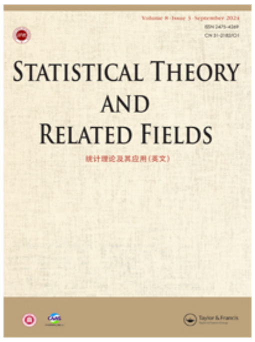

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Bivariate iterated Farlie–Gumbel–Morgenstern stress–strength reliability model for Rayleigh margins: Properties and estimation

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ABSTRACT
In this paper, we propose bivariate iterated Farlie–Gumbel–Morgenstern (FGM) due to [Huang and Kotz (1984). Correlation structure in iterated Farlie-Gumbel-Morgenstern distributions. Biometrika 71(3), 633–636. https://doi.org/10.2307/2336577] with Rayleigh marginals. The dependence stress–strength reliability function is derived with its important reliability characteristics. Estimates of dependence reliability parameters are obtained. We analyse the effects of dependence parameters on the reliability function. We found that the upper bound of the positive correlation coefficient is attaining to 0.41 under a single iteration with Rayleigh marginals. A comprehensive comparison between classical FGM with iterated FGM copulas is graphically examined to assess the over or under estimation of reliability with respect to α and β. We propose a two-phase estimation procedure for estimating the reliability parameters. A Monte-Carlo simulation study is conducted to assess the finite sample behaviour of the proposed reliability estimators. Finally, the proposed estimators are examined and validated with real data sets.

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Iterated FGM; Rayleigh distribution; dependence stress–strength; reliability; Monte-Carlo simulation

1. Introduction

The stress–strength analysis is one of the well-accepted approach invented firstly to assess the physics of the failure of a system due to various natural stresses (e.g., temperature, pressure, voltage, etc.) and subsequently, it has been extensively utilized to evaluate the system reliability from last six decades (see, Birnbaum, 1956) in engineering science. Stress–strength model is capable to elaborate the functional relationship between two or more random variables and hence it has attracted to many other domain areas of research, including engineering, viz., medical, economics, insurance, social and humanities sciences, etc., in the last two decades ago.

A numerous amount work on stress–strength reliability (SSR) model has been cited in the literature, and it has also attracted in modelling with associated ideas, like accelerated life testing, simultaneous and multiple stresses application, interference stress–strength, augmented strength, Fuzzy stress–strength, etc., for estimating the reliability parameters of a system (single component or multi-component). The analysis of SSR models are mostly based on various settings of independence (with identical/non-identical nature) assumptions of stress (Y) and strength (X) random variables, where X and Y are assumed to follow a suitable failure time distribution and hence inferential problems are carried out using parametric (classical and Bayesian), nonparametric and semiparametric methods of estimation. Some recent key works in this direction are refer to Kotz and Pensky (2003), Chandra and Rathaur (2017), Chandra and Rathaur (2020), Pak et al. (2022) and references therein.

However, in the present real-life scenarios of fastest growing productions of highly sophisticated and complex advanced technology used in designing the products, where considering of independence assumptions of underlying variables does not make sense. For instance, components within an electronic system (strength) may share the same or different stresses. A healthy individual (strength) may subjected to various stresses like, diseases of blood pressure and diabetic with (low or high) level of scale measure. In both examples, the stress and strength variables are associated. Consequently, if the dependence between X and Y is not considered, reliability (or survival) may be either over or under estimated, which may mislead the inferences on time to event analysis. Hence, it is vital to consider and model the association between X and Y.

Besides, some works on SSR consider either stresses or strengths as dependent within themselves but independent between each other. These are assumed to follow bivariate family of life distributions (viz., exponential, gamma, lognormal, etc.) (Chandra & Pandey, 2012; Gupta & Gupta, 2012; Nadarajah, 2005) and references therein.

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In the aforesaid attempts, correlation between X and Y have not been considered, and their dependence effects on system's reliability were completely ignored. Hence, a copula-based approach is more appropriate in modelling the association between two or more random variables to address these issues and construct a family of multivariate distribution function. A variety of copula families are available in the literature and each of them having their own unique characteristics and constraints, even there is no hard and fast rule for choosing one, among them, where FGM is one of the most widely used family due to its simplicity. Hence a suitable choice of copula function is also as important for better inference purpose. Morgenstern (1956) proposed the copula function to construct the structure of association between two random variables to measure the dependence relation between them for Cauchy marginals. Gumbel (1960) adopted the same idea for exponential marginals and (Farlie, 1960) independently studies the dependence structure between random variables and also proposed the generalization of bivariate form suggested by Morgenstern. Later, their initiatives becomes more popularized in formulating the dependence structure of multivariate case due to Johnson and Kotz (1975a, 1977) and it is termed as FGM copula function.

In context of dependence SSR modelling, Domma and Giordano (2012) proposed dependence stress–strength model for Dagum margins using Frank copula and gave a nice application for income-consumption data to measure of household financial fragility. Domma and Giordano (2013) attempted dependence SSR for different copula functions, namely, FGM, Generalized FGM and Frank copula with Burr system of margins. Rubio and Steel (2013) studied the posterior analysis of SSR for both independent and dependent assumptions of X and Y, where marginal distributions of $X$and Y belong to the class of distributions obtained by skewing scale mixtures of normal, linked with Gaussian copula. Barbiero (2017) derived the dependence structure of SSR model for exponential margins using the extended FGM and Ali–Mikhail–Haq copulas, it is graphically examined that the dependence parameter is significantly affecting the reliability of a system. It is also interesting to note that lower and upper bounds for reliability is studied when information about dependence parameter is unavailable. Patil and Naik-Nimbalkar (2017) studies estimation of dependence SSR, where X and Y follow two- parameter Pareto distributions, based on four different copula functions under Archimedean type (Ali–Mikhail–Haq, Gumbel's bivariate exponential and Gumbel–Hougaard copulas) as well as non-Archimedean type (FGM copula) for varying ranges of dependence parameter. The effect of dependency on reliability and asymptotic properties of reliability estimators are also reported.

Ahmed et al. (2020) obtained the inferences on Bayes and ML estimates of SSR, where stress and strength follow General Exponential (GE) as well as General Inverse Exponential (GIE) as marginal distributions proposed by Mokhlis et al. (2017), both are linked with FGM copula. Bai et al. (2021) drawn inferences on SSR of multi-state system for both independent and dependent strengths using generalized survival signature tricks for exponential marginal distributions, where Gumbel copula is proposed for dependence structure of strengths.

Recently, James and Chandra (2022) and James et al. (2022) considers the estimation of dependence SSR by assuming that X follows Xgamma distribution and Y follows exponential distribution for farmer attempt as well as both X and Y follow Lindley marginal distributions for later one, for non-identical cases of X and Y are linked with FGM copula function.

However, it is noticed that majority of works cited above on dependence structure of SSR are mainly based on FGM copula due its flexibility in compare to other existing copulas. The FGM copula is only admitted to model low (weak) dependence between X and Y (see, Ahmed et al., 2020, p. 142). Hence, it does not allow to model for higher dependence (see, Durante, 2006) due to limited range of values of correlation coefficients, namely, Spearman's $\rho \in [-0.33, 0.33]$ and Kendall's $\tau \in [-0.22, 0.22]$. Therefore, a number of modifications and extensions of FGM copula has been suggested by several authors in the literature as alternatives of weak dependence and also to improve the correlation between the underlying variables. Johnson and Kotz (1975a) introduced iterative ideas for several generalizations of FGM for multivariate structure of distributions.

Later, J. Huang and Kotz (1984) extended the iteration idea for bivariate FGM by adding one extra parameter for single iteration and observed that maximum positive correlation attaining wider ranges from 0.414 (for normal marginal) to 0.434 (for uniform marginal), even the covariance is increased nearly a 200% for Pareto marginals.

Subsequently, J. S. Huang and Kotz (1999) proposed two different extensions of polynomial type bivariate FGM by adding one additional parameter and shown that some improvements in the positive correlation while lower bounds of negative correlation remains unchanged, i.e. $\rho \in [-0.333, 0.375]$, for one form of extended FGM and $\rho \in [0.333, 0.391]$ for another extended form of FGM. Later, Bairamov and Kotz (2002) introduced another polynomial type generalized FGM copula for more wider range of dependence parameter with $\rho \in [-0.48, 0.502]$.

Some more interesting extensions are cited in the literature, particularly extensions based on a new class of symmetric as well as asymmetric types of bivariate FGM copula with wider range of dependence parameters refer to Bekrizadeh, Parham, and Zadkarmi (2012) and Bekrizadeh, Parham, and Jamshidi (2017) with $\rho \in [-0.50, 0.43]$ and varieties of work based on FGM refer to this paper.

In this paper, we consider the estimation of stress–strength reliability model for generalized FGM copula based on J. Huang and Kotz (1984) with wider range of positive correlation coefficient $\rho \in [-0.33, 0.434]$. This iterated FGM with its wider range of correlation motivates us to use it rather simple FGM copula to this attempt. In this paper we assess the impact of additional parameter β and how affecting the reliability in the presence of baseline dependence parameter (α).

The rest of the paper is organized as follow. We propose a new bivariate Iterated Farlie–Gumbel–Morgenstern (IFGM) bivariate distribution with Rayleigh marginals in Section 2. In Section 3, the dependence measures of the Iterated Farlie–Gumbel–Morgenstern Bivariate Rayleigh (IFGMBR) distribution are discussed. The expressions for dependence stress–strength reliability and some reliability characteristics of reliability measures of the proposed distribution are derived in Section 4. In Section 5, we propose a two-phase method including pseudo likelihood in estimating the parameters of dependence stress–strength reliability model. A numerical comparisons of the reliability estimates with respect to dependence parameter based on Monte-Carlo simulation and its findings are reported in Section 6. Section 7, real data application is utilized to illustrate the proposed estimators.Finally, concluding remarks are given in Section 8.

2. Iterated FGM type bivariate rayleigh distribution

In the original FGM, J. Huang and Kotz (1984) used successive iterations to capture a broad range of correlation between random variables. In this study, we consider bivariate FGM with a single iteration, the joint cumulative distribution function (c.d.f) is given as follows

$$F_{(XY)}(x,y) = F_X(x)G_Y(y)\left[1 + \alpha\overline{F}_X(x)\overline{G}_Y(y) + \beta F_X(x)G_Y(y)\overline{F}_X(x)\overline{G}_Y(y)\right], \tag{1}$$

and the corresponding joint probability density function (p.d.f)

$$\begin{aligned} f_{(XY)}(x,y) =& f_X(x)g_Y(y)[1 + \alpha(1 - 2F_X(x))(1 - 2G_Y(y)) + \beta F_X(x)G_Y(y) \\ & \times (2 - 3F_X(x))(2 - 3G_Y(y))], \end{aligned} \tag{2}$$

where F_X and G_Y are the c.d.f's, $f(x)$ and $g(y)$ are the p.d.f's and $\overline{F}_X(x)$ and $\overline{G}_Y(y)$ denote the survival functions of X and Y, respectively. The natural parameter space Θ is convex, where $\Theta = \left\{(\alpha,\beta) : -1 \le \alpha \le 1; \alpha + \beta \ge -1; \beta \le \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}\right\}$.

In this paper, we proposed a new bivariate IFGM distribution by assuming that both random variables X and Y follow non-identical two parameter Rayleigh distributions with respective c.d.f's are given by

$$F_X(x;\lambda_1,\mu_1) = 1 - \mathrm{e}^{-\lambda_1(x-\mu_1)^2}, \quad x > \mu_1, \quad \lambda_1 \ge 0, \tag{3}$$

$$G_Y(y;\lambda_2,\mu_2) = 1 - \mathrm{e}^{-\lambda_2(y-\mu_2)^2}, \quad y > \mu_2, \quad \lambda_2 \ge 0, \tag{4}$$

and the corresponding and p.d.f's are

$$f_X(x;\lambda_1,\mu_1) = 2\lambda_1(x-\mu_1)\mathrm{e}^{-\lambda_1(x-\mu_1)^2}, \quad x > \mu_1, \quad \lambda_1 \ge 0, \tag{5}$$

$$g_Y(y;\lambda_2,\mu_2) = 2\lambda_2(y-\mu_2)\mathrm{e}^{-\lambda_2(y-\mu_2)^2}, \quad y > \mu_2, \quad \lambda_2 \ge 0, \tag{6}$$

where λ_i and $\mu_i; i = 1, 2$ are the scale and location parameters of marginal distributions. The main advantage of using two parameter distribution over Rayleigh is that when the life time of the unit dose not begin from zero. Moreover, it has an interesting property of unimodal and increasing hazard rate make it more appropriate for modelling the life time distribution of the unit that rapidly age with time.

Then the joint c.d.f and joint p.d.f of IFGMBR distribution is given as follows

$$\begin{aligned} F_{(XY)}(x,y) =& (1 - \mathrm{e}^{-\lambda_1(x-\mu_1)^2})(1 - \mathrm{e}^{-\lambda_2(y-\mu_2)^2})\Big(1 + \alpha\mathrm{e}^{-\lambda_1(x-\mu_1)^2}\mathrm{e}^{-\lambda_2(y-\mu_2)^2} \\ & +\beta(1 - \mathrm{e}^{-\lambda_1(x-\mu_1)^2})(1 - \mathrm{e}^{-\lambda_2(y-\mu_2)^2})\mathrm{e}^{-\lambda_1(x-\mu_1)^2}\mathrm{e}^{-\lambda_2(y-\mu_2)^2}\Big), \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_{(XY)}(x,y) =& 4\lambda_1\lambda_2(x-\mu_1)(y-\mu_2)\mathrm{e}^{-\lambda_1(x-\mu_1)^2}\mathrm{e}^{-\lambda_2(y-\mu_2)^2}\Big(1 + \alpha(2\mathrm{e}^{-\lambda_1(x-\mu_1)^2} - 1) \\ & \times (2\mathrm{e}^{-\lambda_2(y-\mu_2)^2} - 1) + \beta(1 - \mathrm{e}^{-\lambda_1(x-\mu_1)^2})(1 - \mathrm{e}^{-\lambda_2(y-\mu_2)^2}) \\ & \times (3\mathrm{e}^{-\lambda_1(x-\mu_1)^2} - 1)(3\mathrm{e}^{-\lambda_2(y-\mu_2)^2} - 1)\Big). \end{aligned} \tag{8}$$

2.1. Conditional distribution

Let (X, Y) be a two-dimensional random variable follows IFMBR distribution, then the conditional c.d.f of X given $Y = y$ is given by

$$\begin{aligned} F_{X|Y}(x|y) = &\left(1 - e^{-\lambda_1(x-\mu_1)^2}\right)\left[1 + \alpha e^{-\lambda_1(x-\mu_1)^2}\left\{2e^{-\lambda_1(x-\mu_1)^2} - 1\right\}\right] \\ &+ \beta e^{-\lambda_1(x-\mu_1)^2}\left(e^{-2\lambda_1(x-\mu_1)^2} - 2e^{-\lambda_1(x-\mu_1)^2} + 1\right) \\ &\times\left(\frac{3e^{-2\lambda_2(y-\mu_2)^2}}{4\lambda_2(y-\mu_2)} - \frac{2e^{-\lambda_2(y-\mu_2)^2}}{\lambda_2(y-\mu_2)} - y + \mu_2\right), \end{aligned} \tag{9}$$

and then the conditional p.d.f of X given $Y = y$ is given by

$$\begin{aligned} f_{(X|Y)}(x|y) = &2\lambda_1(x-\mu_1)e^{-\lambda_1(x-\mu_1)^2}\left[1 + \alpha\left\{2e^{-\lambda_1(x-\mu_1)^2} - 1\right\}\left\{2e^{-\lambda_2(y-\mu_2)^2} - 1\right\}\right. \\ &+ \beta\left\{1 - e^{-\lambda_1(x-\mu_1)^2}\right\}\left\{1 - e^{-\lambda_2(y-\mu_2)^2})(3e^{-\lambda_1(x-\mu_1)^2} - 1\right\} \\ &\times\left.\left\{3e^{-\lambda_2(y-\mu_2)^2} - 1\right\}\right]. \end{aligned} \tag{10}$$

Further, the conditional expectation of X given $Y = y$ is obtained as

$$\begin{aligned} E[X/Y = y] = &\left[\frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\lambda_1}} + \mu_1\right] + \alpha\left[2e^{-\lambda_2(y-\mu_2)^2} - 1\right]\left[\frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\lambda_1}}\left\{\frac{1-2\sqrt{2}}{2\sqrt{2}}\right\} - \frac{\mu_1}{2}\right] \\ &+ \beta\left\{3e^{-\lambda_2(y-\mu_2)^2} - 1\right\}\left\{1 - e^{-\lambda_2(y-\mu_2)^2}\right\}\left[\frac{\sqrt{\pi}}{\sqrt{2\lambda_1}}\left\{1 - \frac{1}{\sqrt{2}}\right\}\right. \\ &\left. - \frac{\sqrt{\pi}}{2\sqrt{3\lambda_1}}\right]. \end{aligned} \tag{11}$$

In similar manner, the expressions of $F_{Y/X}(y/x), f_{Y/X}(y/x)$ and $E[Y/X = x]$ can also be obtained.

3. Dependence measures

Copulas are used to study the dependency or association between two variables. In the literature, we know several coefficients based on copulas that describe the dependence between random variables. In this section, we will discuss the three well-known measures of the association, including Spearman's rho, Kendall's tau and Blomqvist's beta (known as medial correlation).

3.1. Correlation structure

Let $F_X(x)$ and $G_Y(y)$ be an absolutely continuous marginals defined in (3) and (4), respectively, with finite standard deviation σ_x and σ_y. The correlation coefficient between X and Y for optimal choice of α and β (see, J. Huang & Kotz, 1984) is given by

$$\rho = \frac{Cov(X, Y)}{\sigma_x\sigma_y} = \frac{\alpha}{4}\frac{\mu_{22} - \mu_{12}}{\sigma_x}\frac{\nu_{22} - \nu_{12}}{\sigma_y} + \frac{\beta}{9}\frac{\mu_{33} - \mu_{23}}{\sigma_x}\frac{\nu_{33} - \nu_{23}}{\sigma_y}, \tag{12}$$

where $\mu_{kn} = E[X_{k:n}]$ and $\nu_{kn} = E[Y_{k:n}]$, $X_{k:n}$ and $Y_{k:n}$ are the kth smallest order statistic of sample size n from F and G, respectively. J. Huang and Kotz (1984) studied the one-iteration of original FGM and observed that the correlation between X and Y attaining the wider range for upper bound and lower bound remains unchanged. Hence, the maximal positive correlation coefficient 0.414 for normal marginal and 0.434 for uniform marginal. Moreover, in general, for any marginal choice of F with finite $\sigma(x)$, the maximal upper bound of (12), based on optimal value of α and β, attains to 0.50. The correlation coefficient of (X, Y) for IFGMBR distribution is obtained as

$$\rho = \frac{\alpha}{4}\prod_{i=1}^{2}\frac{\left(\frac{(\sqrt{2}-2)\sqrt{\pi}}{2\sqrt{\lambda_i}}\right)}{\sqrt{\frac{4-\pi}{4\lambda_i}}} + \frac{\beta}{9}\prod_{i=1}^{2}\frac{\left(\frac{(\sqrt{3}-3\sqrt{2}+3)\sqrt{\pi}}{2\sqrt{\lambda_i}}\right)}{\sqrt{\frac{4-\pi}{4\lambda_i}}}. \tag{13}$$

Admissible range of correlation coefficient ρ of IFGMBR distribution are reported in Table 1.

Table 1. Admissible range of correlation coefficient ρ of IFGMBR distribution for $\lambda_1 = 0.2$ and $\lambda_2 = 0.9$.

	α					
β	−1	−0.5	−0.1	0.1	0.5	1
0.1	−0.3042205	−0.1472403	−0.02165604	0.04113607	0.1667203	0.3237006
0.5	−0.2652605	−0.1082800	0.01730404	0.08009615	0.2056804	0.3626606
1	−0.2165604	−0.05958008	0.06600414	0.1287963	0.2543805	0.4113607

Table 2. Admissible range of correlation coefficient ρ of FGMBR distribution for $\lambda_1 = 0.2$ and $\lambda_2 = 0.9$.

α	−1	−0.5	−0.1	0.1	0.5	1
ρ	−0.3139605	−0.1569803	−0.03139605	0.03139605	0.1569803	0.3139605

When the additional parameter $\beta = 0$ in Equation (13) reduces to the correlation coefficient of FGMBR distribution, the admissible range of ρ is reported in Tables 1 and 2.

3.2. *Kendall's tau*

Kendall's tau for a two dimensional random vector (X, Y) with joint distribution function $F_{XY}(xy)$ is defined as follows:

$$\tau = 4\int_x\int_y F_{XY}(xy)f_{XY}(xy)\,\mathrm{d}x\,\mathrm{d}y - 1. \tag{14}$$

Substituting the joint c.d.f and joint p.d.f from Equations (7) and (8) in (14), the expression of Kendall's tau for IFGMBR distribution is obtained as follows

$$\tau = \frac{2\alpha}{9} + \frac{\beta}{18} + \frac{\alpha\beta}{900}. \tag{15}$$

3.3. *Medial correlation*

Medial correlation is also called the Blomqvist's beta is an association measure based on the medians. A population version of Blomqvist's beta is given by

$$\beta(F_{XY}(x,y)) = 4F_{XY}(M_X, M_Y) - 1 = 4C\left(\frac{1}{2},\frac{1}{2}\right) - 1. \tag{16}$$

Substituting the joint c.d.f and joint p.d.f from Equations (7) and (8) in (16), the expression of medial correlation for IFGMBR distribution can be obtained as

$$\beta(F_{XY}(x,y)) = \frac{\alpha}{4} + \frac{\beta}{16}. \tag{17}$$

4. Reliability measures

In this section, we derive stress–strength reliability function, MTTF, MRLF, vitality function and hazard rate function of IFGMRD distribution.

4.1. *Stress–strength reliability measure*

We assume that bivariate random variate (X, Y) follows IFGMBRD distribution with scale (λ_1,λ_2) and location (μ_1, μ_2) parameters, and then dependence stress–strength reliability is defined by

$$R = P(Y < X) = \int_{\mu_1}^{\infty}\int_{\mu_2}^{x} f_{XY}(x,y)\,\mathrm{d}y\,\mathrm{d}x. \tag{18}$$

The structure of the joint density function allows us to express the stress–strength measure as a sum of three components: one in the case of independence ($\alpha = \beta = 0$), one in the case where dependence is a function of α only, and the other in the case where dependence is a function of β only, i.e.,

$$R = R_I + R_\alpha + R_\beta, \tag{19}$$

where

$$\begin{aligned} R_I &= \int_{\mu_1}^{\infty}\int_{\mu_2}^{x} f_X(x)g_Y(y)\,\mathrm{d}x\,\mathrm{d}y,\\ R_\alpha &= \alpha\int_{\mu_1}^{\infty}\int_{\mu_2}^{x} f_X(x)g_Y(y)(1-2F_X(x))(1-2G_Y(y))\,\mathrm{d}x\,\mathrm{d}y,\\ R_\beta &= \beta\int_{\mu_1}^{\infty}\int_{\mu_2}^{x} f_X(x)g_Y(y)F_X(x)G_Y(y)(2-3F_X(x))(2-3G_Y(y))\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

The expressions of R_I, R_α and R_β are given by

$$R_I = 1 - C_1 - \frac{\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)}, \tag{20}$$

$$\begin{aligned} R_\alpha =& \alpha\left[2C_2 + \frac{2\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(2\lambda_1+\lambda_2)} - \frac{\lambda_1 \mathrm{e}^{-2\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)} - 2C_3 - \frac{\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)}\right.\\ &\left. - C_1 + 4C_4 + \frac{\lambda_1 \mathrm{e}^{-2\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+2\lambda_2)}\right], \end{aligned} \tag{21}$$

$$\begin{aligned} R_\beta =& \beta\left[4C_2 + \frac{4\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(2\lambda_1+\lambda_2)} + 12C_5 - 8C_3 + \frac{4\lambda_1 \mathrm{e}^{-3\lambda_2(\mu_1-\mu_2)^2}}{(2\lambda_1+3\lambda_2)} - \frac{4\lambda_1 \mathrm{e}^{-2\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)}\right.\\ & - C_1\frac{\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)} - 3C_6 - \frac{\lambda_1 \mathrm{e}^{-3\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+3\lambda_2)} + 8C_4 + \frac{2\lambda_1 \mathrm{e}^{-2\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+2\lambda_2)}\\ &\left. - 3C_9 - 3C_7 - \frac{3\lambda_1 \mathrm{e}^{-\lambda_2(\mu_1-\mu_2)^2}}{(3\lambda_1+\lambda_2)} - \frac{\lambda_1 \mathrm{e}^{-3\lambda_2(\mu_1-\mu_2)^2}}{(\lambda_1+\lambda_2)} + 12C_8 + \frac{6\lambda_1 \mathrm{e}^{-2\lambda_2(\mu_1-\mu_2)^2}}{(3\lambda_1+2\lambda_2)}\right], \end{aligned} \tag{22}$$

where

$$\begin{aligned} C_1 =& \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(\lambda_1+\lambda_2)\sqrt{\lambda_1+\lambda_2}}\exp\left(\frac{(\lambda_1\mu_1+\lambda_2\mu_2)^2}{(\lambda_1+\lambda_2)} - \lambda_1\mu_1^2 - \lambda_2\mu_2^2\right)\\ &\times\left(1-\operatorname{erf}\left(\frac{\lambda_2(\mu_1-\mu_2)}{\sqrt{\lambda_1+\lambda_2}}\right)\right),\\ C_2 =& \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(2\lambda_1+\lambda_2)\sqrt{2\lambda_1+\lambda_2}}\exp\left(\frac{(2\lambda_1\mu_1+\lambda_2\mu_2)^2}{(2\lambda_1+\lambda_2)} - 2\lambda_1\mu_1^2 - \lambda_2\mu_2^2\right)\\ &\times\left(1-\operatorname{erf}\left(\frac{\lambda_2(\mu_1-\mu_2)}{\sqrt{2\lambda_1+\lambda_2}}\right)\right),\\ C_3 =& \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(\lambda_1+\lambda_2)\sqrt{2\lambda_1+2\lambda_2}}\exp\left(\frac{2(\lambda_1\mu_1+\lambda_2\mu_2)^2}{(\lambda_1+\lambda_2)} - 2\lambda_1\mu_1^2 - 2\lambda_2\mu_2^2\right)\\ &\times\left(1-\operatorname{erf}\left(\frac{2\lambda_2(\mu_1-\mu_2)}{\sqrt{2\lambda_1+2\lambda_2}}\right)\right),\\ C_4 =& \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(\lambda_1+2\lambda_2)\sqrt{\lambda_1+2\lambda_2}}\exp\left(\frac{(\lambda_1\mu_1+2\lambda_2\mu_2)^2}{(\lambda_1+2\lambda_2)} - \lambda_1\mu_1^2 - 2\lambda_2\mu_2^2\right)\\ &\times\left(1-\operatorname{erf}\left(\frac{2\lambda_2(\mu_1-\mu_2)}{\sqrt{\lambda_1+2\lambda_2}}\right)\right),\\ C_5 =& \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(2\lambda_1+3\lambda_2)\sqrt{2\lambda_1+3\lambda_2}}\exp\left(\frac{(2\lambda_1\mu_1+3\lambda_2\mu_2)^2}{(2\lambda_1+3\lambda_2)} - 2\lambda_1\mu_1^2 - 3\lambda_2\mu_2^2\right)\\ &\times\left(1-\operatorname{erf}\left(\frac{3\lambda_2(\mu_1-\mu_2)}{\sqrt{2\lambda_1+3\lambda_2}}\right)\right), \end{aligned}$$

$$C_6 = \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(\lambda_1+3\lambda_2)\sqrt{\lambda_1+3\lambda_2}} \exp\left(\frac{(\lambda_1\mu_1+3\lambda_2\mu_2)^2}{(\lambda_1+3\lambda_2)} - \lambda_1\mu_1^2 - 3\lambda_2\mu_2^2\right) \times \left(1 - \operatorname{erf}\left(\frac{3\lambda_2(\mu_1-\mu_2)}{\sqrt{\lambda_1+3\lambda_2}}\right)\right),$$

$$C_7 = \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(\lambda_1+\lambda_2)\sqrt{3\lambda_1+3\lambda_2}} \exp\left(\frac{3(\lambda_1\mu_1+\lambda_2\mu_2)^2}{(\lambda_1+\lambda_2)} - 3\lambda_1\mu_1^2 - 3\lambda_2\mu_2^2\right) \times \left(1 - \operatorname{erf}\left(\frac{3\lambda_2(\mu_1-\mu_2)}{\sqrt{3\lambda_1+3\lambda_2}}\right)\right),$$

$$C_8 = \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(3\lambda_1+2\lambda_2)\sqrt{3\lambda_1+2\lambda_2}} \exp\left(\frac{(3\lambda_1\mu_1+2\lambda_2\mu_2)^2}{(3\lambda_1+2\lambda_2)} - 3\lambda_1\mu_1^2 - 2\lambda_2\mu_2^2\right) \times \left(1 - \operatorname{erf}\left(\frac{2\lambda_2(\mu_1-\mu_2)}{\sqrt{3\lambda_1+2\lambda_2}}\right)\right),$$

$$C_9 = \frac{\lambda_1\lambda_2\sqrt{\pi}(\mu_2-\mu_1)}{(3\lambda_1+\lambda_2)\sqrt{3\lambda_1+\lambda_2}} \exp\left(\frac{(3\lambda_1\mu_1+\lambda_2\mu_2)^2}{(3\lambda_1+\lambda_2)} - 3\lambda_1\mu_1^2 - \lambda_2\mu_2^2\right) \times \left(1 - \operatorname{erf}\left(\frac{\lambda_2(\mu_1-\mu_2)}{\sqrt{3\lambda_1+\lambda_2}}\right)\right).$$

When the location parameters $\mu_1 = \mu_2 = 0$, Equation (19) will reduces to the reliability function of two non-identical scale parameters of stress and strength for Rayleigh distribution and its expression is given by

$$R = \frac{\lambda_2}{(\lambda_1+\lambda_2)} + \alpha\left(\frac{2\lambda_1}{(2\lambda_1+\lambda_2)} - \frac{2\lambda_1}{(\lambda_1+\lambda_2)} + \frac{\lambda_1}{(\lambda_1+2\lambda_2)}\right) + \beta\left(\frac{4\lambda_1}{(2\lambda_1+\lambda_1)} + \frac{4\lambda_1}{(2\lambda_1+3\lambda_2)} - \frac{6\lambda_1}{(\lambda_1+\lambda_2)} - \frac{\lambda_1}{(\lambda_1+3\lambda_2)} + \frac{2\lambda_1}{(\lambda_1+2\lambda_2)} - \frac{3\lambda_1}{(3\lambda_1+\lambda_2)} + 6\frac{4\lambda_1}{(3\lambda_1+2\lambda_2)}\right). \quad (23)$$

Further, if $\beta = 0$, then Equation (19) is reduced to the reliability function of X and Y linked by FGM copula. In addition, if both $\alpha, \beta = 0$, then Equation (19) reduces to the reliability function when X and Y are independent.

In this paper, our aim is to show that ignoring the dependence between stress and strength, even when it exists, results in higher or lower values of reliability than the actual case.

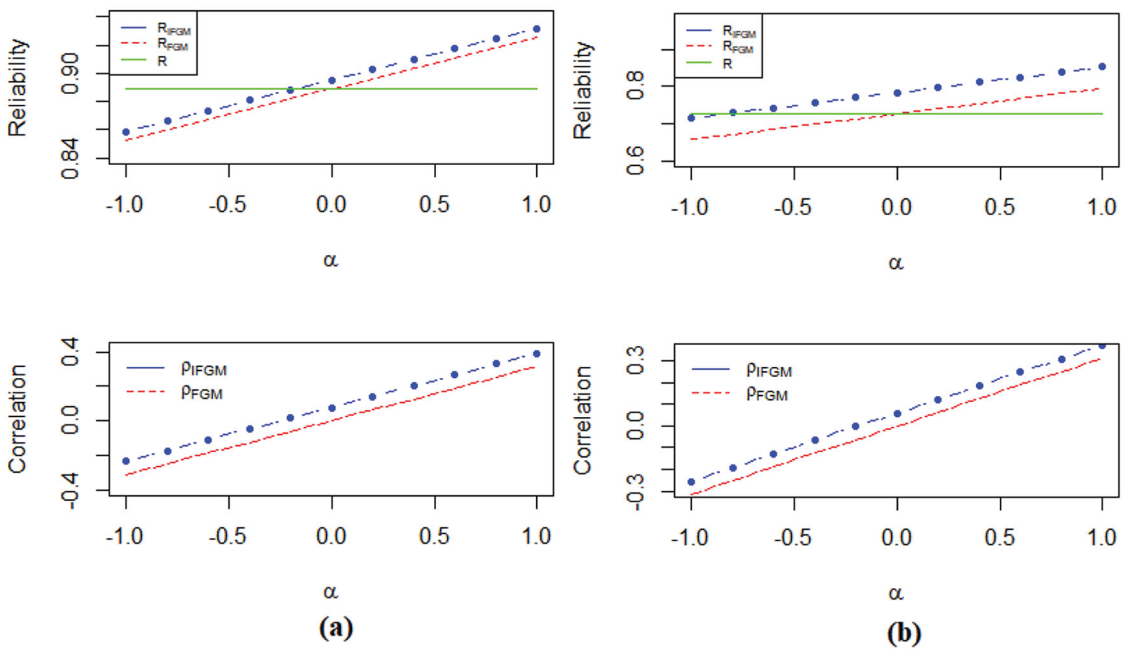

Figure 1. Plot of reliability function and corresponding correlation for some combinations of marginal and copula parameters (a) $\lambda_1 = 0.1$, $\mu_1 = 0$, $\lambda_2 = 0.8$, $\mu_2 = 0$ and $\beta = 0.8$ and (b) $\lambda_1 = 0.2$, $\mu_1 = 0.1$, $\lambda_2 = 0.6$, $\mu_2 = 0.2$ and $\beta = 0.6$.

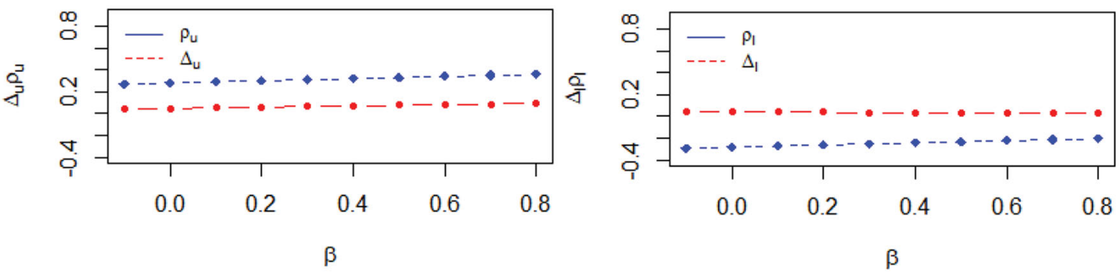

Figure 2. Plot of (c) $\Delta_u = R_u - R_I$, ρ_u vs. β for $\lambda_1 = 0.2$, $\mu_1 = 0.6$, $\lambda_2 = 0.8$, $\mu_2 = 0.4$ and (d) $\Delta_l = R_I - R_l$, ρ_l vs. β for $\lambda_1 = 0.1$, $\mu_1 = 0.3$, $\lambda_2 = 0.9$, $\mu_2 = 0.4$, R_I is the reliability fuction when X and Y are independent.

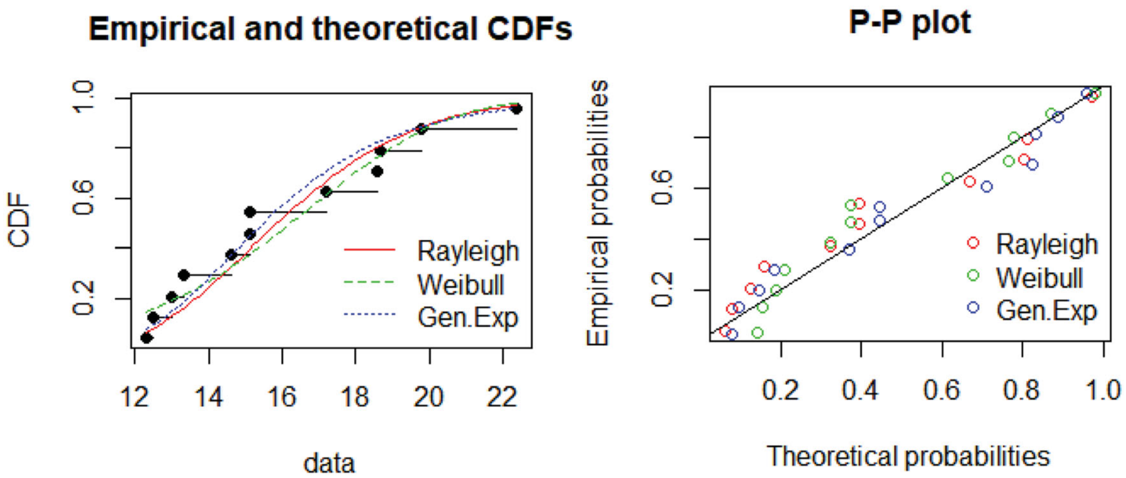

Figure 3. The plot of empirical and theoretical c.d.f's and P–P plot for data X of data 1.

A comparison of R_{IFGM} (X and Y are linked by IFGM), R_{FGM} (X and Y are linked by FGM), and R (X and Y are independent) is presented in Figure 1. In Figure 1, the reliability and corresponding correlation coefficient are given for a specific combination of the parameters λ_1, μ_1, λ_2, μ_2 and β in terms of the variation in the dependence parameter α. Based on the Figures 1(a) and 1(b), it is clear that the assumption of independence between stress and strength leads R to assume higher or lower values of reliability than would be in the actual case.

Let R_l, ρ_l and R_u, ρ_u denote the values of the reliability function and correlation coefficient corresponding to the lower and upper values of the dependence parameter α. Further, from Figure 1(a) it is observed that the reliability $R = 0.8889$ when X and Y are independent, however, it should be $R_u = 0.9318$ ($R_l = 0.8585$) if they are linked by IFGM copula with correlation $\rho_u = 0.3918$ ($\rho_l = -0.2360$). If X and Y are associated with FGM copula $R_u = 0.9254$, with correlation $\rho_u = 0.3139$. Hence IFGM is more relevant on R, since it models higher dependence between X and Y as compared to FGM.

Figure 2 demonstrates the relationship between the distance $\Delta_u = R_u - R_I$ and ρ_u and $\Delta_l = R_I - R_l$ and ρ_l with respect to the variation in the additional parameter β. Based on the Figure 2, we can infer that as distance Δ increases the bound of ρ increases, or in other words, the higher the correlation, between X and Y, the higher the value of Δ.

4.2. *Survival function*

Let (X, Y) be a two-dimensional random variable with p.d.f (8), and then the survival function under IFGMBR distribution is given by

$$S(x,y) = e^{-\lambda_1(x-\mu_1)^2} e^{-\lambda_2(y-\mu_2)^2} \Big(1 + \alpha(1 - e^{-\lambda_1(x-\mu_1)^2})(1 - e^{-\lambda_2(y-\mu_2)^2}) + \beta\,(1 - e^{-\lambda_1(x-\mu_1)^2})^2(1 - e^{-\lambda_2(y-\mu_2)^2})^2\Big). \tag{24}$$

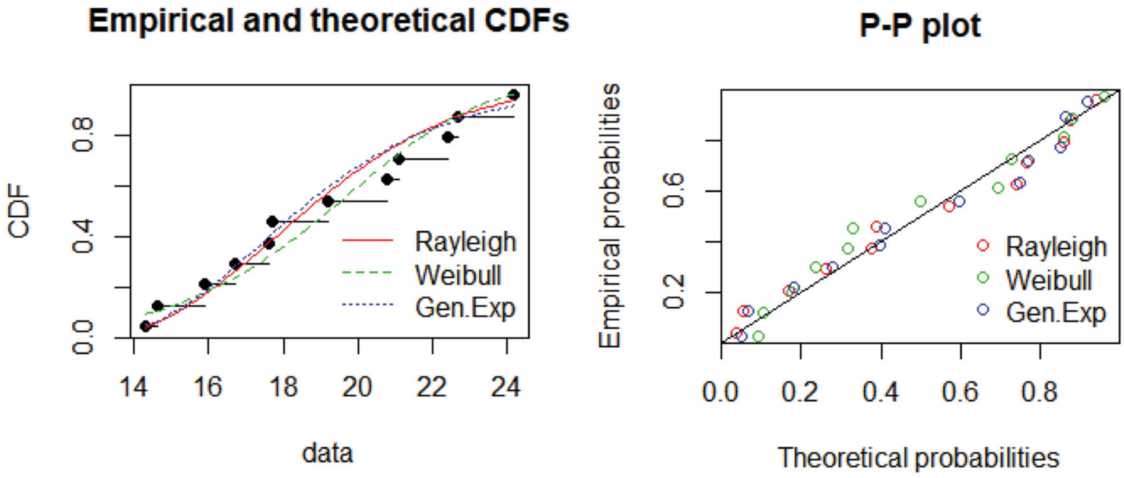

Figure 4. The plot of empirical and theoretical c.d.f.'s and P–P plot for data Y of data 1.

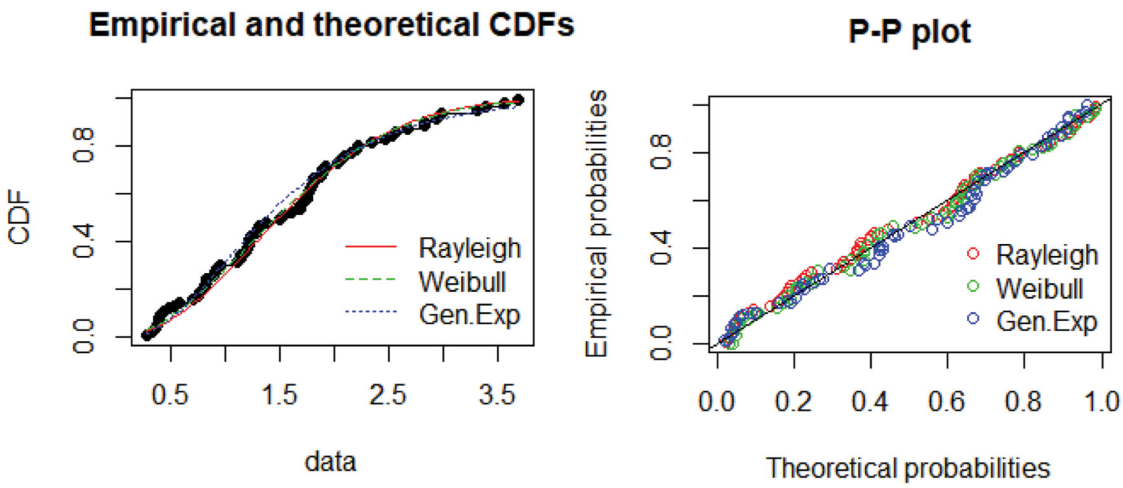

Figure 5. The plot of empirical and theoretical c.d.f.'s and P–P plot for data X of data 2.

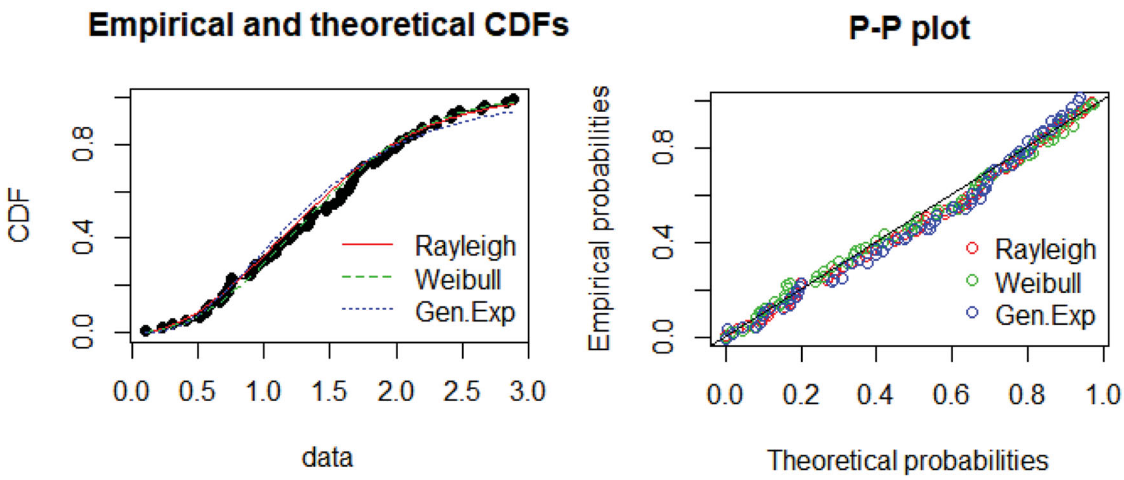

Figure 6. The plot of empirical and theoretical c.d.f.'s and P–P plot for data Y of data 2.

4.3. Mean time to failure

The mean time to failure (MTTF) is defined as the expected mean of the lifetime before a failure occurs. Then the MTTF can be defined as

$$\mathrm{MTTF} = \int_{\mu_1}^{\infty} \int_{\mu_2}^{\infty} S_{(X,Y)}(x, y))\, \mathrm{d}y\, \mathrm{d}x. \tag{25}$$

Then using (24) and (25), MTTF for IFGMBR distribution is obtained as

$$\text{MTTF} = \frac{\pi}{4\sqrt{\lambda_1}\sqrt{\lambda_2}}\left(1+\alpha\left(1-\frac{1}{\sqrt{2}}\right)^2\right)+\beta\left(\frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{\lambda_1}}+\frac{1}{\sqrt{3\lambda_1}}\right)-\frac{\sqrt{\pi}}{\sqrt{2\lambda_1}}\right)$$
$$\times\left(\frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{\lambda_2}}+\frac{1}{\sqrt{3\lambda_2}}\right)-\frac{\sqrt{\pi}}{\sqrt{2\lambda_2}}\right). \tag{26}$$

4.4. Mean residual life function

We consider bivariate mean residual life (m.r.l) function of a new iterated bivariate FGM Rayleigh distribution. Shanbhag and Kotz (1987) proposed the following form of bivariate m.r.l. function

$$r(x,y) = (r_1(x,y), r_2(x,y)), \tag{27}$$

where

$$r_1(x,y) = E(X-x|X\geq x, Y\geq y), \tag{28}$$

and

$$r_2(x,y) = E(Y-y|X\geq x, Y\geq y). \tag{29}$$

The expressions for $r_1(x,y)$ and $r_2(x,y)$ of IFGMBR distribution are obtained as

$$r_1(x,y) = \frac{\sqrt{\pi}\left[\left(1-\text{erf}\left(\sqrt{\lambda_1}(x-\mu_1)\right)\right)+\alpha K_1\right]}{2\sqrt{\lambda_1}e^{-\lambda_1(x-\mu_1)^2}[[1+\alpha u(x)u(y)]}$$
$$+\frac{\beta(u(y))^2K_3}{e^{-\lambda_1(x-\mu_1)^2}(1+\alpha u(x)u(y)+\beta(u(x))^2(u(y))^2)}, \tag{30}$$

and

$$r_2(x,y) = \frac{\sqrt{\pi}\left[\left(1-\text{erf}\left(\sqrt{\lambda_2}(y-\mu_2)\right)\right)+\alpha K_2\right]}{2\sqrt{\lambda_2}e^{-\lambda_2(y-\mu_2)^2}\left[1+\alpha u(x)u(y)\right]}$$
$$+\frac{\beta(u(x))^2K_4}{e^{-\lambda_2(y-\mu_2)^2}(1+\alpha u(x)u(y)+\beta(u(x))^2(u(y))^2)}, \tag{31}$$

where

$$K_1 = u(y)\left(\left(1-\text{erf}\left(\sqrt{\lambda_1}(x-\mu_1)\right)\right)-\frac{1}{\sqrt{2}}\left(1-\text{erf}\left(\sqrt{2\lambda_1}(x-\mu_1)\right)\right)\right),$$
$$K_2 = u(x)\left(\left(1-\text{erf}\left(\sqrt{\lambda_2}(y-\mu_2)\right)\right)-\frac{1}{\sqrt{2}}\left(1-\text{erf}\left(\sqrt{2\lambda_2}(y-\mu_2)\right)\right)\right),$$
$$K_3 = \frac{\sqrt{\pi}}{2\sqrt{\lambda_1}}\left(1-\text{erf}\left(\sqrt{\lambda_1(x-\mu_1)}\right)\right)-\frac{\sqrt{\pi}}{\sqrt{2\lambda_1}}\left(1-\text{erf}\left(\sqrt{2\lambda_1(x-\mu_1)}\right)\right)$$
$$+\frac{\sqrt{\pi}}{2\sqrt{3\lambda_1}}\left(1-\text{erf}\left(\sqrt{3\lambda_1(x-\mu_1)}\right)\right),$$

and

$$K_4 = \frac{\sqrt{\pi}}{2\sqrt{\lambda_2}}\left(1-\text{erf}\left(\sqrt{\lambda_2(y-\mu_2)}\right)\right)-\frac{\sqrt{\pi}}{\sqrt{2\lambda_2}}\left(1-\text{erf}\left(\sqrt{2\lambda_2(y-\mu_2)}\right)\right)+\frac{\sqrt{\pi}}{2\sqrt{3\lambda_2}}\left(1-\text{erf}\left(\sqrt{3\lambda_2(y-\mu_2)}\right)\right).$$

By combining (27), (30) and (31), which give the m.r.l for IFGMBR distribution.

4.5. Vitality function

Sankaran and Nair (1991) proposed the bivariate vitality function as

$$V(x,y) = \left(V_1(x,y), V_2(x,y)\right), \tag{32}$$

where

$$V_1(x,y) = E[X \mid X\geq x, Y\geq y], \tag{33}$$

$$V_2(x,y) = E[Y \mid X \geq x, Y \geq y]. \tag{34}$$

$V(x,y)$ measures the system's expected lifetime based on its current age. $V_1(x,y)$ represents the expected lifetime of a system based on the assumption that the first and second components have lived beyond x and y, respectively. A similar interpretation can be given to $V_2(x,y)$.

Further, the bivariate vitality function $V_i(x,y)$ is related to the mean residual life function $r(x,y)$ with the following relation as

$$V_i(x,y) = x + r_i(x,y), \quad i = 1,2. \tag{35}$$

Using (30) and (31) in (35), the expressions of $V_1(x,y)$ and $V_2(x,y)$ are obtained as

$$\begin{aligned} V_1(x,y) =& x + \frac{\sqrt{\pi}\left[\left(1 - \operatorname{erf}\left(\sqrt{\lambda_1}(x-\mu_1)\right)\right) + \alpha K_1\right]}{2\sqrt{\lambda_1}\mathrm{e}^{-\lambda_1(x-\mu_1)^2}[[1+\alpha u(x)u(y)]} \\ &+ \frac{\beta(u(y))^2 K_3}{\mathrm{e}^{-\lambda_1(x-\mu_1)^2}(1+\alpha u(x)u(y)+\beta(u(x))^2(u(y))^2)}, \end{aligned} \tag{36}$$

$$\begin{aligned} V_2(x,y) =& y + \frac{\sqrt{\pi}\left[\left(1 - \operatorname{erf}\left(\sqrt{\lambda_2}(y-\mu_2)\right)\right) + \alpha K_2\right]}{2\sqrt{\lambda_2}\mathrm{e}^{-\lambda_2(y-\mu_2)^2}\left[1+\alpha u(x)u(y)\right]} \\ &+ \frac{\beta(u(x))^2 K_4}{\mathrm{e}^{-\lambda_2(y-\mu_2)^2}(1+\alpha u(x)u(y)+\beta(u(x))^2(u(y))^2)}. \end{aligned} \tag{37}$$

The vitality function of IFGMBR distribution can be obtained by substituting (36) and (37) in (32).

4.6. Hazard rate function

The bivariate hazard rate function due to Basu (1971) is defined by

$$h(x,y) = \frac{f_{(XY)}(x,y)}{S(x,y)}. \tag{38}$$

Using (8) and (24) with (38) the hazard rate function of IFGMBR distribution is obtained as

$$h(x,y) = \frac{4\lambda_1\lambda_2(x-\mu_1)(y-\mu_2)(1+\alpha M)}{(1+\alpha uv+\beta u^2v^2)} + \frac{\beta uvwz}{(1+\alpha uv+\beta u^2v^2)}, \tag{39}$$

where $M = (2\mathrm{e}^{-\lambda_1(x-\mu_1)^2} - 1)(2\mathrm{e}^{-\lambda_2(y-\mu_2)^2} - 1)$ $w = (3\mathrm{e}^{-\lambda_1(x-\mu_1)^2} - 1)$ and $z = (3\mathrm{e}^{-\lambda_2(y-\mu_2)^2} - 1)$.

Johnson and Kotz (1975b) defined a hazard rate function in a vector form, as shown below

$$h_V(x,y) = \left(\frac{-\partial \log S(x,y)}{\partial x}, \frac{-\partial \log S(x,y)}{\partial y}\right), \tag{40}$$

where $S(x,y)$ denote the bivariate survival function.

$$h_1 = \frac{-\partial \log S(x,y)}{\partial x} = 2\lambda_1(x-\mu_1) - \frac{\lambda_1(x-\mu_1)\mathrm{e}^{-\lambda_1(x-\mu_1)^2} v\,(2\alpha+4\beta uv)}{(1+\alpha uv+\beta u^2v^2)}, \tag{41}$$

$$h_2 = \frac{-\partial \log S(x,y)}{\partial y} = 2\lambda_2(y-\mu_2) - \frac{\lambda_2(y-\mu_2)\mathrm{e}^{-\lambda_2(y-\mu_2)^2} u\,(2\alpha+4\beta uv)}{(1+\alpha uv+\beta u^2v^2)}. \tag{42}$$

Using the expressions (41) and (42) with (40) yields the vector hazard rate function of IFGMBR distribution.

5. Parameter estimation

In this section, we propose a two-phase estimation method to estimate the marginal and dependence parameters. In the first phase we estimate the dependence parameters α and β using the Blomqvist's beta and Kendall's tau measures of association, which is given by

$$M_{XY} = \frac{\alpha}{4} + \frac{\beta}{16}, \tag{43}$$

$$\tau_{XY} = \frac{2\alpha}{9} + \frac{\beta}{18} + \frac{\alpha\beta}{900}. \tag{44}$$

Then, solving the equations $M_{XY} = \widetilde{M}_{XY}$ and $\tau_{XY} = \tilde{\tau}_{XY}$ simultaneously we get the sample estimates of the dependence parameters as $\tilde{\alpha}$ and $\tilde{\beta}$, where $\widetilde{M}_{XY}$ and $\tilde{\tau}_{XY}$ be the sample version of Blomqvist's beta and Kendall's tau, respectively.

Assume that $\{(x_i, y_i), i = 1, 2, \ldots, n\}$ be n independent pairs of bivariate random sample from the joint distribution function $F_{XY}(x, y)$. Now, in the second phase, using the moment estimates $\tilde{\alpha}$ and $\tilde{\beta}$ of dependence parameters, the marginal parameter estimates are obtained by maximizing the following pseudo-likelihood function

$$\begin{aligned}\tilde{\ell} =& \log(f_X(x)) + \log(g_Y(y)) + \log\big(1 + \tilde{\alpha}(1 - 2F_X(x))(1 - 2G_Y(y)) \\ &+ \tilde{\beta}F_X(x)G_Y(y)(2 - 3F_X(x))(2 - 3G_Y(y))\big).\end{aligned} \tag{45}$$

Using Equations (3)–(6) and (45), we have

$$\begin{aligned}\tilde{\ell} \propto& n\log(\lambda_1) + n\log(\lambda_2) + \sum_{i=1}^{n}\log(x_i - \mu_1) - \lambda_1\sum_{i=1}^{n}(x_i - \mu_1)^2 + \sum_{i=1}^{n}\log(y_i - \mu_2) \\ &- \lambda_2\sum_{i=1}^{n}(y_i - \mu_2)^2 + \sum_{i=1}^{n}\log\Big((1 + \tilde{\alpha}(2e^{-\lambda_1(x_i-\mu_1)^2} - 1)(2e^{-\lambda_2(y_i-\mu_2)^2} - 1) \\ &+ \tilde{\beta}(1 - e^{-\lambda_1(x_i-\mu_1)^2})(1 - e^{-\lambda_2(y_i-\mu_2)^2}(3e^{-\lambda_1(x-\mu_1)^2} - 1)(3e^{-\lambda_2(y_i-\mu_2)^2} - 1)\Big).\end{aligned} \tag{46}$$

Then the normal equations are given as follows

$$\begin{aligned}\frac{\partial\tilde{\ell}}{\partial\lambda_1} =& \frac{n}{\lambda_1} - \sum_{i=1}^{n}(x_i - \mu_1)^2 - \sum_{i=1}^{n}\left(\frac{2\tilde{\alpha}(x_i - \mu_1)^2 e^{-\lambda_1(x_i-\mu_1)^2}D(y_i)}{(1 + \tilde{\alpha}D(x_i)D(y_i)}\right) \\ &+ \sum_{i=1}^{n}\frac{1}{H(x,y)(P_1 + H(x,y)}\Big(H(x,y)(4\tilde{\beta}\lambda_2(x_i - \mu_1)(y_i - \mu_2)e^{-\lambda_2(y_i-\mu_2)^2}(1 - e^{-\lambda_2(y_i-\mu_2)^2})) \\ &\times (3e^{-\lambda_2(y_i-\mu_2)^2} - 1)M_1) - K_1(4\lambda_2(x_i - \mu_1)(y_i - \mu_2)e^{-\lambda_2(y_i-\mu_2)^2}M_2\Big) = 0,\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial\tilde{\ell}}{\partial\lambda_2} =& \frac{n}{\lambda_2} - \sum_{i=1}^{n}(y_i - \mu_2)^2 - \sum_{i=1}^{n}\left(\frac{2\tilde{\alpha}(y_i - \mu_2)^2 e^{-\lambda_2(y_i-\mu_2)^2}D(x_i)}{(1 + \tilde{\alpha}D(x_i)D(y_i)}\right) \\ &+ \sum_{i=1}^{n}\frac{1}{H(x,y)(P_1 + H(x,y)}\Big(H(x,y)(4\tilde{\beta}\lambda_1(y_i - \mu_2)(x_i - \mu_1)e^{-\lambda_1(x_i-\mu_1)^2}(1 - e^{-\lambda_1(x_i-\mu_1)^2})) \\ &\times (3e^{-\lambda_1(x_i-\mu_1)^2} - 1)M_3) - K_1(4\lambda_1(x_i - \mu_1)(y_i - \mu_2)e^{-\lambda_1(x_i-\mu_1)^2}M_4\Big) = 0,\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial\tilde{\ell}}{\partial\mu_1} =& -\sum_{i=1}^{n}\frac{1}{x_i - \mu_1} + \lambda_1\sum_{i=1}^{n}2(x_i - \mu_1) + 4\tilde{\alpha}\lambda_1 \times \sum_{i=1}^{n}\left(\frac{(x_i - \mu_1)e^{-\lambda_1(x_i-\mu_1)^2}D(y_i)}{1 + \tilde{\alpha}D(y_i)D(x_i)}\right) \\ &+ \sum_{i=1}^{n}\frac{1}{H(x,y)(P_1 + H(x,y)}\Big(H(x,y)(4\tilde{\beta}\lambda_1\lambda_2(y_i - \mu_2)e^{-\lambda_2(y_i-\mu_2)^2} \\ &\times (1 - e^{-\lambda_2(y_i-\mu_2)^2}))(3e^{-\lambda_2(y_i-\mu_2)^2} - 1)M_5) - K_1(4\lambda_1\lambda_2(y_i - \mu_2)e^{-\lambda_2(y_i-\mu_2)^2}M_6\Big) = 0,\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial\tilde{\ell}}{\partial\mu_2} =& -\sum_{i=1}^{n}\frac{1}{y_i - \mu_2} + \lambda_2\sum_{i=1}^{n}2(y_i - \mu_2) + 4\tilde{\alpha}\lambda_2 \times \sum_{i=1}^{n}\left(\frac{(y_i - \mu_2)e^{-\lambda_2(y_i-\mu_2)^2}D(x_i)}{1 + \tilde{\alpha}D(y_i)D(x_i)}\right) \\ &+ \sum_{i=1}^{n}\frac{1}{H(x,y)(P_1 + H(x,y)}\Big(H(x,y)(4\tilde{\beta}\lambda_1\lambda_2(x_i - \mu_1)e^{-\lambda_1(yxi-\mu_1)^2} \\ &\times (1 - e^{-\lambda_1(x_i-\mu_1)^2}))(3e^{-\lambda_1(x_i-\mu_1)^2} - 1)M_7) - K_1(4\lambda_1\lambda_2(x_i - \mu_1)e^{-\lambda_1(x_i-\mu_1)^2}M_8\Big) = 0,\end{aligned} \tag{50}$$

where

$$
\begin{aligned}
D(x_i) =&(2e^{-\lambda_1(x_i-\mu_1)^2}-1), \quad D(y_i)=(2e^{-\lambda_2(y_i-\mu_2)^2}-1),\\
H(x,y) =&4\lambda_1\lambda_2(x_i-\mu_1)(y_i-\mu_2)e^{-\lambda_1(x_i-\mu_1)^2}e^{-\lambda_1(y_i-\mu_2)^2}(1+\alpha(2e^{-\lambda_1(x_i-\mu_1)^2}-1)\\
&(2e^{-\lambda_2(y_i-\mu_2)^2}-1)),\\
P_1 =&4\lambda_1\lambda_2(x_i-\mu_1)(y_i-\mu_2)e^{-\lambda_1(x_i-\mu_1)^2}e^{-\lambda_1(y_i-\mu_2)^2}\times\beta(1-e^{-\lambda_1(x_i-\mu_1)^2})\\
&\times(1-e^{-\lambda_2(y_i-\mu_2)^2})(3e^{-\lambda_1(x_i-\mu_1)^2}-1)(3e^{-\lambda_2(y_i-\mu_2)^2}-1),\\
M_1 =&4e^{-2\lambda_1(x_i-\mu_1)^2}(1-2\lambda_1(x_i-\mu_1)^2)-e^{-\lambda_1(x_i-\mu_1)^2}\\
&(1-\lambda_1(x_i-\mu_1)^2)-3e^{-3\lambda_1(x_i-\mu_1)^2}(1-3\lambda_1(x_i-\mu_1)^2),\\
M_2 =&e^{-\lambda_1(x_i-\mu_1)^2}\Big(1-\lambda_1(x_i-\mu_1)^2+\alpha(2e^{-\lambda_2(y_i-\mu_2)^2}-1)(2e^{-\lambda_1(x_i-\mu_1)^2}\\
&-4\,\lambda_1(x_i-\mu_1)^2e^{-\lambda_1(x_i-\mu_1)^2}-(1-\lambda_1(x_i-\mu_1)^2))\Big),\\
M_3 =&4e^{-2\lambda_2(y_i-\mu_2)^2}(1-2\lambda_2(y_i-\mu_2)^2)-e^{-\lambda_2(y_i-\mu_2)^2}\\
&(1-\lambda_2(y_i-\mu_2)^2)-3e^{-3\lambda_2(y_i-\mu_2)^2}(1-3\lambda_2(y_i-\mu_2)^2),\\
M_4 =&e^{-\lambda_2(y_i-\mu_2)^2}\Big(1-\lambda_2(y_i-\mu_2)^2+\alpha(2e^{-\lambda_1(x_i-\mu_1)^2}-1)(2e^{-\lambda_2(y_i-\mu_2)^2}\\
&-4\lambda_2(y_i-\mu_2)^2e^{-\lambda_2(y_i-\mu_2)^2})-(1-\lambda_2(y_i-\mu_2)^2))\Big),\\
M_5 =&4e^{-2\lambda_1(x_i-\mu_1)^2}(4\lambda_1(x_i-\mu_1)^2-1)-e^{-\lambda_1(x_i-\mu_1)^2}
\end{aligned}
$$

Table 3. Estimates of R when $\mu_1=\mu_2=0$ and $\beta=0.2$ with different combinations of (λ_1,λ_2) with varying values of α.

		α					
n	(λ_1,λ_2)	−0.9	−0.5	−0.1	0.5	0.1	0.9
	(0.2, 0.6)	0.7146	0.7318	0.7489	0.7575	0.7746	0.7918
		0.7021	0.7235	0.7389	0.7519	0.7877	0.7908
		0.0051	0.0044	0.0032	0.0056	0.0038	0.0046
20	(0.5, 1.1)	0.6578	0.6723	0.6869	0.6942	0.7087	0.7233
		0.6678	0.6703	0.6799	0.6812	0.7008	0.7120
		0.0068	0.0055	0.0048	0.0036	0.0058	0.0041
	(1.1, 2.8)	0.6851	0.7011	0.7171	0.7251	0.7412	0.7572
		0.6701	0.6941	0.7021	0.7171	0.7322	0.7472
		0.0058	0.0039	0.0051	0.0044	0.0035	0.0062
	(0.2, 0.6)	0.7146	0.7318	0.7489	0.7575	0.7746	0.7918
		0.7081	0.7245	0.7411	0.7587	0.7887	0.7990
		0.0044	0.0038	0.0024	0.0040	0.0028	0.0033
50	(0.5, 1.1)	0.6578	0.6723	0.6869	0.6942	0.7087	0.7233
		0.6699	0.6777	0.6812	0.6899	0.7070	0.7190
		0.0048	0.0039	0.0033	0.0020	0.0049	0.0035
	(1.1, 2.8)	0.6851	0.7011	0.7171	0.7251	0.7412	0.7572
		0.6761	0.6988	0.7098	0.7190	0.7402	0.7492
		0.0044	0.0028	0.0030	0.0035	0.0022	0.0042
	(0.2, 0.6)	0.7146	0.7318	0.7489	0.7575	0.7746	0.7918
		0.7111	0.7295	0.7488	0.7687	0.7889	0.8010
		0.0025	0.0021	0.0018	0.0029	0.0015	0.0020
100	(0.5, 1.1)	0.6578	0.6723	0.6869	0.6942	0.7087	0.7233
		0.6611	0.6780	0.6898	0.6901	0.7058	0.7214
		0.0030	0.0029	0.0018	0.0015	0.0033	0.0025
	(1.1, 2.8)	0.6851	0.7011	0.7171	0.7251	0.7412	0.7572
		0.6822	0.7018	0.7118	0.7199	0.7421	0.7511
		0.0031	0.0019	0.0022	0.0025	0.0011	0.0035
	(0.2, 0.6)	0.7146	0.7318	0.7489	0.7575	0.7746	0.7918
		0.7121	0.7302	0.7499	0.7627	0.7819	0.8001
		0.0011	0.0018	0.009	0.0020	0.0006	0.0014
200	(0.5, 1.1)	0.6578	0.6723	0.6869	0.6942	0.7087	0.7233
		0.6600	0.6755	0.6860	0.6933	0.7078	0.7250
		0.0018	0.0020	0.0011	0.0009	0.0028	0.0015
	(1.1, 2.8)	0.6851	0.7011	0.7171	0.7251	0.7412	0.7572
		0.6852	0.7026	0.7133	0.7239	0.7451	0.7551
		0.0022	0.0011	0.0009	0.0016	0.0008	0.0025

First row: true values, second row: estimates, third row: MSE for R.

$$
\begin{aligned}
&(2\lambda_1(x_i-\mu_1)^2-1)-3\mathrm{e}^{-3\lambda_1(x_i-\mu_1)^2}(6\lambda_1(x_i-\mu_1)^2-1),\\
M_6 =&2\lambda_1(x_i-\mu_1)^2-\mathrm{e}^{-\lambda_1(x_i-\mu_1)^2}+\alpha(2\mathrm{e}^{-\lambda_2(y_i-\mu_2)^2}-1)\Big((4\lambda_1(x_i-\mu_1)^2\mathrm{e}^{-2\lambda_1(x_i-\mu_1)^2}\\
&-\mathrm{e}^{-2\lambda_1(x_i-\mu_1)^2})-(2\lambda_1(x_i-\mu_1)^2\mathrm{e}^{-\lambda_1(x_i-\mu_1)^2}-\mathrm{e}^{-\lambda_1(x_i-\mu_1)^2})\Big),\\
M_7 =&4\mathrm{e}^{-2\lambda_2(y_i-\mu_2)^2}(4\lambda_2(y_i-\mu_2)^2-1)-\mathrm{e}^{-\lambda_2(y_i-\mu_2)^2}(2\lambda_2(y_i-\mu_2)^2-1)\\
&-3\mathrm{e}^{-3\lambda_2(y_i-\mu_2)^2}(6\lambda_2(y_i-\mu_2)^2-1)\\
M_8 =&2\lambda_2(y_i-\mu_2)^2-\mathrm{e}^{-\lambda_2(y_i-\mu_2)^2}+\alpha(2\mathrm{e}^{-\lambda_1(x_i-\mu_1)^2}-1)\Big((4\lambda_2(y_i-\mu_2)^2\mathrm{e}^{-2\lambda_2(y_i-\mu_2)^2}\\
&-\mathrm{e}^{-2\lambda_2(y_i-\mu_2)^2})-(2\lambda_2(y_i-\mu_2)^2\mathrm{e}^{-\lambda_2(y_i-\mu_2)^2}-\mathrm{e}^{-\lambda_2(y_i-\mu_2)^2})\Big).
\end{aligned}
$$

Because the aforementioned pseudo-likelihood equations are nonlinear and cannot be solved analytically, it must be solved numerically using any iterative approach. Further, substituting the estimates $\widehat{\lambda}_1, \widehat{\mu}_1, \widehat{\lambda}_2, \widehat{\mu}_2, \tilde{\alpha}$ and $\tilde{\beta}$ in (19) results the estimate of stress–strength reliability R.

6. Simulation study

In this Section, we investigate the influence of the dependence parameters α and β on R numerically by using a two-phase estimation approach for different sample sizes using the criteria of mean square error (MSE) with respect to different combinations of stress and strength parameters. For mathematical simplicity, here we assume that the location parameters μ_1 and μ_2 are assumed to be known. A Monte-Carlo simulation study is carried out to compare the estimates numerically based on the samples generated from IFGMBR distribution for different sample

Table 4. Estimates of R when $\mu_1 = \mu_2 = 0$ and $\beta = 0.5$ with different combinations of (λ_1,λ_2) with varying values of α.

		α					
n	(λ_1, λ_2)	−0.9	−0.5	−0.1	0.5	0.1	0.9
	(0.2, 0.6)	0.7242	0.7414	0.7585	0.7671	0.7842	0.8014
		0.7078	0.7201	0.7314	0.7598	0.7807	0.7878
		0.0061	0.0041	0.0048	0.0036	0.0044	0.0055
20	(0.5, 1.1)	0.6669	0.6815	0.6960	0.7033	0.7178	0.7324
		0.6544	0.6743	0.6789	0.6892	0.7098	0.7190
		0.0058	0.0044	0.0040	0.0049	0.0058	0.0061
	(1.1, 2.8)	0.6946	0.7107	0.7267	0.7347	0.7507	0.7668
		0.6701	0.6941	0.7081	0.7179	0.7388	0.7582
		0.0051	0.0044	0.0041	0.0064	0.0049	0.0055
	(0.2, 0.6)	0.7242	0.7414	0.7585	0.7671	0.7842	0.8014
		0.7108	0.7291	0.7398	0.7601	0.7847	0.7987
		0.0042	0.0033	0.0030	0.0022	0.0031	0.0045
50	(0.5,1.1)	0.6669	0.6815	0.6960	0.7033	0.7178	0.7324
		0.6598	0.6781	0.6800	0.6904	0.7111	0.7201
		0.0035	0.0024	0.0033	0.0040	0.0045	0.0055
	(1.1, 2.8)	0.6946	0.7107	0.7267	0.7347	0.7507	0.7668
		0.6791	0.7044	0.7100	0.7198	0.7403	0.7502
		0.0044	0.0032	0.0035	0.0055	0.0038	0.0044
	(0.2, 0.6)	0.7242	0.7414	0.7585	0.7671	0.7842	0.8014
		0.7187	0.7381	0.7418	0.7641	0.7888	0.8013
		0.0030	0.0023	0.0025	0.0016	0.0021	0.0031
100	(0.5, 1.1)	0.6669	0.6815	0.6960	0.7033	0.7178	0.7324
		0.6654	0.6798	0.6865	0.6988	0.7154	0.7245
		0.0028	0.0015	0.0022	0.0030	0.0039	0.0042
	(1.1, 2.8)	0.6946	0.7107	0.7267	0.7347	0.7507	0.7668
		0.6801	0.7094	0.7150	0.7210	0.7463	0.7592
		0.0033	0.0025	0.0015	0.0030	0.0021	0.0026
	(0.2, 0.6)	0.7242	0.7414	0.7585	0.7671	0.7842	0.8014
		0.7217	0.7410	0.7498	0.7684	0.7893	0.8035
		0.0022	0.0018	0.0015	0.0009	0.0010	0.0011
200	(0.5, 1.1)	0.6669	0.6815	0.6960	0.7033	0.7178	0.7324
		0.6688	0.6848	0.6895	0.7088	0.7187	0.7295
		0.0015	0.0006	0.0012	0.0014	0.0020	0.0018
	(1.1, 2.8)	0.6946	0.7107	0.7267	0.7347	0.7507	0.7668
		0.6891	0.7102	0.7250	0.7289	0.7513	0.7652
		0.0012	0.0016	0.0008	0.0019	0.0011	0.0015

First row: true values, second row: estimates, third row: MSE for R.

Table 5. Estimates of R when $\mu_1 = \mu_2 = 0.5$ and $\beta = 0.8$ with different combinations of (λ_1,λ_2) with varying values of α.

		α					
n	(λ_1,λ_2)	−0.9	−0.5	−0.1	0.5	0.1	0.9
	(0.2, 0.65)	0.7380	0.7554	0.7729	0.7817	0.7991	0.8166
		0.7218	0.7452	0.7654	0.7878	0.8017	0.8278
		0.0181	0.0165	0.0148	0.0154	0.0134	0.0197
20	(0.3,1.1)	0.7579	0.7756	0.7934	0.8022	0.8200	0.8377
		0.7544	0.7746	0.7844	0.8019	0.8198	0.8250
		0.0188	0.0166	0.0198	0.0169	0.0158	0.0111
	(0.6, 1.5)	0.6912	0.7071	0.7230	0.7309	0.7468	0.7627
		0.6701	0.6941	0.7081	0.7179	0.7388	0.7582
		0.0154	0.0188	0.0241	0.0122	0.0199	0.0125
	(0.2, 0.65)	0.7380	0.7554	0.7729	0.7817	0.7991	0.8166
		0.7354	0.7466	0.7699	0.7887	0.8098	0.8221
		0.0125	0.0133	0.0115	0.0140	0.0101	0.0151
50	(0.3, 1.1)	0.7579	0.7756	0.7934	0.8022	0.8200	0.8377
		0.7524	0.7687	0.7811	0.7819	0.8192	0.8298
		0.0122	0.0138	0.0118	0.0146	0.0118	0.0101
	(0.6, 1.5)	0.6912	0.7071	0.7230	0.7309	0.7468	0.7627
		0.6855	0.6992	0.7023	0.7189	0.7254	0.7599
		0.0122	0.0149	0.0181	0.0106	0.0159	0.0110
	(0.2, 0.65)	0.7380	0.7554	0.7729	0.7817	0.7991	0.8166
		0.7372	0.7522	0.7787	0.7898	0.7951	0.8106
		0.0100	0.0086	0.0074	0.0111	0.0062	0.0082
100	(0.3, 1.1)	0.7579	0.7756	0.7934	0.8022	0.8200	0.8377
		0.7521	0.7785	0.7902	0.8085	0.8264	0.8370
		0.0086	0.0103	0.0071	0.0066	0.0089	0.0073
	(0.6, 1.5)	0.6912	0.7071	0.7230	0.7309	0.7468	0.7627
		0.6998	0.7021	0.7322	0.7351	0.7433	0.7663
		0.0062	0.0112	0.0125	0.0073	0.0055	0.0092
	(0.2, 0.65)	0.7380	0.7554	0.7729	0.7817	0.7991	0.8166
		0.7311	0.7598	0.7617	0.7808	0.7998	0.8136
		0.0084	0.0045	0.0052	0.0061	0.0043	0.0065
200	(0.3, 1.1)	0.7579	0.7756	0.7934	0.8022	0.8200	0.8377
		0.7506	0.7801	0.7980	0.8036	0.8255	0.8310
		0.0071	0.0098	0.0045	0.0055	0.0063	0.0058
	(0.6, 1.5)	0.6912	0.7071	0.7230	0.7309	0.7468	0.7627
		0.6903	0.7083	0.7301	0.7366	0.7462	0.7633
		0.0052	0.0088	0.0095	0.0056	0.0042	0.0079

First row: true values, second row: estimates, third row: MSE for R.

sizes $n = 20$, 50, 100, and 200. We use the following steps to generate the bivariate samples (x_i, y_i), $i = 1, 2, \ldots, n$ from IFGMBR distribution with parameters $\lambda_1, \mu_1, \lambda_2, \mu_2, \beta$ and α.

(1) Generate two independent random samples u_i and t_i for $i = 1, 2, \ldots, n$, from $U(0, 1)$ distribution.
(2) Compute v_i using the equation $C(v_i/u_i) = t_i$ and where $C(v_i/u_i)$ represents the conditional copula of IFGMBR distribution.
(3) Set $x_i = \sqrt{\frac{-\log(1-u_i)}{\lambda_1}} + \mu_1$ and $y_i = \sqrt{\frac{-\log(1-v_i)}{\lambda_2}} + \mu_2$.
(4) The bivariate vector (x_i, y_i) for $i = 1, 2, \ldots, n$, is the random pair from the IFGMBR distribution.

The numerical investigations of derived reliability estimators are carried out with the help of the R software. The simulation results are used to compare the reliability estimates on the basis of average estimates and MSEs. The main purpose of such investigation is to assess the pattern of dependence relations between stress and strength and also to observe their effects on R. Such investigations are performed for different sample sizes with variations of α and the additional parameter β. The average estimates and MSEs of stress–strength reliability based on 10,000 replications are presented in Tables 3–7.

(1) It is observed that as the dependence parameter increases, the reliability R also increases as well, i.e., the higher value of the association parameter results in a higher value of reliability. Also, as the sample size increases, the MSEs of the estimates gradually decrease.
(2) It is observed that when the additional parameter β increases, the reliability R is also increased. Further, the value of R is attaining maximum (or minimum) while the dependence parameter α is maximum ($\alpha = 0.9$) (or minimum ($\alpha = -0.9$)).
(3) The value of R shows increasing trends when the dependence parameter α increases while (λ_1, λ_2) parameter values are varying with two different values of β.

Table 6. Estimates of MTTF when $\beta = 0.2$ with different combinations of (λ_1,λ_2) with varying values of α.

		α					
n	(λ_1, λ_2)	−0.9	−0.5	−0.1	0.5	0.1	0.9
	(0.2, 0.6)	2.1090	2.1879	2.2653	2.3820	2.3942	2.4598
		2.0123	2.1456	2.2456	2.2907	2.3212	2.3435
		0.0056	0.0049	0.0039	0.0060	0.0048	0.0051
20	(0.5, 1.1)	0.9854	1.0218	1.0581	1.0826	1.1163	1.1489
		0.9809	1.0438	1.0522	1.1056	1.1450	1.1476
		0.0055	0.0054	0.0046	0.0056	0.0055	0.0043
	(1.1, 2.8)	0.4164	0.4317	0.4471	0.4548	0.4701	0.4855
		0.4088	0.4222	0.4287	0.4343	0.4445	0.4780
		0.0055	0.0045	0.0034	0.0048	0.0038	0.0053
	(0.2, 0.6)	2.1090	2.1879	2.2653	2.3820	2.3942	2.4598
		2.0197	2.1175	2.2544	2.2921	2.3202	2.3650
		0.0044	0.0038	0.0024	0.0040	0.0028	0.0033
50	(0.5, 1.1)	0.9854	1.0218	1.0581	1.1126	1.1363	1.1489
		0.9573	0.9876	1.0112	1.0652	1.1003	1.1622
		0.0040	0.0043	0.0034	0.0038	0.0044	0.0024
	(1.1, 2.8)	0.4164	0.4317	0.4471	0.4548	0.4701	0.4855
		0.4164	0.4211	0.4292	0.4433	0.4565	0.4700
		0.0044	0.0036	0.0029	0.0041	0.0030	0.0048
	(0.2, 0.6)	2.1090	2.1879	2.2653	2.3820	2.3942	2.4598
		2.1264	2.1975	2.2765	2.3987	2.3955	2.4655
		0.0033	0.0032	0.0019	0.0031	0.0023	0.0023
100	(0.5, 1.1)	0.9854	1.0218	1.0581	1.1126	1.1363	1.1489
		0.9644	0.9800	1.0234	1.0564	1.1122	1.1822
		0.0036	0.0035	0.0028	0.0028	0.0044	0.0024
	(1.1, 2.8)	0.4164	0.4317	0.4471	0.4548	0.4701	0.4855
		0.4164	0.4321	0.4213	0.4547	0.4575	0.4732
		0.0036	0.0022	0.0015	0.0032	0.0021	0.0031
	(0.2, 0.6)	2.1090	2.1879	2.2653	2.3820	2.3942	2.4598
		2.1086	2.1891	2.2611	2.3834	2.3941	2.4511
		0.0019	0.0012	0.0011	0.0021	0.0014	0.0014
200	(0.5, 1.1)	0.9854	1.0218	1.0581	1.1126	1.1363	1.1489
		0.9811	1.0212	1.0595	1.1100	1.1359	1.1491
		0.0021	0.0022	0.0019	0.0017	0.0025	0.0019
	(1.1, 2.8)	0.4164	0.4317	0.4471	0.4548	0.4701	0.4855
		0.4154	0.4213	0.4488	0.4533	0.4791	0.4852
		0.0022	0.0012	0.0006	0.0014	0.0011	0.0016

First row: true values, second row: estimates, third row: MSE for MTTF.

(4) The proposed reliability estimators behave consistently while increasing sample sizes through simulated samples.
(5) It is observed that when the dependence parameter α increases the MTTF function also increases. Furthermore, when β increases MTTF function also increases.
(6) Also, as the sample size increases, the MSEs of the estimates of MTTF gradually decrease.

7. Real life application

In this section, two different real data sets are analysed to illustrate the applications of the proposed model in various research fields. First, we examined the data set from the field of medicine (data I). This data represents the human heart condition in response to external/emotional stress in a controlled environment used by Pham (2020) for two-parameter exponential margins.

The second data set (data II) is an application to meteorology and includes rainfall data at the Los Angeles Civic Center between 1943 and 2018 shown in Pak et al. (2022). Based on Weibull record data in the presence of inter-record times, the data was used to estimate the stress–strength reliability R. We transformed data II by taking the square root of the data. The correlation coefficient and test of correlation for real data sets are presented in Table 8.

The first step is to perform a K–S test on each data set separately to determine the goodness of fit for the marginal distributions. The results are shown in Tables 9 and 10, respectively. The Rayleigh distribution has a lower AIC and BIC than the other life distributions, as shown by Table 9. Table 10 further suggests that the Rayleigh distribution fits the data set well. Moreover, the Rayleigh distribution is known for its simplicity and flexibility in modelling various types of data. This makes it a popular choice in survival analysis and reliability studies.

Finally, the a bivariate Rayleigh distribution based on IFGM copula are fitted to the data sets. A comparison of IFGMBR has been carried out with the FGMBR distribution (James et al., 2023), FGMBW distribution (Almetwally et al., 2020) and FGMBGE distribution (Lutfiah et al., 2017) on the basis of AIC and BIC. It is observe that AIC and

Table 7. Estimates of MTTF when $\beta = 0.6$ with different combinations of (λ_1,λ_2) with varying values of α.

		α					
n	(λ_1, λ_2)	−0.9	−0.5	−0.1	0.5	0.1	0.9
	(0.2, 0.6)	2.1448	2.2226	2.3004	2.3393	2.4171	2.4949
		2.0123	2.2356	2.2326	2.2417	2.3212	2.4415
		0.0037	0.0041	0.0030	0.0020	0.0033	0.0044
20	(0.5, 1.1)	1.0018	1.0381	1.0745	1.0927	1.1290	1.1653
		1.0018	1.0234	1.0567	1.0887	1.1270	1.1441
		0.0045	0.0031	0.0038	0.0026	0.0044	0.0043
	(1.1, 2.8)	0.4234	0.4387	0.4540	0.4617	0.4771	0.4924
		0.4123	0.4265	0.4347	0.4555	0.4655	0.4810
		0.0036	0.0028	0.0040	0.0022	0.0044	0.0031
	(0.2, 0.6)	2.1448	2.2226	2.3004	2.3393	2.4171	2.4949
		2.1328	2.3222	2.3765	2.3654	2.5663	2.5433
		0.0024	0.0033	0.0018	0.0011	0.0023	0.0032
50	(0.5,1.1)	1.0018	1.0381	1.0745	1.0927	1.1290	1.1653
		1.0018	1.0271	1.0654	1.0876	1.1540	1.1876
		0.0033	0.0018	0.0022	0.0015	0.0030	0.0026
	(1.1,2.8)	0.4234	0.4387	0.4540	0.4617	0.4771	0.4924
		0.4453	0.4456	0.4654	0.4765	0.4854	0.4984
		0.0017	0.0009	0.0014	0.0008	0.0019	0.0015
	(0.2, 0.6)	2.1448	2.2226	2.3004	2.3393	2.4171	2.4949
		2.1328	2.3542	2.3865	2.3704	2.5342	2.6758
		0.0018	0.0022	0.0011	0.0008	0.0015	0.0012
100	(0.5,1.1)	1.0018	1.0381	1.0745	1.0927	1.1290	1.1653
		1.0018	1.0231	1.0456	1.0765	1.0977	1.1544
		0.0023	0.0008	0.0014	0.0006	0.0016	0.0019
	(1.1,2.8)	0.4234	0.4387	0.4540	0.4617	0.4771	0.4924
		0.4123	0.4221	0.4435	0.4564	0.4721	0.4901
		0.0011	0.0007	0.0016	0.0009	0.0022	0.0014
	(0.2, 0.6)	2.1448	2.2226	2.3004	2.3393	2.4171	2.4949
		2.1345	2.2231	2.3011	2.3365	2.4161	2.4876
		0.0009	0.0014	0.0006	0.0005	0.0007	0.0006
200	(0.5,1.1)	1.0018	1.0381	1.0745	1.0927	1.1290	1.1653
		1.0018	1.0241	1.0654	1.0876	1.1321	1.1675
		0.0011	0.0004	0.0010	0.0004	0.0012	0.0008
	(1.1,2.8)	0.4234	0.4387	0.4540	0.4617	0.4771	0.4924
		0.4234	0.4365	0.4498	0.4654	0.4691	0.4960
		0.0008	0.0004	0.0011	0.0007	0.0013	0.0010

First row: true values, second row: estimates, third row: MSE for MTTF.

Table 8. The correlation coefficient and test of correlation for real data sets.

Data	Correlation measure	Correlation	*P*-value
Data I	Pearson's	−0.0280	0.9311
	Kendall's	−0.0458	0.8366
Data II	Pearson's	0.1757	0.1397
	Kendall's	0.1064	0.1861

Table 9. Goodness of fit test for Data 1.

	X				*Y*			
	D	*P*-value	AIC	BIC	D	*P*-value	AIC	BIC
Rayleigh	0.1888	0.7858	63.1693	64.1391	0.1563	0.1563	64.6732	65.6430
Weibull	0.2091	0.6703	66.2331	67.2029	0.1715	0.8151	65.8901	66.8599
Gen.Exp	0.1573	0.9277	63.8134	64.7832	0.1640	0.8526	65.97461	66.9444

BIC of IFGMBR distribution is least in compare to other FGMB distributions, the results are presented in Tables 11 and 12, respectively.

8. Concluding remarks

Huang and Kotz (1984) investigated the single iterated FGM distribution and found that the maximum positive correlation is higher than the usual FGM distribution. Also he showed that a single iteration can triple the covariance for a certain marginal distributions. But iteration based FGM copula and its application has not discussed in the literature. This present study is an attempt on this direction.

Table 10. Goodness of fit test for Data 2.

	X				Y			
	D	*P*-value	AIC	BIC	D	*P*-value	AIC	BIC
Rayleigh	0.0623	0.9425	177.4054	181.9588	0.0780	0.7726	148.5079	153.0613
Weibull	0.0658	0.9144	177.2219	181.7752	0.0652	0.919	147.4094	151.9627
Gen.Exp	0.1021	0.4402	180.0294	184.5827	0.0937	0.5519	156.7723	161.3257

Table 11. The estimates of the parameters of FGM distributions for data 1.

	$\hat{\lambda}_1$	$\hat{\mu}_1$	$\hat{\lambda}_2$	$\hat{\mu}_2$	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC	MTTF
IFGM-Rayleigh	0.0093	0.2345	0.0079	1.2771	−0.5302	0.9011	121.2547	124.164	90.6575
FGM-Rayleigh	0.0098	8.0036	0.0024	0.0675	−0.4649	–	152.4928	154.9174	–
FGM-Weibull	4.4980	18.816	5.701	19.311	−0.2940	–	137.8986	140.3232	–
FGM-Gen.Exp	0.0714	0.0042	0.4085	0.0203	−0.3536	–	267.2368	269.6613	–

Table 12. The estimates of the parameters of FGM distributions for data 2.

	$\hat{\lambda}_1$	$\hat{\mu}_1$	$\hat{\lambda}_2$	$\hat{\mu}_2$	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC	
IFGM-Rayleigh	1.0903	0.4045	0.3735	0.0334	0.4745	0.0239	325.6484	326.5404	1.2819
FGM-Rayleigh	0.3165	0.0220	0.5009	0.1775	0.5647	–	330.937	334.3204	–
FGM-Weibull	1.9266	1.7818	2.2579	1.6015	0.5755	–	331.465	334.8484	–
FGM-Gen.Exp	0.13599	0.0159	0.8388	1.4317	0.4556	–	619.7982	631.1815	–

The main objective of the study is to develop an IFGM based SSR model using Rayleigh marginal as a baseline distribution. We derived the expression for the correlation coefficient ρ and we found that IFGMBR distribution is more suitable in compared to the FGMBR distribution for modelling higher positive association. In addition, we found that, increasing the extra parameter β boosts the upper bound of correlation coefficient ρ.

Further, a graphical comparison of SSR using IFGM and FGM based on Rayleigh distribution is performed with respect to the dependence parameter α and the additional parameter β. From the graphical representation it is clear that the if the association between X and Y is ignored then reliability may be either over or under estimated. Moreover IFGMBR is better than FGMBR, because it capture higher dependence between X and Y .

The performance of dependence SSR is studied by Monte-Carlo simulation using a two-phase estimation method. We investigated the expression of R with respect to the variation in the dependence parameter α as well as in additional parameter β while fixing others parameters. From the numerical results we found that as the dependence parameter increases, the reliability R is also increases as well, i.e. higher value of the association parameter, higher the value of reliability. Also, as the sample size increases, the MSEs of estimates obtained are decresing. Furthermore, as future study objectives, one may think other life distributions as well as the second iterated FGM family for modelling SSR.

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