[illegible]

$$\begin{aligned} R_0(t) &= 1 - \Psi_{01}(t) \\ &= 1 - (A_0^{01} + A_1^{01} e^{\sigma_1 t} + A_2^{01} e^{\sigma_2 t} \\ &\quad + \cdots + A_6^{01} e^{\sigma_6 t}), \end{aligned} \quad (28)$$

Using the eigenvalues presented in Table 5 for  $k = 0, 1, 2, \dots, 6$ , we can obtain the constant coefficients in Equation (28) (see, Table 6). When the values in Tables 5 and 6 are used in Equation (28), we plotted the reliability  $R_0(t)$  against time  $t$  for the circular

$\sigma$	$\lambda = 0.5, \mu = 1.5$	$\lambda = 0.5, \mu = 3.0$	$\lambda = 0.25, \mu = 1.5$
$\sigma_1$	-8.2922	-11.1417	-5.5709
$\sigma_2$	-6.0051	-8.0973	-4.0486
$\sigma_3$	-4.2440	-5.7354	-2.8677
$\sigma_4$	-2.9306	-3.9541	-1.9770
$\sigma_5$	-1.8225	-2.2057	-1.1028
$\sigma_6$	-0.7057	-0.3659	-0.1830

$A_k^{01}$	$\lambda = 0.5, \mu = 1.5$	$\lambda = 0.5, \mu = 3.0$	$\lambda = 0.25, \mu = 1.5$
$A_0^{01}$	1.0000	1.0000	1.0000
$A_1^{01}$	-0.0060	-0.0026	-0.0026
$A_2^{01}$	0.0051	-0.0025	-0.0025
$A_3^{01}$	-0.0890	-0.0165	-0.0165
$A_4^{01}$	-0.1391	-0.0212	-0.0212
$A_5^{01}$	1.2074	0.3139	0.3140
$A_6^{01}$	-1.9784	-1.2712	-1.2710





repairable  $C(3, 8 : F)$  system in Figure 2 based on the three parameter sets.

## 5. Concluding remarks

In this paper, a repairable  $C(k, n : F)$  system with  $n$  linearly and circularly arranged components is studied. A novel and simple method for determining the time-dependent transition performance probability and time-dependent reliability of the system when the state of the system is known within a certain of time is developed. Some numerical examples are used to illustrate the usefulness of the proposed method. Computer code written in R Core Team (2019) to compute the time-dependent transition performance probability can be obtained from the authors upon request. The proposed method can serve as a simple alternative to the Laplace transform and the Runge-Kutta method which are commonly used to determine the time-dependent reliability of the repairable systems.

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