# Bayesian-inspired minimum contamination designs under a double-pair conditional effect model 

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# Bayesian-inspired minimum contamination designs under a double-pair conditional effect model 

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#### Abstract

In two-level fractional factorial designs, conditional main effects can provide insights by which to analyze factorial effects and facilitate the de-aliasing of fully aliased two-factor interactions. Conditional main effects are of particular interest in situations where some factors are nested within others. Most of the relevant literature has focused on the development of data analysis tools that use conditional main effects, while the issue of optimal factorial design for a given linear model involving conditional main effects has been largely overlooked. Mukerjee, Wu and Chang [Statist. Sinica 27 (2017) 997-1016] established a framework by which to optimize designs under a conditional effect model. Although theoretically sound, their results were limited to a single pair of conditional and conditioning factors. In this paper, we extend the applicability of their framework to double pairs of conditional and conditioning factors by providing the corresponding parameterization and effect hierarchy. We propose a minimum contamination-based criterion by which to evaluate designs and develop a complementary set theory to facilitate the search of minimum contamination designs. The catalogues of 16 - and 32 -run minimum contamination designs are provided. For five to twelve factors, we show that all 16-run minimum contamination designs under the conditional effect model are also minimum aberration according to Fries and Hunter [Technometrics 22 (1980) 601-608].


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## 1. Introduction

Factorial designs have been widely used in industry and academia in the past decades. Two-level fractional factorial designs have proven particularly effective in situations in which the purpose of experiments is to screen out inactive factors. Researchers have dedicated considerable effort to the evaluation of two-level fractional factorial designs. Fries and Hunter (1980) proposed a model robust criterion named minimum aberration tailored specifically to two-level regular fractional factorial designs. The minimum aberration criterion minimizes the wordlength patterns of two-level regular designs in a sequential manner from lower-order factorial effects to higher-order ones. It is established for design selection under the assumption that lower-order effects are more important than higherorder effects and effects of the same order are equally important. The minimum aberration criterion has since been adaptive to accommodate nonregular fractional factorial designs (Cheng et al., 2002; Tang \& Deng, 1999; Xu \& Wu, 2001) and multi-stratum factorial designs (Chang, 2022; Chang \& Cheng, 2018). All the aforementioned minimum aberration criteria were developed under an orthogonal parameterization of factorial effects (Cheng, 2014, chapter 6). Accordingly, a defining word of length four, say $F_{1} F_{2} F_{3} F_{4}$, creates three pairs of fully aliased twofactor interactions as follows: $F_{1} F_{2}=F_{3} F_{4}, F_{1} F_{3}=F_{2} F_{4}$ and $F_{1} F_{4}=F_{2} F_{3}$. The two-factor interactions in each pair are completely mixed up and cannot be estimated simultaneously. Interested readers may refer to Mukerjee and Wu (2006), Cheng (2014) and Wu and Hamada (2021) for further details.

In some practical situations, it is preferable to investigate a two-factor interaction via two conditional main effects, with each one conditionally defined according to another factor of a specific level. For example, the two-factor interaction $F_{1} F_{2}$ can be decomposed as the difference between a conditional main effect $F_{1}$ conditioned on the low level of $F_{2}$ and that conditioned on the high level of $F_{2}$. Sliding level experiments in engineering are structured in this way (Wu \& Hamada, 2021, p. 343), where the interest is on the conditional main effects conditioned on the slid factors. Mukerjee et al. (2017) reported on an industrial experiment involving motor and speed as two factors. The objective of the experiment was to assess the comparison of the motors separately at each speed; therefore, the conditional main effects of the motors conditioned on each level of speed are of particular interest. Other intriguing examples pertaining to the use of conditional main effects are outlined in Wu (2015) and Wu (2018).

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It is known, also mentioned in Wu (2015), that for 2 two-level factors $F_{1}$ and $F_{2}$, the main effect $F_{1}$ in conjunction with the interaction $F_{1} F_{2}$ spans the same vector space as that spanned by the conditional main effects $F_{1}$ respectively conditioned on the two levels of $F_{2}$. Mukerjee et al. (2017) referred to $F_{1}$ and $F_{2}$ as a pair of conditional and conditioning factors. To deal with a single pair of conditional and conditioning factors, Mukerjee et al. (2017) proposed a model in which the main effect of $F_{1}$ and the two-factor interaction $F_{1} F_{2}$ are replaced with the two associated conditional main effects. This can be viewed as an alternative to the aforementioned orthogonal parameterization, under which the defining relation $I=F_{1} F_{2} F_{3} F_{4}$ produces partially aliased effects (neither fully aliased nor orthogonal). In this paper, the model proposed by Mukerjee et al. (2017) is referred to as a conditional effect model. The fact that the parameterization of a single-pair conditional effect model differs from that of a model tailored to the minimum aberration necessitates a new aberration criterion applicable to conditional effect models. Mukerjee et al. (2017) proposed a minimum aberration criterion as well as a strategy by which to search for designs under a single-pair conditional effect model. Apart from Mukerjee et al. (2017), an alternative approach by the use of the indicator function in Pistone and Wynn (1996) is given by Sabbaghi (2020), who developed an algebra for conditional effect models. From the data analysis aspect, Mak and Wu (2019) proposed a bi-level variable selection of conditional main effects in observational data using a penalty function with two layers: the outer one controlling between-group selection, and the inner one controlling within-group selection.

It is important to consider that the parameterization under a condition effect model destroys the fully aliased relationship between two-factor interactions. Thus, one application of conditional main effects involves the de-aliasing of two-factor interactions in regular fractional factorial designs of resolution four. This involves identifying significant but aliased two-factor interactions and then transforming the model to the corresponding conditional effect model. Afterwards, the de-aliasing strategies in Wu (2015), Su and Wu (2017), Chang (2019) and Lawson (2020) can then be used to facilitate subsequent data analysis.

In the current study, we consider two-level factorial designs in conjunction with conditional effect models involving two pairs of conditional and conditioned factors. The remaining factors, not involving the two pairs of conditional and conditioning factors, are called traditional factors. Wu and Hamada (2021, p. 347) described an experiment involving the sealing of a light bulb, in which the outcomes were determined mainly by two pairs of conditional and conditioning factors $(H, G)$ and $(J, I)$, respectively. The corresponding linear model comprised the conditional main effects of $H$ and $J$ respectively conditioned on $G$ and $I$. In the current study, we extend the work of Mukerjee et al. (2017) to double-pair conditional effect models. The minimum aberration was based on the effect hierarchy principle (Wu \& Hamada, 2021, p. 168); however, it is difficult to artificially argue an order among the effects in a conditional effect model. In accordance with Mukerjee et al. (2017), we adopt the Bayesian approach proposed in Mitchell et al. (1995), later popularized by Kerr (2001), Joseph (2006), Joseph and Delaney (2007), Ai et al. (2009), Joseph et al. (2009), Kang and Joseph (2009), Chang and Cheng (2018) and Chang (2019, 2022), to derive an order of factorial effects based on their prior variances. Once this order has been established, we then define a criterion for design evaluation by sequentially minimizing the bias (contamination) caused by lower-order interactions to higher-order interactions. We refer to the proposed criterion as the minimum contamination criterion. We provide the catalogues of 16 -run and 32 -run minimum contamination designs for various factor numbers at the end of this paper.

The remainder of this paper is organized as follows. Section 2 introduces the parametrization and a Bayesianinspired hierarchical order of effects under a double-pair conditional effect model. Some sufficient conditions for a design to be universally optimal under the conditional effect model involving only main effects are given in Section 3. Section 4 presents a new minimum contamination criterion defined according to the Bayesian-inspired hierarchical order. In addition, we develop a complementary set theory to guide the search for minimum contamination designs involving a large number of factors. An efficient computational procedure is developed in Section 5 to search for minimum contamination regular/nonregular designs for an arbitrary number of factors. Numerical examples and a real experiment are discussed. Conclusions are drawn in Section 6.

## 2. Conditional effect model and Bayesian-inspired effect hierarchy

We give the details regarding the parameterization under a double-pair conditional effect model. We adopt the Bayesian approach in Mitchell et al. (1995) to derive an effect order, which serves as the building block for the minimum contamination criterion introduced in Section 4.

### 2.1. Conditional effect model

A double-pair conditional effect model is a linear model with reparameterization using two pairs of conditional and conditioning factors. Consider a $2^{n}$ full factorial design with $n(\geq 4)$ factors $F_{1}, \ldots, F_{n}$, each at levels 0 and

1. Without loss of generality, let $F_{1}, F_{2}$ be one pair of conditional and conditioned factors and $F_{3}, F_{4}$ be the other pair. The main effect and interaction effects involving $F_{1}$ (respectively, $F_{3}$ ) are defined conditionally on each fixed level of $F_{2}$ (respectively, $F_{4}$ ). Define $\Omega$ as the set of $v=2^{n}$ binary $n$-tuples representing the $2^{n}$ treatment combinations of the full factorial design. For $\left(i_{1}, \ldots, i_{n}\right) \in \Omega$, let $\tau\left(i_{1} \cdots i_{n}\right)$ be the treatment effect of treatment combination $i_{1} \cdots i_{n}$. Under a linear model, $\tau\left(i_{1} \cdots i_{n}\right)$ is the expectation of the response measured at the treatment combination $i_{1} \cdots i_{n}$. Similarly, we write $\theta\left(i_{1} \cdots i_{n}\right)$ for the factorial effect $F_{1}^{i_{1}} \cdots F_{n}^{i_{n}}$ using the conventional orthogonal parameterization (Cheng, 2014, chapter 6) when $i_{1} \cdots i_{n}$ is nonnull, and $\theta(0 \cdots 0)$ for the grand mean. With $n=3$ for illustration,

$$
\begin{aligned}
& \theta(100)=\frac{1}{8}\{\tau(100)+\tau(110)+\tau(101)+\tau(111)-\tau(000)-\tau(010)-\tau(001)-\tau(011)\}, \\
& \theta(110)=\frac{1}{8}\{\tau(000)+\tau(110)+\tau(001)+\tau(111)-\tau(100)-\tau(101)-\tau(010)-\tau(011)\}, \\
& \theta(111)=\frac{1}{8}\{\tau(100)+\tau(010)+\tau(001)+\tau(111)-\tau(000)-\tau(110)-\tau(101)-\tau(011)\}
\end{aligned}
$$

separately represent the main effect of $F_{1}$, the two-factor interaction of $F_{1}$ and $F_{2}$, and the three-factor interaction of $F_{1}, F_{2}, F_{3}$. Each $\theta\left(i_{1} i_{2} i_{3}\right)$ can be interpreted using the mean responses measured at different levels of factors. For example, the main effect $\theta(100)$ is the average of the difference between the mean responses measured at the levels 1 and 0 of $F_{1}$. Let $\boldsymbol{\tau}$ and $\boldsymbol{\theta}$ be $v \times 1$ vectors with elements $\tau\left(i_{1} \cdots i_{n}\right)$ and $\theta\left(i_{1} \cdots i_{n}\right)$ arranged in the lexicographic order, respectively, where for $n=3$, we have

$$
\boldsymbol{\theta}=(\theta(000), \theta(001), \theta(010), \theta(011), \theta(100), \theta(101), \theta(110), \theta(111))^{\top} .
$$

Then, the linear model using the orthogonal parameterization under the full factorial design is given by

$$
\begin{equation*}
\left.\boldsymbol{\tau}=\mathbf{H}^{\otimes n} \boldsymbol{\theta} \quad \text { (or equivalently, } \boldsymbol{\theta}=v^{-1} \mathbf{H}^{\otimes n} \boldsymbol{\tau}\right) \tag{1}
\end{equation*}
$$

where $\otimes$ represents the Kronecker product and $\mathbf{H}^{\otimes n}$ denotes the $n$-fold Kronecker product of

$$
\mathbf{H}=\left(\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right)
$$

a Hadamard matrix of order two. When interpreted under the linear model, the columns of $\mathbf{H}$ respectively correspond to the grand mean and a contrast of the mean responses measured at the two levels of the factor. Since a $2^{n}$ full factorial design has a cross-product structure, the associated matrix of the grand mean and contrasts can be obtained via the $n$-fold Kronecker product of $\mathbf{H}$ as in (1). We refer to the model in (1) as a traditional model under the full factorial design.

Let $\mathbf{H}(0)=(1,1)$ and $\mathbf{H}(1)=(1,-1)$ be the top and bottom rows of $\mathbf{H}$, respectively. Emphasizing $F_{1}$ and $F_{3}$, we express Equation (1) as

$$
\boldsymbol{\theta}=v^{-1}\left(\begin{array}{l}
\mathbf{H}(0) \otimes \mathbf{H} \otimes \mathbf{H}(0) \otimes \mathbf{H}  \tag{2}\\
\mathbf{H}(1) \otimes \mathbf{H} \otimes \mathbf{H}(0) \otimes \mathbf{H} \\
\mathbf{H}(0) \otimes \mathbf{H} \otimes \mathbf{H}(1) \otimes \mathbf{H} \\
\mathbf{H}(1) \otimes \mathbf{H} \otimes \mathbf{H}(1) \otimes \mathbf{H}
\end{array}\right) \mathbf{H}^{\otimes(n-4)} \boldsymbol{\tau}
$$

Let $\beta\left(j_{1} \cdots j_{n}\right)$ be the factorial effect $F_{1}^{i_{1}} \cdots F_{n}^{i_{n}}$ under a conditional effect model with two pairs conditional and conditioning factors $F_{1}, F_{2}$ and $F_{3}, F_{4}$, respectively. Denote the vector with the $v$ elements $\beta\left(j_{1} \cdots j_{n}\right)$ 's by $\boldsymbol{\beta}$ in the same lexicographic order as $\boldsymbol{\theta}$. Under the double-pair conditional effect model, the factorial effects involving $F_{1}$ and $F_{3}$ are defined conditionally on the levels of $F_{2}$ and $F_{4}$, respectively. Thus in (2), $\mathbf{H}$ is replaced with $\sqrt{2} \mathbf{I}_{2}$ whenever $\mathbf{H}(1)$ precedes it. We can reparametrize $\boldsymbol{\theta}$ by $\boldsymbol{\beta}$ with

$$
\begin{equation*}
\boldsymbol{\beta}=v^{-1} \mathbf{W} \otimes \mathbf{H}^{\otimes(n-4)} \boldsymbol{\tau} \tag{3}
\end{equation*}
$$

where

$$
\mathbf{W}=\left(\begin{array}{l}
\mathbf{H}(0) \otimes \mathbf{H} \otimes \mathbf{H}(0) \otimes \mathbf{H} \\
\mathbf{H}(1) \otimes \sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H}(0) \otimes \mathbf{H} \\
\mathbf{H}(0) \otimes \mathbf{H} \otimes \mathbf{H}(1) \otimes \sqrt{2} \mathbf{I}_{2} \\
\mathbf{H}(1) \otimes \sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H}(1) \otimes \sqrt{2} \mathbf{I}_{2}
\end{array}\right)
$$

in which $\mathbf{I}_{2}$ is the identity matrix of order two. The model in (3) is called a double-pair conditional effect model in this paper. The factorial effects involving $F_{1}$ and $F_{3}$ are referred to as conditional (factorial) effects, while those
not involving them are referred to as unconditional (factorial) effects. We cluster $\boldsymbol{\beta}$ into various groups of vectors representing unconditional and conditional factorial effects. Define

$$
\begin{aligned}
\Omega_{0 l}= & \left\{\left(j_{1}, \ldots, j_{n}\right): j_{1}=j_{3}=0, \text { and } l \text { of } j_{2}, j_{4}, \ldots, j_{n} \text { equal } 1\right\}, \\
\Omega_{1 l}= & \left\{\left(j_{1}, \ldots, j_{n}\right): j_{1}=1, j_{2}=0,1, j_{3}=0, \text { and } l-1 \text { of } j_{4}, \ldots, j_{n} \text { equal } 1\right\} \\
& \cup\left\{\left(j_{1}, \ldots, j_{n}\right): j_{3}=1, j_{4}=0,1, j_{1}=0, \text { and } l-1 \text { of } j_{2}, j_{5}, \ldots, j_{n} \text { equal } 1\right\}, \\
\Omega_{2 l}= & \left\{\left(j_{1}, \ldots, j_{n}\right): j_{1}=j_{3}=1, j_{2}=0,1, j_{4}=0,1, \text { and } l-2 \text { of } j_{5}, \ldots, j_{n} \text { equal } 1\right\},
\end{aligned}
$$

where $1 \leq l \leq n-2$. It is apparent that $\beta\left(j_{1} \cdots j_{n}\right)=\theta\left(j_{1} \cdots j_{n}\right)$ if $\left(j_{1}, \ldots, j_{n}\right) \in \Omega_{0 l}$. Let $\boldsymbol{\beta}_{s l}$ be the vector with elements $\beta\left(j_{1} \cdots j_{n}\right)$, where $\left(j_{1}, \ldots, j_{n}\right) \in \Omega_{s l}$. Later we will see that the effects associated with the same $\Omega_{s l}$ have the same importance using the Bayesian-inspired effect hierarchy introduced in the next subsection.

### 2.2. Bayesian-inspired effect hierarchy

Chipman et al. (1997) proposed a Bayesian variable selection for designed experiments with complex aliasing. Rather than independence prior distribution on the factorial effects, Chipman et al. (1997) used a hierarchical prior that is consistent with the effect heredity principle (Wu \& Hamada, 2021, p. 169). A basic idea of their approach is to assign a larger prior variance to a more important factorial effect. This idea is compatible with the Bayesian (functional) prior distribution derived by Mitchell et al. (1995), Kerr (2001), Joseph (2006), Joseph and Delaney (2007), Ai et al. (2009), Joseph et al. (2009), Kang and Joseph (2009), Chang and Cheng (2018) and Chang (2019, 2022). With the derived prior variances, one can readily define an effect hierarchical order and an aberration-like criterion for design evaluation.

In this paper, we adopt the Bayesian approach in Mitchell et al. (1995), who set up a functional prior by regarding $\tau$ as a realization of a stationary Gaussian random function. Then the prior distribution of factorial effects is induced by the relation in (1). When applied to (3), the Bayesian approach induces an effect hierarchy of the $\beta\left(j_{1} \cdots j_{n}\right)$ 's via a prior specification on $\boldsymbol{\tau}$ in terms of a zero-mean Gaussian random function such that $\operatorname{cov}(\boldsymbol{\tau})=\sigma^{2} \mathbf{R}^{\otimes n}$, where $\sigma^{2}>$ 0 and the $2 \times 2$ matrix $\mathbf{R}$ has diagonal elements 1 and off-diagonal elements $\rho, 0<\rho<1$. This covariance structure is equivalent to Equation (4) of Joseph (2006). It follows by expanding $\mathbf{R}^{\otimes n}$ that the correlation of $\tau\left(i_{1} \cdots i_{n}\right)$ and $\tau\left(j_{1} \cdots j_{n}\right)$ is

which is equal to $\rho^{\sum_{l: 1 \leq l \leq n, i} \neq j_{l}}$ and only depends on the total number of components where $\left(i_{1} \cdots i_{n}\right)$ and $\left(j_{1} \cdots j_{n}\right)$ differ. Thus, such a covariance structure can be interpreted as treating every factor equally, which is reasonable since there is usually no knowledge about the importance of the factors at the experimentation stage. In conjunction with (3), the prior covariance matrix of $\boldsymbol{\beta}$ is given by

$$
\begin{aligned}
\operatorname{cov}(\boldsymbol{\beta}) & =v^{-2}\left\{\mathbf{W} \otimes \mathbf{H}^{\otimes(n-4)}\right\} \operatorname{cov}(\boldsymbol{\tau})\left\{\mathbf{W} \otimes \mathbf{H}^{\otimes(n-4)}\right\}^{\top} \\
& =\sigma^{2} v^{-2}\left\{\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right\} \otimes\{\mathbf{H R} \mathbf{H}\}^{\otimes(n-4)} .
\end{aligned}
$$

The following result gives the variances of the $\beta\left(j_{1} \cdots j_{n}\right)$ 's.
Theorem 2.1: For a $\left(j_{1}, \ldots, j_{n}\right) \in \Omega_{s l}$, we have

$$
\operatorname{var}\left(\beta\left(j_{1} \cdots j_{n}\right)\right)=\sigma^{2} v^{-1}(1+\rho)^{n-l-s}(1-\rho)^{l}
$$

Proof: By the identity $\operatorname{cov}(\boldsymbol{\beta})=\sigma^{2} v^{-2}\left\{\mathbf{W R}^{\otimes 4} \mathbf{W}^{\top}\right\} \otimes\{\mathbf{H R H}\}^{\otimes(n-4)}$, one can easily verify that HRH $=2 \operatorname{diag}(1+$ $\rho, 1-\rho$ ),

$$
\mathbf{W}=\left(\begin{array}{cccc}
\mathbf{H}^{\otimes 2} & \mathbf{H}^{\otimes 2} & \mathbf{H}^{\otimes 2} & \mathbf{H}^{\otimes 2} \\
\sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H} & \sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H} & -\sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H} & -\sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H} \\
\sqrt{2} \mathbf{H} \otimes \mathbf{I}_{2} & -\sqrt{2} \mathbf{H} \otimes \mathbf{I}_{2} & \sqrt{2} \mathbf{H} \otimes \mathbf{I}_{2} & -\sqrt{2} \mathbf{H} \otimes \mathbf{I}_{2} \\
2 \mathbf{I}_{2}^{\otimes 2} & -2 \mathbf{I}_{2}^{\otimes 2} & -2 \mathbf{I}_{2}^{\otimes 2} & 2 \mathbf{I}_{2}^{\otimes 2}
\end{array}\right),
$$

and

$$
\mathbf{R}^{\otimes 4}=\left(\begin{array}{cccc}
\mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} & \rho^{2} \mathbf{R}^{\otimes 2} \\
\rho \mathbf{R}^{\otimes 2} & \mathbf{R}^{\otimes 2} & \rho^{2} \mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} \\
\rho \mathbf{R}^{\otimes 2} & \rho^{2} \mathbf{R}^{\otimes 2} & \mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} \\
\rho^{2} \mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} & \rho \mathbf{R}^{\otimes 2} & \mathbf{R}^{\otimes 2}
\end{array}\right)
$$

$\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}$ is a $16 \times 16$ matrix, which can be regarded as a $4 \times 4$ block-matrix with each block being a $4 \times 4$ matrix. Denote the $(i, j)$ th block-matrix of $\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}$ by $\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{i j}$. Then by calculation, we get $\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{i j}=\mathbf{0}$ when $i \neq j$; for $i=j$, we obtain $\left[\mathbf{W R}^{\otimes 4} \mathbf{W}^{\top}\right]_{11}=4(1+\rho)^{2}(\mathbf{H R H})^{\otimes 2},\left[\mathbf{W R}^{\otimes 4} \mathbf{W}^{\top}\right]_{22}=8\left(1-\rho^{2}\right) \mathbf{R} \otimes(\mathbf{H R H})$, $\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{33}=8\left(1-\rho^{2}\right)(\mathbf{H R H}) \otimes \mathbf{R}$ and $\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{44}=16(1-\rho)^{2} \mathbf{R} \otimes \mathbf{R}$.

The diagonal elements of $(\mathbf{H R H})^{\otimes 2}$, denoted by $\operatorname{diag}\left((\mathbf{H R H})^{\otimes 2}\right)$, can be obtained as $4(1+\rho)^{2}, 4(1-\rho)(1+$ $\rho), 4(1-\rho)(1+\rho), 4(1-\rho)^{2}$. We also have $\operatorname{diag}(\mathbf{R} \otimes(\mathbf{H R H}))=2(1+\rho, 1-\rho, 1+\rho, 1-\rho), \operatorname{diag}((\mathbf{H R H}) \otimes$ $\mathbf{R})=2(1+\rho, 1+\rho, 1-\rho, 1-\rho) \quad$ and $\operatorname{diag}(\mathbf{R} \otimes \mathbf{R})=(1,1,1,1)$. Thus, we get $\operatorname{diag}\left(\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{11}\right)=$ $16\left((1+\rho)^{4},(1-\rho)(1+\rho)^{3},(1-\rho)(1+\rho)^{3},(1-\rho)^{2}(1+\rho)^{2}\right), \quad \operatorname{diag}\left(\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{22}\right)=16\left((1+\rho)^{2}(1-\rho)\right.$, $\left.(1+\rho)(1-\rho)^{2},(1+\rho)^{2}(1-\rho),(1+\rho)(1-\rho)^{2}\right), \quad \operatorname{diag}\left(\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{33}\right)=16\left((1+\rho)^{2}(1-\rho),(1+\rho)^{2}(1-\right.$ $\left.\rho),(1+\rho)(1-\rho)^{2},(1+\rho)(1-\rho)^{2}\right)$ and $\operatorname{diag}\left(\left[\mathbf{W} \mathbf{R}^{\otimes 4} \mathbf{W}^{\top}\right]_{44}\right)=16\left((1-\rho)^{2},(1-\rho)^{2},(1-\rho)^{2},(1-\rho)^{2}\right)$.

Since the variances of $\beta\left(j_{1} \cdots j_{n}\right)$ 's are only related to the diagonal elements of $\operatorname{cov}(\boldsymbol{\beta})$, one can easily check for $\mathrm{a}\left(j_{1}, \ldots, j_{n}\right) \in \Omega_{s l}, \operatorname{var}\left(\beta\left(j_{1} \cdots j_{n}\right)\right)=\sigma^{2} v^{-1}(1+\rho)^{n-l-s}(1-\rho)^{l}$ based on the above calculation.

Let $V_{s l}=\operatorname{var}\left(\beta\left(j_{1} \cdots j_{n}\right)\right)$ for $\left(j_{1}, \ldots, j_{n}\right) \in \Omega_{s l}$. From Theorem 2.1, it is clear that $V_{0 l}>V_{1 l}>V_{2 l}$ for $2 \leq l \leq$ $n-2$. Because $V_{2 l} / V_{0, l+1}=1 /\left(1-\rho^{2}\right)>1$ for all $0<\rho<1$, we have

$$
\begin{equation*}
V_{01}>V_{11}>V_{02}>V_{12}>V_{22}>V_{03}>V_{13}>V_{23}>\cdots>V_{0, n-2}>V_{1, n-2}>V_{2, n-2} . \tag{4}
\end{equation*}
$$

In view of (4), we define the following effect hierarchy under the conditional effect model (3) as follows. The unconditional main effects have the largest variance $V_{01}$ and are the most important, while the conditional main effects with variance $V_{11}$ are positioned next; then come the unconditional two-factor interactions ( $V_{02}$ ), followed by the one-pair conditional two-factor interactions ( $V_{12}$ ), then two-pair conditional two-factor interactions ( $V_{22}$ ), and so on. This effect hierarchy order is not surprising since a conditional main effect is proportional to the average of a unconditional main effect and two-factor interaction, resulting in a variance in-between.

## 3. Universally optimal designs for main effect model

As mentioned in Mukerjee et al. (2017), a justifiable criterion for design evaluation is to identify a class of designs which ensure optimal inference on the $\beta\left(j_{1} \cdots j_{n}\right)$ 's corresponding to $\Omega_{01}$ and $\Omega_{11}$ in the absence of all interactions. Then, in order to possess model robustness, among these designs we find one which sequentially minimizes a suitably defined measure of bias caused by successive interactions in the effect hierarchy. In this section, we connect the conditional effect model with the traditional model. Then we provide some sufficient conditions for a design to be universally optimal under a main-effect conditional effect model.

The connection between conditional effects $\boldsymbol{\beta}$ and traditional factorial effects $\boldsymbol{\theta}$ can be established by (1) and (3) as follows:

$$
\begin{aligned}
\boldsymbol{\beta} & =v^{-1} \mathbf{W} \otimes \mathbf{H}^{\otimes(n-4)} \boldsymbol{\tau} \\
& =v^{-1} \mathbf{W} \otimes \mathbf{H}^{\otimes(n-4)} \mathbf{H}^{\otimes n} \boldsymbol{\theta} \\
& =v^{-1}\left\{\mathbf{W} \mathbf{H}^{\otimes 4}\right\} \otimes\{\mathbf{H} \mathbf{H}\}^{\otimes(n-4)} \boldsymbol{\theta} .
\end{aligned}
$$

By using the fact $\mathbf{H H}=2 \mathbf{I}_{2}$ and

$$
\mathbf{W} \mathbf{H}^{\otimes 4}=4\left(\begin{array}{cccc}
\mathbf{H}^{\otimes 2} \mathbf{H}^{\otimes 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \sqrt{2} \mathbf{I}_{2} \otimes \mathbf{H} \mathbf{H}^{\otimes 2} & \mathbf{0} \\
\mathbf{0} & \sqrt{2} \mathbf{H} \otimes \mathbf{I}_{2} \mathbf{H}^{\otimes 2} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 2 \mathbf{I}_{2}^{\otimes 2} \mathbf{H}^{\otimes 2}
\end{array}\right)
$$

we obtain

$$
\boldsymbol{\beta}=\left(\begin{array}{cccc}
\mathbf{I}_{2}^{\otimes(n-2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{1}{\sqrt{2}} \mathbf{H} \otimes \mathbf{I}_{2} \otimes \mathbf{I}_{2}^{\otimes(n-4)} & \mathbf{0} \\
\mathbf{0} & \frac{1}{\sqrt{2}} \mathbf{I}_{2} \otimes \mathbf{H} \otimes \mathbf{I}_{2}^{\otimes(n-4)} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2} \mathbf{H}^{\otimes 2} \otimes \mathbf{I}_{2}^{\otimes(n-2)}
\end{array}\right) \boldsymbol{\theta}
$$

which implies

$$
\boldsymbol{\theta}=\left(\begin{array}{cccc}
\mathbf{I}_{2}^{\otimes(n-2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{1}{\sqrt{2}} \mathbf{H} \otimes \mathbf{I}_{2} \otimes \mathbf{I}_{2}^{\otimes(n-4)} & \mathbf{0} \\
\mathbf{0} & \frac{1}{\sqrt{2}} \mathbf{I}_{2} \otimes \mathbf{H} \otimes \mathbf{I}_{2}^{\otimes(n-4)} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2} \mathbf{H}^{\otimes 2} \otimes \mathbf{I}_{2}^{\otimes(n-2)}
\end{array}\right) \boldsymbol{\beta}
$$

because $\mathbf{H}^{-1}=(1 / 2) \mathbf{H}$. This yields

$$
\begin{aligned}
\theta\left(0 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right)= & \beta\left(0 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right) \\
\theta\left(1 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{\sqrt{2}}\left\{\beta\left(100 j_{4} j_{5} \cdots j_{n}\right)+\delta\left(j_{2}\right) \beta\left(110 j_{4} j_{5} \cdots j_{n}\right)\right\} \\
\theta\left(0 j_{2} 1 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{\sqrt{2}}\left\{\beta\left(0 j_{2} 10 j_{5} \cdots j_{n}\right)+\delta\left(j_{4}\right) \beta\left(0 j_{2} 11 j_{5} \cdots j_{n}\right)\right\} \\
\theta\left(1 j_{2} 1 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{2}\left\{\beta\left(1010 j_{5} \cdots j_{n}\right)+\delta\left(j_{4}\right) \beta\left(1011 j_{5} \cdots j_{n}\right)\right. \\
& \left.+\delta\left(j_{2}\right) \beta\left(1110 j_{5} \cdots j_{n}\right)+\delta\left(j_{2}\right) \delta\left(j_{4}\right) \beta\left(1111 j_{5} \cdots j_{n}\right)\right\}
\end{aligned}
$$

where $\delta(j)=-2 j+1$. In view of $\boldsymbol{\theta}$, the first of above identities shows that $2^{n-2} / 2^{n}=1 / 4$ of the $\theta\left(j_{1} \cdots j_{n}\right)$ 's remain unconditional effects, while the other $3 / 4$ of the $\theta\left(j_{1} \cdots j_{n}\right)$ 's involve $F_{1}$ and $F_{3}$ and hence are a combination of the conditional effects. These equations uncover the connection between factorial effects under the traditional model and under the conditional effect model. Take $n=5$ for example. We have $\theta(10000)=\{\beta(10000)+\beta(11000)\} / \sqrt{2}$. Recall that $\beta(10000)$ and $\beta(11000)$ are the conditional main effects of $F_{1}$ conditioned on levels 0 and 1 of $F_{2}$, respectively. Thus, the unconditional main effect of $F_{1}$ is proportional to the average of the two conditional main effects of $F_{1}$. Likewise, we have $\theta(11000)=\{\beta(10000)-\beta(11000)\} / \sqrt{2}$, which means that the unconditional twofactor interaction of $F_{1}$ and $F_{2}$ is proportional to the difference between the two conditional main effects of $F_{1}$.

Consider an $N$-run design represented by the $N \times n$ design matrix $\mathbf{D}$ with elements 1 (high level) and -1 (low level). Denote the corresponding $N \times 2^{n}$ full model matrix under (1) by $\mathbf{X}$, each column corresponding to one $\theta\left(j_{1} \cdots j_{n}\right)$. Note that $\mathbf{X}$ can be obtained by deleting the rows of $\mathbf{H}^{\otimes n}$ that are not in $\mathbf{D}$. With the vector of responses $\boldsymbol{y}$, we have $E(\boldsymbol{y})=\mathbf{X} \boldsymbol{\theta}$. Each column of $\mathbf{X}$ is represented by $\boldsymbol{x}\left(j_{1} \cdots j_{n}\right)$ with $\left(j_{1}, \ldots, j_{n}\right) \in \Omega$. The connection between $\theta\left(j_{1} \cdots j_{n}\right)$ 's and $\beta\left(j_{1} \cdots j_{n}\right)$ 's suggests

$$
\begin{aligned}
z\left(0 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right)= & x\left(0 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right) \\
z\left(1 j_{2} 0 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{\sqrt{2}}\left\{x\left(100 j_{4} j_{5} \cdots j_{n}\right)+\delta\left(j_{2}\right) x\left(110 j_{4} j_{5} \cdots j_{n}\right)\right\} \\
z\left(0 j_{2} 1 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{\sqrt{2}}\left\{x\left(0 j_{2} 10 j_{5} \cdots j_{n}\right)+\delta\left(j_{4}\right) x\left(0 j_{2} 11 j_{5} \cdots j_{n}\right)\right\} \\
z\left(1 j_{2} 1 j_{4} j_{5} \cdots j_{n}\right)= & \frac{1}{2}\left\{x\left(1010 j_{5} \cdots j_{n}\right)+\delta\left(j_{4}\right) x\left(1011 j_{5} \cdots j_{n}\right)\right. \\
& \left.+\delta\left(j_{2}\right) x\left(1110 j_{5} \cdots j_{n}\right)+\delta\left(j_{2}\right) \delta\left(j_{4}\right) x\left(1111 j_{5} \cdots j_{n}\right)\right\}
\end{aligned}
$$

where $\delta(j)=-2 j+1$. Let $\mathbf{Z}_{s l}$ and $\mathbf{X}_{s l}$ consist of the $\boldsymbol{z}\left(j_{1} \cdots j_{n}\right)$ 's and $\boldsymbol{x}\left(j_{1} \cdots j_{n}\right)$ 's respectively, where $\left(j_{1}, \ldots, j_{n}\right) \in$ $\Omega_{s l}$. Then the conditional effect model under $\mathbf{D}$ can be represented by

$$
\begin{equation*}
E(\boldsymbol{y})=\boldsymbol{z}(0 \cdots 0) \beta(0 \cdots 0)+\sum_{s=0}^{2} \sum_{l=1}^{n-2} \mathbf{Z}_{s l} \boldsymbol{\beta}_{s l} \tag{5}
\end{equation*}
$$

We follow the convention that the random observational errors are uncorrelated and homogeneous with equal variance.

### 3.1. Universally optimal designs

If all interactions are absent, then the model (5) reduces to

$$
\begin{equation*}
E(\boldsymbol{y})=\boldsymbol{z}(0 \cdots 0) \beta(0 \cdots 0)+\mathbf{Z}_{01} \boldsymbol{\beta}_{01}+\mathbf{Z}_{11} \boldsymbol{\beta}_{11} \tag{6}
\end{equation*}
$$

consisting of only unconditional and conditional main effects. In the following, we present a theorem which gives some requirements for a design to be universally optimal under model (6).

## Theorem 3.1: Suppose an $N-r u n$ design $\mathbf{D}$ satisfies

(i) $\mathbf{D}$ is an orthogonal array of strength two;
(ii) all eight triples of symbols occur equally often when $\mathbf{D}$ is projected onto $F_{1}, F_{2}, F_{j}, j \in\{4,5, \ldots, n\}$;
(iii) all eight triples of symbols occur equally often when $\mathbf{D}$ is projected onto $F_{3}, F_{4}, F_{j}, j \in\{2,5, \ldots, n\}$;
(iv) all sixteen triples of symbols occur equally often when $\mathbf{D}$ is projected onto $F_{1}, F_{2}, F_{3}, F_{4}$.

Then $\mathbf{D}$ is universally optimal among all $N$-run designs for inference on $\boldsymbol{\beta}_{01}$ and $\boldsymbol{\beta}_{11}$ under model (6).
Proof: Let $\mathbf{Z}_{1}=\left(\mathbf{Z}_{01}, \mathbf{Z}_{11}\right)$. Denote the information matrix of $\boldsymbol{\beta}_{01}$ and $\boldsymbol{\beta}_{11}$ under model (6) by $\mathbf{M}$. Note that $\mathbf{M}$ can be obtained by the Schur complement

$$
\mathbf{M}=\mathbf{Z}_{1}^{\top}\left\{\mathbf{I}_{N}-\boldsymbol{z}(0 \cdots 0)\left[\boldsymbol{z}(0 \cdots 0)^{\top} \boldsymbol{z}(0 \cdots 0)\right]^{-1} \boldsymbol{z}(0 \cdots 0)^{\top}\right\} \mathbf{Z}_{1}
$$

which can be simplified as $\mathbf{M}=\mathbf{Z}_{1}^{\top}\left\{\mathbf{I}_{N}-\frac{1}{N} \mathbf{1}_{N} \mathbf{1}_{N}^{\top}\right\} \mathbf{Z}_{1}$ because $\boldsymbol{z}(0 \cdots 0)=\mathbf{1}_{N}$. Because $\mathbf{Z}_{1}^{\top} \mathbf{Z}_{1}-\mathbf{M}$ is nonnegative definite, we have

$$
\begin{equation*}
\operatorname{tr}[\mathbf{M}] \leq \operatorname{tr}\left[\mathbf{Z}_{1}^{\top} \mathbf{Z}_{1}\right]=N(n-2)+4 N=N(n+2) \tag{7}
\end{equation*}
$$

for every $N$-run design. Under the conditions (i),...,(iv), it is easy to verify that $\mathbf{Z}_{1}^{\top} \mathbf{1}_{N}=\mathbf{0}$ and $\mathbf{Z}_{1}^{\top} \mathbf{Z}_{1}=N \mathbf{I}_{n+2}$. Thus $\mathbf{M}=N \mathbf{I}_{n+2}$ and $\operatorname{tr}[\mathbf{M}]$ reaches the upper bound in (7). The result now follows from Kiefer (1975).

By Theorem 3.1, a necessary condition for universally optimal designs is $N \geq 16$. Therefore, if $n=4$, the model (5) can only involve the two pairs of conditional and conditioning factors ( $F_{1}, F_{2}$ ) and ( $F_{3}, F_{4}$ ). Then the universally optimal design is exactly the $2^{4}$ full factorial design. Thus, to avoid trivialities, we let $n \geq 5$ in the discussion of design selection in the next section.

## 4. Minimum contamination and complementary set theory

The designs meeting the conditions (i),...,(iv) of Theorem 3.1 are universally optimal under model (6). In addition, $\widehat{\boldsymbol{\beta}}_{h 1}=N^{-1} \mathbf{Z}_{h 1}^{\top} \boldsymbol{y}$ is the best linear unbiased estimator of $\boldsymbol{\beta}_{h 1}, h=0,1$. However, nonnegligible interactions may exist and $\widehat{\boldsymbol{\beta}}_{h 1}$ is no longer unbiased in this case. We revert back to the model (5), which includes all interactions, to assess the impact of possible presence of interactions on $\widehat{\boldsymbol{\beta}}_{h 1}$. Under model (5), $\widehat{\boldsymbol{\beta}}_{h 1}$ has bias $N^{-1} \sum_{s=0}^{2} \sum_{l=2}^{n-2} \mathbf{Z}_{h 1}^{\top} \mathbf{Z}_{s l} \boldsymbol{\beta}_{s l}$. The matrix $N^{-1} \sum_{s=0}^{2} \sum_{l=2}^{n-2} \mathbf{Z}_{h 1}^{\top} \mathbf{Z}_{s l}$ is referred to as an alias matrix in Wu and Hamada (2021, p. 419). A reasonable measure of the bias in $\widehat{\boldsymbol{\beta}}_{h 1}$ caused by the interactions, as in Tang and Deng (1999), is

$$
K_{s l}(h)=N^{-2} \operatorname{tr}\left[\mathbf{Z}_{h 1}^{\top} \mathbf{Z}_{s l} \mathbf{Z}_{s l}^{\top} \mathbf{Z}_{h 1}\right]=N^{-2} \operatorname{tr}\left[\mathbf{X}_{h 1}^{\top} \mathbf{X}_{s l} \mathbf{X}_{s l}^{\top} \mathbf{X}_{h 1}\right],
$$

where the last equality holds because $\mathbf{X}_{s l}$ is an orthogonal transform of $\mathbf{Z}_{s l}$. Based on the effect hierarchy in (4), one should minimize the bias successively in order of priority. Thus, we define a minimum contamination design as the one which minimizes the terms of

$$
\begin{equation*}
K=\left\{K_{02}(0), K_{02}(1), K_{12}(0), K_{12}(1), K_{22}(0), K_{22}(1), K_{03}(0), K_{03}(1), \ldots\right\} \tag{8}
\end{equation*}
$$

in a sequential manner from left to right. In (8), $K_{s l}(0)$ appears before $K_{s l}(1)$ because the contamination or bias in $\widehat{\boldsymbol{\beta}}_{01}$ is deemed more severe than in $\widehat{\boldsymbol{\beta}}_{11}$. The concept of minimizing contamination due to the existence of interactions is not new. We note that Cheng and Tang (2005) used this idea to develop a general theory for minimum aberration.

The minimum contamination criterion in (8) induces a ranking of designs of the same run size. It is time consuming, however, to find the minimum contamination design via complete search using (8) if $n$ is large. A useful technique of design construction is via complementary designs. Tang and Wu (1996) provided identities related to the wordlength pattern of a regular two-level design to that of its complementary design. Suen et al. (1997) extended these identities to regular $s^{n-p}$ designs. Cheng (2014, p. 179) reviewed the design construction using complementary designs.

We now focus on regular designs under the conditional effect model due to their nice properties and popularity among practitioners. Let $\Delta_{r}$ be the set of nonnull $r \times 1$ binary vectors. All operations with these vectors are over
the finite field GF(2). Regarding the notation, we do not apply bold font style to these binary vectors to distinguish them from the vectors with the elements belonging to real numbers. A regular design in $N=2^{r}(r<n)$ runs is given by $n$ distinct vectors $b_{1}, \ldots, b_{n}$ from $\Delta_{r}$ such that the matrix $B=\left(b_{1}, \ldots, b_{n}\right)$ has full row rank. The design consists of the $N$ treatment combinations of the form $a^{\top} B$, where $a \in \Delta_{r} \cup\{0\}$.

In the following, we define some useful quantities to represent $K_{s l}(h)$. Let $A_{l}^{(1)}$ be the number of ways of choosing $l$ out of $b_{2}, b_{4}, \ldots, b_{n}$ such that the sum of the chosen $l$ equals $0 ; A_{l}^{(21)}$ be the number of ways of choosing $l$ out of $b_{4}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}, b_{1}+b_{2}\right\} ; A_{l}^{(22)}$ be the number of ways of choosing $l$ out of $b_{2}, b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{3}, b_{3}+b_{4}\right\} ; A_{l}^{(2)}=A_{l}^{(21)}+A_{l}^{(22)} ; A_{l}^{(31)}$ be the number of ways of choosing $l$ out of $b_{4}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{0, b_{2}\right\} ; A_{l}^{(32)}$ be the number of ways of choosing $l$ out of $b_{2}, b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{0, b_{4}\right\}$; $A_{l}^{(3)}=A_{l}^{(31)}+A_{l}^{(32)} ; A_{l}^{(42)}$ be the number of ways of choosing $l$ out of $b_{2}, b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}+b_{3}, b_{1}+b_{3}+b_{4}\right\} ; A_{l}^{(43)}$ be the number of ways of choosing $l$ out of $b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}+b_{3}, b_{1}+b_{3}+b_{4}\right\} ; A_{l}^{(52)}$ be the number of ways of choosing $l$ out of $b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}+b_{2}+b_{3}, b_{1}+b_{2}+b_{3}+b_{4}\right\} ; A_{l}^{(7)}$ be the number of ways of choosing $l$ out of $b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}+b_{3}, b_{1}+b_{2}+b_{3}, b_{1}+b_{3}+b_{4}, b_{1}+b_{2}+\right.$ $\left.b_{3}+b_{4}\right\} ; A_{l}^{(8)}$ be the number of ways of choosing $l$ out of $b_{5}, \ldots, b_{n}$ such that the sum of the chosen $l$ is in the set $\left\{b_{1}, b_{3}, b_{1}+b_{2}, b_{1}+b_{4}, b_{2}+b_{3}, b_{3}+b_{4}, b_{1}+b_{2}+b_{4}, b_{2}+b_{3}+b_{4}\right\}$. The next result, with the proof deferred to the appendix, gives expressions for $K_{s l}(h)$ in terms of the quantities just introduced.

Theorem 4.1: For $2 \leq l \leq n-2$, we have
(a) $K_{0 l}(0)=(l+1) A_{l+1}^{(1)}+(n-l-1) A_{l-1}^{(1)}$;
(b) $K_{0 l}(1)=A_{l-1}^{(2)}+A_{l}^{(2)}$;
(c) $K_{l l}(0)=(n-l-1) A_{l-2}^{(2)}+A_{l-1}^{(2)}+l A_{l}^{(2)}$;
(d) $K_{1 l}(1)=2 A_{l-1}^{(3)}+2\left\{A_{l-1}^{(42)}+A_{l-2}^{(43)}+A_{l-1}^{(52)}\right\}$;
(e) $K_{2 l}(0)=2 A_{l-2}^{(7)}+(n-l-1) A_{l-3}^{(7)}+(l-1) A_{l-1}^{(7)}$;
(f) $K_{2 l}(1)=2 A_{l-2}^{(8)}$.

In view of Theorem 4.1, sequential minimization of $K$ is equivalent to that of the terms of $A=\left\{A_{3}^{(1)}, A_{2}^{(2)}, A_{1}^{(42)}+\right.$ $\left.A_{1}^{(52)}, A_{1}^{(7)}, A_{4}^{(1)}, A_{3}^{(2)}, \ldots\right\}$, which is reduced to

$$
A=\left\{A_{3}^{(1)}, A_{2}^{(2)}, A_{1}^{(7)}, A_{4}^{(1)}, A_{3}^{(2)}, \ldots\right\}
$$

because $F_{1}, F_{2}, F_{3}, F_{4}$ form a complete factorial, implying $A_{1}^{(42)}+A_{1}^{(52)}=A_{1}^{(7)}$.
We now develop a complementary set theory for the first four terms in the sequence $A$. Let $\widetilde{T}$ be the complement of $\left\{b_{2}, b_{4}, \ldots, b_{n}\right\}$ in $\Delta_{r} ; A_{l}(\widetilde{T})$ be the number of ways of choosing $l$ members of $\widetilde{T}$ such that the sum of the chosen $l$ equals 0 . Let $T_{12}=\widetilde{T} \backslash\left\{b_{1}, b_{1}+b_{2}\right\} ; T_{34}=\widetilde{T} \backslash\left\{b_{3}, b_{3}+b_{4}\right\} ; A_{l}^{(12)}\left(T_{12}\right)$ be the number of ways of choosing $l$ members of $T_{12}$ such that the sum of the chosen $l$ is in $\left\{b_{1}, b_{1}+b_{2}\right\} ; A_{l}^{(34)}\left(T_{34}\right)$ be the number of ways of choosing $l$ members of $T_{34}$ such that the sum of the chosen $l$ is in $\left\{b_{3}, b_{3}+b_{4}\right\}$.

Theorem 4.2: Let $c_{j}, j=1, \ldots, 5$, be constants irrelevant to designs and $T=\Delta_{r} \backslash\left\{b_{5}, \ldots, b_{n}\right\}$. Define $H_{i}(\cdot, \cdot)$ as Equation (2) in Mukerjee and Wu (2001). We have
(a) $A_{3}^{(1)}=c_{1}-A_{3}(\widetilde{T})$;
(b) $A_{4}^{(1)}=c_{2}+A_{3}(\widetilde{T})+A_{4}(\widetilde{T})$;
(c) $A_{2}^{(2)}=c_{3}+A_{2}^{(12)}\left(T_{12}\right)+A_{2}^{(34)}\left(T_{34}\right)$;
(d) $A_{1}^{(7)}=B_{1}+B_{2}+B_{3}+B_{4}$, where $B_{1}=c_{41}+H_{1}\left(\left\{b_{1}+b_{3}\right\}\right.$, $\left.T\right)$ if $b_{1}+b_{3}=b_{j}$ for some $j \in\{5, \ldots, n\}$ and zero otherwise; $B_{2}=c_{42}+H_{1}\left(\left\{b_{1}+b_{2}+b_{3}\right\}\right.$, T) if $b_{1}+b_{2}+b_{3}=b_{j}$ for some $j \in\{5, \ldots, n\}$ and zero otherwise; $B_{3}=c_{43}+H_{1}\left(\left\{b_{1}+b_{3}+b_{4}\right\}, T\right)$ if $b_{1}+b_{3}+b_{4}=b_{j}$ for some $j \in\{5, \ldots, n\}$ and zero otherwise; $B_{4}=c_{44}+H_{1}\left(\left\{b_{1}+b_{2}+b_{3}+b_{4}\right\}, T\right)$ if $b_{1}+b_{2}+b_{3}+b_{4}=b_{j}$ for some $j \in\{5, \ldots, n\}$ and zero otherwise. $c_{4 j}$ 's are constants for every design.

Proof: Parts (a) and (b) are evident from Tang and Wu (1996).

For (c), note that $A_{2}^{(21)}=H_{2}\left(\left\{b_{1}\right\},\left\{b_{4}, \ldots, b_{n}\right\}\right)+H_{2}\left(\left\{b_{1}+b_{2}\right\},\left\{b_{4}, \ldots, b_{n}\right\}\right)$, which can be simplified as $A_{2}^{(21)}=c+H_{2}\left(\left\{b_{1}\right\},\left\{b_{2}, b_{1}+b_{2}\right\} \cup T_{12}\right)+H_{2}\left(\left\{b_{1}+b_{2}\right\},\left\{b_{1}, b_{2}\right\} \cup T_{12}\right)$ by Lemmas 1 and 3 in Mukerjee and Wu (2001), where $c$ is a constant for every design. Because the design is an orthogonal array of strength two, we have $H_{2}\left(\left\{b_{1}\right\},\left\{b_{2}, b_{1}+b_{2}\right\} \cup T_{12}\right)=1+H_{2}\left(\left\{b_{1}\right\}, T_{12}\right)$ and $H_{2}\left(\left\{b_{1}+b_{2}\right\},\left\{b_{1}, b_{2}\right\} \cup T_{12}\right)=1+H_{2}\left(\left\{b_{1}+\right.\right.$ $\left.\left.b_{2}\right\}, T_{12}\right)$. Hence $A_{2}^{(21)}=c+2+H_{2}\left(\left\{b_{1}\right\}, T_{12}\right)+H_{2}\left(\left\{b_{1}+b_{2}\right\}, T_{12}\right)=c+2+A_{2}^{(12)}\left(T_{12}\right)$. Similarly, $A_{2}^{(22)}=c^{\prime}+$ $2+A_{2}^{(34)}\left(T_{34}\right)$, where $c^{\prime}$ is a constant for every design. Therefore, we have $A_{2}^{(2)}=c_{3}+A_{2}^{(12)}\left(T_{12}\right)+A_{2}^{(34)}\left(T_{34}\right)$ by letting $c_{3}=c+c^{\prime}+4$.

For (d), note that $A_{1}^{(7)}=H_{1}\left(\left\{b_{1}+b_{3}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)+H_{1}\left(\left\{b_{1}+b_{3}+b_{4}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)+H_{1}\left(\left\{b_{1}+b_{2}+b_{3}\right\}\right.$, $\left.\left\{b_{5}, \ldots, b_{n}\right\}\right)+H_{1}\left(\left\{b_{1}+b_{2}+b_{3}+b_{4}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)$. Let $F=\Delta_{r} \backslash\left\{b_{1}+b_{3}, b_{5}, \ldots, b_{n}\right\}$. If $b_{1}+b_{3} \neq b_{j}$ for $j=5, \ldots, n$, then $H_{1}\left(\left\{b_{1}+b_{3}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)=0$. If $b_{1}+b_{3}=b_{j}$ for some $j \in\{5, \ldots, n\}$, then $H_{1}\left(\left\{b_{1}+\right.\right.$ $\left.\left.b_{3}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)=c_{41}+H_{1}\left(\left\{b_{1}+b_{3}\right\}, F\right)$ by Lemmas 1 and 3 in Mukerjee and Wu (2001), where $c_{41}$ is a constant for every design. Since $b_{1}+b_{3}=b_{j}$ for some $j \in\{5, \ldots, n\}$, we have $F=T$ and $H_{1}\left(\left\{b_{1}+b_{3}\right\}, F\right)=H_{1}\left(\left\{b_{1}+\right.\right.$ $\left.\left.b_{3}\right\}, T\right)$. Thus $H_{1}\left(\left\{b_{1}+b_{3}\right\},\left\{b_{5}, \ldots, b_{n}\right\}\right)=B_{1}$. Similarly, we have $H_{1}\left(\left\{b_{1}+b_{2}+b_{3}\right\}, T\right)=B_{2}, H_{1}\left(\left\{b_{1}+b_{3}+\right.\right.$ $\left.\left.b_{4}\right\}, T\right)=B_{3}$ and $H_{1}\left(\left\{b_{1}+b_{2}+b_{3}+b_{4}\right\}, T\right)=B_{4}$. So, $A_{1}^{(7)}=B_{1}+B_{2}+B_{3}+B_{4}$.

Theorem 4.2 provides a way to evaluate designs using the sequence $A$, and equivalently sequence $K$, with the number of factors $n=(N-1)+2-\tilde{t}=N-1-\tilde{t}$, where $\tilde{t}$ is the cardinality of $\widetilde{T}$. Once $\widetilde{T}$ is constructed, one can quickly get $\left\{b_{2}, b_{4}, \ldots, b_{n}\right\}$ by the identity $\Delta_{r}=\widetilde{T} \cup\left\{b_{2}, b_{4}, \ldots, b_{n}\right\}$, and construct the design $\left\{b_{1}, b_{2}, b_{3}, b_{4}, \ldots, b_{n}\right\}$. Constructing $\widetilde{T}$ with minimum contamination given a large $\tilde{t}$ is usually time-consuming. It is more practical to find a small $\widetilde{T}$ with minimum contamination, leading to a large $n$. Thus, Theorem 4.2 helps find large minimum contamination designs. For example, consider $\tilde{t}=5$. The set $\widetilde{T}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}, \alpha_{1}+\alpha_{2}\right\}$ can be verified to have maximal $A_{3}(\widetilde{T})=1$, where $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ are four linearly independent vectors from $\Delta_{r}$. Moreover, assigning $\underset{\sim}{\alpha}=b_{1}, \alpha_{2}=b_{3}+b_{4}, \alpha_{3}=b_{3}$ and $\alpha_{4}=b_{1}+b_{2}$ results in minimal $A_{2}^{(12)}\left(T_{12}\right)=A_{2}^{(34)}\left(T_{34}\right)=0$. Thus, we have $\widetilde{T}=\left\{b_{1}, b_{3}+b_{4}, b_{3}, b_{1}+b_{2}, b_{1}+b_{3}+b_{4}\right\}$ and construct a minimum contamination design with the number of factors $N-6$, which equals $10,26,58$ for 16 -, 32 - and 64 -run designs.

## 5. Efficient design search and examples

Finding minimum contamination designs using (8) is a daunting task for even moderate run size and number of factors. This section presents an extension of a searching procedure given by Mukerjee et al. (2017) to the current setting and provides examples for illustration.

### 5.1. A procedure for efficient design search

The minimum contamination criterion (8) can be applied to regular and nonregular designs, but requires heavy computation of $K_{s l}(h)=N^{-2} \operatorname{tr}\left[\mathbf{X}_{h 1}^{\top} \mathbf{X}_{s l} \mathbf{X}_{s l}^{\top} \mathbf{X}_{h 1}\right]$. By noting that $\mathbf{X}_{s l} \mathbf{X}_{s l}^{\top}$ is reminiscent of minimum moment aberration in Xu (2003), Mukerjee et al. (2017) developed an efficient computational procedure for $K_{s l}(h)$. We now extend this procedure for computing $\mathbf{X}_{s l} \mathbf{X}_{s l}^{\top}$ to double-pair conditional effect models. For $0 \leq c \leq n-2$, let $Q_{0}(c)=1$, $Q_{1}(c)=2 c-(n-4)$, and

$$
\begin{equation*}
Q_{l}(c)=l^{-1}\left\{[2 c-(n-4)] Q_{l-1}(c)-(n-l-2) Q_{l-2}(c)\right\} \tag{9}
\end{equation*}
$$

where $2 \leq l \leq n-2$. Write $\widetilde{\mathbf{D}}$ for the subarray given by the last $n-4$ columns of $\mathbf{D}$ (i.e. only consisting of traditional factors). For $1 \leq u, w \leq N$, let $c_{u w}$ be the number of positions where the $u$ th and $w$ th rows of $\widetilde{\mathbf{D}}$ have the same entry, and $q_{s l}(u, w)$ be the $(u, w)$ th element of $\mathbf{X}_{s l} \mathbf{X}_{s l}^{\top}$. Denote the $(u, j)$ th element of $\mathbf{D}$ by $d_{u j}$. Then the following result holds.

Theorem 5.1: For $1 \leq u, w \leq N$ and $2 \leq l \leq n-2$, we have
(a) $q_{0 l}(u, w)=\left(d_{u 2} d_{w 2} d_{u 4} d_{w 4}\right) Q_{l-2}\left(c_{u, w}\right)+\left(d_{u 2} d_{w 2}+d_{u 4} d_{w 4}\right) Q_{l-1}\left(c_{u, w}\right)+Q_{l}\left(c_{u, w}\right)$;
(b) $q_{1 l}(u, w)=\left(d_{u 1} d_{w 1}+d_{u 1} d_{w 1} d_{u 2} d_{w 2}+d_{u 3} d_{w 3}+d_{u 3} d_{w 3} d_{u 4} d_{w 4}\right) Q_{l-1}\left(c_{u, w}\right)$;
(c) $q_{2 l}(u, w)=d_{u 1} d_{w 1} d_{u 3} d_{w 3}\left(1+d_{u 2} d_{w 2}+d_{u 4} d_{w 4}+d_{u 2} d_{w 2} d_{u 4} d_{w 4}\right) Q_{l-2}\left(c_{u, w}\right)$.

Proof: For $2 \leq l \leq n-2$, let $\Sigma^{(l)}$ be the sum over binary tuples $j_{5} \cdots j_{n}$ such that $l$ of $j_{5}, \ldots, j_{n}$ equal 1 . We have

$$
\begin{aligned}
q_{0 l}(u, w)= & \Sigma^{(l)} \boldsymbol{x}\left(u ; 0000 j_{5} \cdots j_{n}\right) \boldsymbol{x}\left(w ; 0000 j_{5} \cdots j_{n}\right) \\
& +\Sigma^{(l-1)} \boldsymbol{x}\left(u ; 0100 j_{5} \cdots j_{n}\right) \boldsymbol{x}\left(w ; 0100 j_{5} \cdots j_{n}\right) \\
& +\Sigma^{(l-1)} \boldsymbol{x}\left(u ; 0001 j_{5} \cdots j_{n}\right) \boldsymbol{x}\left(w ; 0001 j_{5} \cdots j_{n}\right) \\
& +\Sigma^{(l-2)} \boldsymbol{x}\left(u ; 0101 j_{5} \cdots j_{n}\right) \boldsymbol{x}\left(w ; 0101 j_{5} \cdots j_{n}\right) \\
= & \left(d_{u 2} d_{w 2} d_{u 4} d_{w 4}\right) \Psi_{l-2}\left(c_{u, w}\right)+\left(d_{u 2} d_{w 2}+d_{u 4} d_{w 4}\right) \Psi_{l-1}\left(c_{u, w}\right)+\Psi_{l}\left(c_{u, w}\right),
\end{aligned}
$$

where $\Psi_{l}(u, w)=\Sigma^{(l)} \prod_{s=5}^{n}\left(d_{u} d_{w}\right)^{j_{s}}$. Similarly, we have

$$
\begin{aligned}
& q_{1 l}(u, w)=\left(d_{u 1} d_{w 1}+d_{u 1} d_{w 1} d_{u 2} d_{w 2}+d_{u 3} d_{w 3}+d_{u 3} d_{w 3} d_{u 4} d_{w 4}\right) \Psi_{l-1}\left(c_{u, w}\right) \\
& q_{2 l}(u, w)=d_{u 1} d_{w 1} d_{u 3} d_{w 3}\left(1+d_{u 2} d_{w 2}+d_{u 4} d_{w 4}+d_{u 2} d_{w 2} d_{u 4} d_{w 4}\right) \Psi_{l-2}\left(c_{u, w}\right) .
\end{aligned}
$$

The result will follow if $\Psi_{l}(u, w)=Q_{l}\left(c_{u w}\right)$. It is clear that $\Psi_{0}(u, w)=1$ and $\Psi_{1}(u, w)=c_{u w}+(-1)(n-4-$ $\left.c_{u w}\right)=2 c_{u w}-(n-4)$. It remains to show $\Psi_{l}(u, w)$ satisfies the recursion relation (9).

Let $\Phi(\xi)=\prod_{j=5}^{n}\left(1+\xi d_{u j} d_{w j}\right)$ and let $\Phi_{l}(\xi)$ be the $l$ th derivative of $\Phi(\xi)$. Note that $\Psi_{l}(u, w)=\Phi_{l}(0) / l$ !. Differentiation of $\log \Phi(\xi)$ yields

$$
\begin{aligned}
\Phi_{1}(\xi) & =\left(\sum_{j=5}^{n} \frac{d_{u j} d_{w j}}{1+\xi d_{u j} d_{w j}}\right) \Phi(\xi) \\
& =\left(\frac{c_{u w}}{1+\xi}-\frac{(n-4)-c_{u w}}{1-\xi}\right) \Phi(\xi)
\end{aligned}
$$

that is, $\left(1-\xi^{2}\right) \Phi_{1}(\xi)=\left\{2 c_{u w}-(n-4)(1+\xi)\right\} \Phi(\xi)$. Differentiating this $l-1$ and taking $\xi=0$, we get

$$
\Phi_{l}(0)=\left[2 c_{u w}-(n-4)\right] \Phi_{l-1}(0)-(l-1)(n-l-2) \Phi_{l-2}(0) .
$$

This leads to (9) by using $\Psi_{l}(u, w)=\Phi_{l}(0) / l$ !.
With the help of Theorem 5.1 and suggested by Mukerjee et al. (2017), an algorithm is provided as follows.
(1) Search for minimum contamination designs by first listing of all nonisomorphic regular designs for given run size $N(\geq 16)$ and number of factors $n(\geq 5)$.
(2) For each nonisomorphic regular design, permute its columns such that the resulting design satisfies the conditions in Theorem 3.1. Let the first four columns represent the two pairs of conditional and conditioned factors, that is, $F_{1}, F_{2}$ and $F_{3}, F_{4}$.
(3) Calculate the criterion in (8) by using Theorem 5.1, and hence find a minimum contamination design.

This procedure mostly consumes affordable computational time. For $N=16$ and $n=10$, for example, it takes around 3.94 minutes to find a minimum contamination design on a desktop with 3.8 GHz CPU and 64 GB of RAM.

### 5.2. Examples

We apply the algorithm presented in Section 5.1 to 16 - and 32 -run designs. Afterwards, the light bulb experiment (Wu \& Hamada, 2021, p. 347) mentioned in Section 1 is revisited.

For $N=16$ and 32, a list of all nonisomorphic regular designs are given in the catalogues in Chen et al. (1993). Table 1 exhibits the results for $N=16$ and $5 \leq n \leq 12$. In the table, the numbers $1,2,4,8$ represent basic factors in a design. The other numbers represent added factors. For example, for $n=5$, if the five factors are denoted by $A, B, C, D, E$, then the minimum contamination design is the one with the defining relation $E=A B C D$ because $15=1+2+4+8$. We can see that all minimum contamination designs under conditional effect models are also minimum aberration under traditional models. The finding supports using minimum aberration designs under traditional models to perform subsequent de-aliasing analysis in Su and Wu (2017). Table 2 exhibits the results for $N=32$ and $6 \leq n \leq 18$. Same as Table 1, the numbers $1,2,4,8,16$ represent basic factors in a design. The other numbers represent added factors. For example, for $n=6$, if the five factors are denoted by $A, B, C, D, E, F$, then the minimum contamination design is the one with the defining relation $F=A B C D E$ because $31=1+2+4+8+16$. The R codes for generating these designs are attached to the supplementary material.

Table 1. Regular minimum contamination designs for $N=32$.

| $n$ | $(1,2,4,8,16,31)$ |
| :--- | :---: |
| 6 | $(1,8,16,7,2,4,27)$ |
| 7 | $(4,16,7,29,1,2,8,11)$ |
| 8 | $(1,4,7,29,2,8,16,11,19)$ |
| 9 | $(4,8,7,19,1,2,16,11,29,30)$ |
| 10 | $(16,11,14,19,1,2,4,8,7,13,21)$ |
| 11 | $(16,11,13,19,1,2,4,8,7,14,21,22)$ |
| 12 | $(16,11,13,19,1,2,4,8,7,14,21,22,25)$ |
| 13 | $(1,4,7,11,2,8,16,13,14,19,21,22,25,26)$ |
| 14 | $(1,2,4,8,16,32,7,11,13,14,19,22,25,26,28)$ |
| 15 | $(1,2,4,8,16,7,11,13,14,19,21,22,25,26,28,31)$ |
| 16 |  |

Table 2. Regular minimum contamination designs for $N=16$.

| $n$ | Minimum aberration design |
| :--- | ---: |
| 5 | $(1,2,4,8,15)$ |
| 6 | $(1,8,2,4,7,11)$ |
| 7 | $(1,2,4,8,7,11,13)$ |
| 8 | $(1,2,4,8,7,11,13,14)$ |
| 9 | $(2,4,8,3,1,5,9,14,15)$ |
| 10 | $(1,6,2,8,4,3,5,9,14,15)$ |
| 11 | $(4,8,5,10,1,2,3,6,9,13,14)$ |
| 12 | $(2,5,6,10,1,4,8,3,9,13,14,15)$ |

The light bulb experiment mentioned in Wu and Hamada (2021, p. 347) studied a light bulb sealing process performed to improve a cosmetic problem that was frequently occurring (Taguchi, 1987, Section 17.6). The outcomes were determined mainly by six two-level traditional factors $A, B, C, D, E, F$ and two pairs of conditional and conditioning factors $(H, G)$ and $(J, I)$, respectively. A 16-run regular fractional factorial design is to be conducted to study this light bulb sealing process. With the use of the algorithm in Section 5.1, we obtain the minimum contamination design with the design matrix given in Table 3 (the column indices also provided in Table 2). The first four columns are assigned to the two pairs of conditional and conditioning factors $(H, G)$ and ( $J, I$ ), respectively; the remaining columns are assigned to the six traditional factors. This $2^{10-6}$ design has the defining generators $B=H I, C=A H$, $A=G I, D=H J, E=A I J$ and $F=A H I J$. The corresponding $K$-sequence in (8) has $2 \times 3 \times(10-2+1)=42$ elements given as follows:

$$
\begin{aligned}
K= & \{9,10,17,4,2,0,28,16,21,12,12,6,35,16,54,16,30,18,28,12,18,24,40,20, \\
& 19,6,17,4,30,12,0,4,1,0,12,6,1,0,0,0,2,2\} .
\end{aligned}
$$

We also evaluate the design provided by Taguchi (1987, Section 17.6) (see Table 7.13 of Wu and Hamada (2021, p.347)) by the minimum contamination criterion in (8). The resulting $K$-sequence is given by

$$
\begin{aligned}
K= & \{15,7,11,6,3,0,20,19,30,10,10,6,35,18,38,22,31,18,32,10,32,18,40,20, \\
& 13,7,15,4,29,12,4,3,2,0,14,6,1,0,0,0,1,2\} .
\end{aligned}
$$

The first element of the $K$-sequence of the design in Table 3 is 9 , smaller than that of the design provided by Taguchi (1987, Section 17.6). Thus, the proposed algorithm found a better design in terms of the minimization of contamination caused by interactions under the double-pair conditional effect model.

## 6. Concluding remarks

This paper extends the work of Mukerjee et al. (2017) to two pairs of conditional and conditioning factors. The conditional effect model in (3) is an orthogonal reparameterization of the traditional model in (1). Such a reparameterization introduces a new effect hierarchy order of factorial effects, resulting in a different design evaluation from the minimum aberration due to Fries and Hunter (1980). We note that alternative reparameterization of (1) is required if the topic of interest does not depend on conditional effects. For example, Yang and Speed (2002) proposed to define the factorial effects with reference to natural baseline levels of the factors, referred to as baseline parameterization. Later, Mukerjee and Tang (2012), Mukerjee and Huda (2016) and Sun and Tang (2022) discussed design optimality and construction under the baseline parameterization.

Mukerjee et al. (2017) claimed that, in practice, the number of conditional and conditioned pairs seldom exceeds two. On the other hand, the effect hierarchy order in (4) is only irrelevant to the value of $r$ when the number of pairs

Table 3. Sixteen-run minimum contamination design with ten factors for the light bulb experiment.

|  | H | G | 1 | $J$ | A | B | C | D | E | F |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 2 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | -1 |
| 3 | 1 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | -1 | -1 |
| 4 | 1 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | -1 | -1 |
| 5 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | 1 | -1 |
| 6 | -1 | 1 | -1 | 1 | -1 | 1 | 1 | -1 | 1 | -1 |
| 7 | -1 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | -1 |
| 8 | -1 | -1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 |
| 9 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 | -1 | 1 |
| 10 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | 1 |
| 11 | -1 | 1 | 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 |
| 12 | -1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 | -1 | 1 |
| 13 | 1 | -1 | 1 | -1 | -1 | 1 | -1 | -1 | 1 | 1 |
| 14 | 1 | -1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 |
| 15 | 1 | 1 | -1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

is not greater than two. When dealing with more than two pairs, the effect hierarchy order is a nontrivial function of $r$ and deriving useful results can be exceedingly difficult. Alternatively, we note that the minimum contamination criterion in (8) can be applied to regular as well as nonregular designs; however, this paper focuses on regular designs owing to their popularity and theoretical underpinnings. The application of this framework to nonregular designs is left for future research. Another interesting future direction suggested by a reviewer is to involve qualitative fourlevel conditional/conditioning factors. It is known that a qualitative four-level factor can be decomposed into three orthogonal main effect components with two levels each (Wu \& Hamada, 2021, chapter 7). However, our theory cannot be directly applied since the three two-level main effect components do not share the same properties as a real two-level factor. The covariance structure for the main effect components is different from that for two-level factors; see Joseph et al. (2009) for a discussion. The resulting effect hierarchy order of conditional factorial effects may be complicated with vague interpretations. Since this topic requires a nontrivial extension, we leave it for future research.

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## Appendix. Proof of Theorem 4.1

In this proof, a traditional factorial effect is represented by a word, i.e. a subset of $\{1, \ldots, n\}$. For two words $W_{1}$ and $W_{2}$, we define $W_{1} \Delta W_{2}$ to be $\left(W_{1} \cup W_{2}\right) \backslash\left(W_{1} \cap W_{2}\right)$. Note that $K_{s l}(h)=N^{-2} \operatorname{tr}\left[\mathbf{X}_{h 1}^{\top} \mathbf{X}_{s l} \mathbf{X}_{s l}^{\top} \mathbf{X}_{h 1}\right]$, which is the sum of squared entries of $N^{-2} \mathbf{X}_{h 1}^{\top} \mathbf{X}_{s l}$. Because the design is regular, each squared entry is either one or zero according to whether the corresponding effects are aliased.

Part (a) is evident from Tang and Deng (1999) except that the number of factors considered in the computation is $n-2$ (exclude $F_{1}$ and $F_{3}$ ). So we have $K_{0 l}(0)=(l+1) A_{l+1}^{(1)}+(n-l-1) A_{l-1}^{(1)}$.

For (b), let $S_{l}$ be the set of all words of length $l$ not containing any word involving 1 and 3 . Let $S_{l 2}$ be a subset of $S_{l}$ and 2 belongs to each word in $S_{l 2}$. Then $\{1\} \Delta W, W \in S_{l 2}$, is of the form $\{1,2\} \cup(W \backslash\{2\})$, where ( $W \backslash\{2\}$ ) is of length $l-1$. Similarly, $\{1\} \Delta W, W \in\left(S \backslash S_{l 2}\right)$, is of the form $\{1\} \cup W$, where $W$ is of length $l$. Similar argument can be made when the roles of $F_{1}$ and $F_{3}, F_{2}$ and $F_{4}$ are interchanged, respectively. By the definition of $A_{l}^{(21)}$ and $A_{l}^{(22)}$, we obtain $K_{0 l}(1)=A_{l-1}^{(2)}+A_{l}^{(2)}$.

For (c), first consider $\{2\} \Delta(\{1\} \cup W)$ and $\{2\} \Delta(\{1,2\} \cup W)$, where $W$ runs through all the words not involving $F_{1}, F_{2}, F_{3}$ and has length $l-1$. It is equivalent to consider $\{1,2\} \cup W$ and $\{1\} \cup W$ for such $W$ 's. This yields $A_{l-1}^{(21)}$ in $K_{1 l}(0)$. Next we consider $\{j\} \triangle(\{1\} \cup W)$ and $\{j\} \triangle(\{1,2\} \cup W)$, where $j=4, \ldots, n$ and $W$ runs through all the words not involving $F_{1}, F_{2}, F_{3}$ and has length $l-1$. By Tang and Deng (1999), this yields $(n-l-1) A_{l-2}^{(21)}+l A_{l}^{(21)}$ in $K_{1 l}(0)$. Similar argument can be made when the roles of $F_{1}$ and $F_{3}, F_{2}$ and $F_{4}$ are interchanged, respectively. By the definition of $A_{l}^{(21)}$ and $A_{l}^{(22)}$, we obtain $K_{1 l}(0)=$ $(n-l-1) A_{l-2}^{(2)}+A_{l-1}^{(2)}+l A_{l}^{(2)}$.

For (d), first consider $\{1\} \Delta(\{1\} \cup W),\{1\} \Delta(\{1,2\} \cup W),\{1,2\} \Delta(\{1\} \cup W)$ and $\{1,2\} \Delta(\{1,2\} \cup W)$, where $W$ runs through all the words not involving $F_{1}, F_{2}, F_{3}$ and has length $l-1$. It is equivalent to consider $W,\{2\} \cup W,\{2\} \cup W$ and $W$ for such $W$ 's. This yields $2 A_{l-1}^{(31)}$ in $K_{1 l}(1)$. Next consider $\{1\} \Delta(\{3\} \cup W),\{1\} \Delta(\{3,4\} \cup W),\{1,2\} \Delta(\{3\} \cup W)$ and $\{1,2\} \Delta(\{3,4\} \cup W)$, where $W$ runs through all the words not involving $F_{1}, F_{3}, F_{4}$ and has length $l-1$. For such $W$ 's, $\{1\} \Delta(\{3\} \cup W)$ and $\{1\} \Delta(\{3,4\} \cup W)$ yield $A_{l-1}^{(42)}$ in $K_{1 l}(1) ;\{1,2\} \Delta(\{3\} \cup W)$ and $\{1,2\} \Delta(\{3,4\} \cup W)$ yield $A_{l-2}^{(43)}+A_{l-1}^{(52)}$ in $K_{1 l}(1)$. Similar argument can be made when the roles of $F_{1}$ and $F_{3}, F_{2}$ and $F_{4}$ are interchanged, respectively. By the definition of $A_{l}^{(3)}$, we obtain $K_{1 l}(1)=2 A_{l-1}^{(3)}+$ $2\left\{A_{l-1}^{(42)}+A_{l-2}^{(43)}+A_{l-1}^{(52)}\right\}$.

For $(e)$, first consider $\{2\} \Delta(\{1,3\} \cup W),\{2\} \Delta(\{1,3,4\} \cup W),\{2\} \Delta(\{1,2,3\} \cup W),\{2\} \Delta(\{1,2,3,4\} \cup W)$ and $\{4\} \Delta(\{1,3\} \cup$ $W),\{4\} \Delta(\{1,3,4\} \cup W),\{4\} \Delta(\{1,2,3\} \cup W),\{4\} \Delta(\{1,2,3,4\} \cup W)$, where $W$ runs through all the words not involving $F_{1}, F_{2}, F_{3}, F_{4}$ and has length $l-2$. This yields $2 A_{l-2}^{(7)}$ in $K_{2 l}(0)$. Next consider $\{j\} \Delta(\{1,3\} \cup W),\{j\} \Delta(\{1,3,4\} \cup W)$, $\{j\} \triangle(\{1,2,3\} \cup W),\{j\} \triangle(\{1,2,3,4\} \cup W)$ for $j=5, \ldots, n$. By Tang and Deng (1999), this yields $(n-l-1) A_{l-3}^{(7)}+(l-1) A_{l-1}^{(7)}$. So we obtain $K_{2 l}(0)=2 A_{l-2}^{(7)}+(n-l-1) A_{l-3}^{(7)}+(l-1) A_{l-1}^{(7)}$.

For $(f)$, consider $\{j\} \Delta(\{1,3\} \cup W),\{j\} \triangle(\{1,3,4\} \cup W), \quad\{j\} \Delta(\{1,2,3\} \cup W), \quad\{j\} \Delta(\{1,2,3,4\} \cup W)$ for $j=1,3$, and $\{i, j\} \Delta(\{1,3\} \cup W),\{i, j\} \Delta(\{1,3,4\} \cup W),\{i, j\} \Delta(\{1,2,3\} \cup W),\{i, j\} \Delta(\{1,2,3,4\} \cup W)$ for $(i, j)=(1,2),(3,4)$, where $W$ runs through all the words not involving $F_{1}, F_{2}, F_{3}, F_{4}$ and has length $l-2$. This yields $2 A_{l-2}^{(8)}$. So we obtain $K_{2 l}(1)=2 A_{l-2}^{(8)}$.

