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Increasing convex order of capital allocation with dependent assets under threshold model

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ABSTRACT
In this manuscript, we consider a risk-preference investor allocating some amount of capital to the dependent risky asset, where the responding asset will occur default if the stochastic return is less than some predetermined threshold. Then, we present sufficient conditions of the increasing convex order on capital allocation with dependent risky assets when the stochastic return is right tail weakly stochastic arrangement increasing. Finally, some numerical examples are given as illustrations.

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1. Introduction

Aggregation of the random variables arises naturally in actuarial science and reliability theory (cf. Rinott et al., 2012), a lot of work devotes to investigating stochastic properties of the aggregation risks in terms of the various stochastic orders when the vector of coefficients is characterized by the majorization order. For example, Ma (2000) studied the linear aggregation of random variables in the sense of the decreasing convex order and the Laplace order by the majorization order. Xu and Hu (2012) investigated stochastic comparisons of capital allocation problems using a general loss function. Zhang and Zhao (2015) provided sufficient conditions for comparing the aggregate risks arising from two sets of heterogeneous portfolios with claims having gamma distributions. Zhang and Cheung (2020) presented stochastic properties of the generalized sum of right tail weakly stochastic arrangement increasing nonnegative random variables accompanied with stochastic arrangement increasing Bernoulli variables. Zhang et al. (2023b) investigated the usual stochastic and hazard rate orders between the largest claim amounts from two sets of heterogeneous and dependent insurance portfolios. For more stochastic comparisons and applications of aggregation, one can refer to the works of Barmalzan et al. (2015), Ariyafar et al. (2020), Zhang (2022), Ding et al. (2021), Yan et al. (2021), Zhang et al. (2022) and references therein.

Understanding risk-management technology can provide some insights into asset returns (cf. Scholes, 2000). In the insurance engineering and actuarial sciences, the risk of the initial asset allocation problem can be modeled by the aggregation of non-negative random variables. It is one main concern to reasonably allocate the initial wealth to concerned risk assets to pursue maximal return in the market, and the capital allocation has been gaining quite a lot of attention in the past several decades. Traditionally, such issues are investigated under the framework of expected utility theory. That is, the investors focus on allocating the initial wealth to the concerned assets to optimize the expected utility of the aggregate stochastic return. In general, the default risk is the possibility that a borrower is incapable of paying the interest or the principal repayment obligations on a loan agreement in the future, and the default risk has a significant impact on the expected total return. The past several years have witnessed fast development in theoretical properties and applications of the asset allocation problems. There are two directions of the related discussions.

On the one hand, for the financial portfolio analysis, the default risks are always not taken into consideration. Suppose that the investor allocates the amount w_i of the entire wealth w to the risk asset with non-negative random potential return $X_i, i = 1, 2, \dots, n$. If the wealth allocation vector is

$$\boldsymbol{w} \in \mathcal{W} = \left\{(w_1, w_2, \dots, w_n) : \sum_{i=1}^{n} w_i = w,\ \text{for fixed } w > 0 \text{ and } w_i \geq 0\right\},$$

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then, for an increasing and concave utility function, the responding aggregate stochastic return of the asset allocation problem can be expressed as

$$\max_{w\in\mathcal{W}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} w_i X_i\right)\right], \text{where } u \text{ is increasing and concave.}$$

To the best of our knowledge, for the aggregate stochastic return from mutually independent assets $X = (X_1, X_2, \ldots, X_n)$, Hadar and Seo (1988) might be among the first to prove that the risk-averse (i.e., the utility function u is increasing and concave) investor would like to invest a larger amount in the asset with larger stochastic return in terms of the first order stochastic dominance. Landsberger and Meilijson (1990) showed that the investor with increasing utility obtains the optimal asset allocation putting more wealth into the asset allocation with the larger stochastic return in terms of the likelihood ratio order. Kijima and Ohnishi (1996) established the similar result for the asset allocation with stochastic returns arrayed in the sense of the reversed hazard rate order. Hennessy and Lapan (2002) found that the risk-averse investor allocates more to the asset with a larger stochastic return in the sense of the reversed hazard rate order, and for investors with increasing utility (Li & You, 2014) provide for the optimal shares of assets with their potential returns arrayed in the sense of the likelihood ratio order. Li and You (2015) showed that the optimal allocation vector should be correspondingly arranged in the ascending order whenever potential returns of the assets have an arrangement increasing joint density for the investors with increasing utility. Besides, Cai and Wei (2015) verified that the optimal allocation vector should be arranged in ascending order whenever the assets have stochastic arrangement increasing potential returns for certain utility functions.

On the other hand, assets with default risks are a very interesting topic in actuarial science. For some portfolios of n assets with stochastic returns $\boldsymbol{X}$, let $\boldsymbol{I} = (I_1, I_2, \ldots, I_n)$ be the indicator vector of the default risks, that is, for $i = 1, 2, \ldots, n$,

$$I_i = \begin{cases} 1, & \text{if the default of the } i\text{-th asset does not occur,} \\ 0, & \text{if the default of the } i\text{-th asset occurs,} \end{cases}$$

and we suppose that $\boldsymbol{X}$ and $\boldsymbol{I}$ are independent. Then, under the framework of expected utility theory, an investor with increasing utility will face the following optimization issue

$$\max_{w\in\mathcal{W}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} w_i I_i X_i\right)\right], \text{ where } u \text{ is increasing.}$$

For the stochastic assets allocation with exchangeable stochastic returns, Cheung and Yang (2004) might take the first to establish that for the investor with increasing utility the optimal allocation assigns more wealth to the asset with smaller default probability in the context of some specific dependence structure of the indicator vector. Chen and Hu (2008) studied the ordering of the optimal asset allocation under some specific dependence structure of the indicator vector and some specific utility functions. Meanwhile, modeling the stochastic returns and the indicator vector respectively by using the weakly stochastic arrangement increasing and the weakly stochastic arrangement increasing through left tail probability distributions, Cai and Wei (2015) gave the ordering of the optimal allocations for investors with specific utility functions. Later, for the following asset allocation problem with default risks,

$$\max_{w\in\mathcal{W}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} w_i I_i X_i\right)\right],$$

where u is increasing and concave.

Li and Li (2016) studied how the allocation impacts the expected stochastic return of the portfolio of risk assets with some new dependence structures characterized through the orthant probability of their stochastic returns. Amini-Seresht et al. (2019) investigated the asset allocation with dependent stochastic returns under a threshold model when assets with stochastic returns were left tail weakly stochastic arrangement increasing. They considered some portfolios consisting of n assets, where the i-th asset will default if X_i is less than some predetermined threshold level $l_i \geq 0$, for $i = 1, 2, \ldots, n$. Then the stochastic return (per share) for the i-th asset can be denoted as $X_i I(X_i > l_i)$, for $i = 1, 2, \ldots, n$, where the indicator random variable $I(\mathrm{A})$ associated with event A has value 1 if event A occurs and has value 0 otherwise.

On the one hand, as discussed by Hagen (1979), Markowitz noted the presence of risk-seeking in preferences among positive as well as among negative prospects, and he proposed a utility function that has convex and concave regions in both the positive and the negative domains. For instance, entrepreneurs are often considered risk lovers because they are willing to invest money, time, and effort into a new venture in the hope of achieving success and high returns. They often take on uncertainty and potential risks because they believe that doing so will lead to innovation and business opportunities (cf. Shane & Venkataraman, 2000). High-yield investors: certain investors seek high-risk and high return investment opportunities, and they may invest in high-risk asset classes (such as stocks, options, cryptocurrencies, etc.) in the hope of achieving higher profits (cf. Frazzini & Pedersen, 2014). On the other hand, Boonen et al. (2021) stated that the reinsurer does not know the preferences of the insurer. Hence, in addition to the risk-averse investors, there are also risk-preference investors, that is, the utility function u is increasing, and convex (cf. Li & You, 2014) in practice market. Chen (2003) pointed the utility of the risk-preference increases with the increase of income, but their marginal utility shows an increasing trend, which is an important difference between risk-preference and risk-averse investors. The risk-preference investors are more willing to accept stochastic returns with risks than determinate returns, and the risk-preference investor always chooses the one with less certainty rather than greater certainty when faced with multiple forms of speculation with the same expected return.

However, the existing studies have studied the optimal asset allocation problem for risk-averse investors. Therefore, motivated by the works of Chen (2003), Li and Li (2016) and Amini-Seresht et al. (2019), for the risk-preference investors, we will study the following asset allocation problem

$$\max_{w\in\mathcal{W}} \mathbb{E}\left[u\left(\sum_{i=1}^{n} w_i X_i I(X_i > l_i)\right)\right], \text{ where } u \text{ is increasing and convex.} \quad (1)$$

This paper further exploits the optimal asset allocation problems (1) in the context of stochastic returns under a threshold model. For a risk-preference investor, the optimal and the worst allocation policies are given when assets with stochastic returns are the right tail weakly stochastic arrangement increasing, respectively. These results complement the corresponding ones of Cheung and Yang (2004) and Amini-Seresht et al. (2019).

The remainder of this work is organized as follows. Some definitions and terminologies are recalled in Section 2. Section 3 establishes the optimal and the worst allocation policies when assets with stochastic returns are right tail weakly stochastic arrangement increasing, respectively. Section 4 provides some numerical examples to verify the theoretical findings. Section 5 presents the theoretical contributions, the potential managerial implications and the future interesting topics.

2. Preliminaries

In this section, we recall some pertinent definitions, notations and useful lemmas used in the sequel. Throughout, the terms 'increasing' and 'decreasing' are used in a non-strict sense. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathcal{D}_+^n = \{\boldsymbol{x} : x_1 \geq x_2 \geq \cdots \geq x_n\}$, $\mathcal{I}_+^n = \{\boldsymbol{x} : x_1 \leq x_2 \leq \cdots \leq x_n\}$.

First, let us recall the definitions of some useful stochastic orders to stochastically compare two random variables. Let $F[\overline{F}]$ and $G[\overline{G}]$ be the distribution[survival] functions of the random variables X and Y, respectively.

Definition 2.1: A random variable X is said to be smaller than Y in the

(i) increasing convex order (denoted by $X \leq_{\rm icx} Y$) if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for any increasing convex function $\phi : \mathbb{R} \to \mathbb{R}$, or equivalently, $\int_t^{\infty} \overline{F}(u)\mathrm{d}u \leq \int_t^{\infty} \overline{G}(u)\mathrm{d}u$, for $t \in \mathbb{R}$;
(ii) increasing concave order (denoted by $X \leq_{\rm icv} Y$) if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for any increasing concave function $\phi : \mathbb{R} \to \mathbb{R}$, or equivalently, $\int_0^{t} \overline{F}(u)\mathrm{d}u \leq \int_0^{t} \overline{G}(u)\mathrm{d}u$, for $t \in \mathbb{R}$.

The increasing convex/concave order is also called the second degree of stochastic dominance. For two non-negative random variables X and Y, denote F^{-1} and G^{-1} the right continuous inverses, respectively. Then, $X \leq_{\rm icx} Y$ if and only if

$$\int_{\alpha}^{1} F^{-1}(t)\mathrm{d}t \leq \int_{\alpha}^{1} G^{-1}(t)\mathrm{d}t, \text{ where } \alpha \in [0,1],$$

and according to the view of Giovagnoli and Wynn (2011), $X \leq_{\rm icv} Y$ if and only if

$$\int_{0}^{\beta} F^{-1}(t)\mathrm{d}t \leq \int_{0}^{\beta} G^{-1}(t)\mathrm{d}t, \text{ where } \beta \in [0,1].$$

Hence, these equivalent definitions are very helpful for numerical studies. For more details on the properties and applications of these stochastic orders, interested readers may refer to the excellent monographs by Shaked and Shanthikumar (2007), Li and Li (2013) and Zhang et al. (2023a).

For any two real-valued vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, let $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ and $y_{1:n} \le y_{2:n} \le \cdots \le y_{n:n}$ be their increasing arrangements, respectively.

Definition 2.2: A vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to majorize $\boldsymbol{y} \in \mathbb{R}^n$, denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{i:n} \le \sum_{i=1}^{j} y_{i:n}$, for all $j = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$.

Majorization order is a useful tool for establishing various inequalities in applied probability and risk management. For more detailed discussions on the theory of majorization and its applications, one may refer to Marshall et al. (1979), Balakrishnan and Zhao (2013) and Zhang and Zhang (2022, 2023).

For a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, let $\boldsymbol{x}_{(i,j)}$ be the sub-vector with x_i and x_j deleted and $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ be any permutation of $\{1, 2, \ldots, n\}$ such that $\pi(\boldsymbol{x}) = (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$. In particular, we denote $\pi_{i,j}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$. For any (i,j) with $1 \le i < j \le n$, let $\Delta_{i,j} g(\boldsymbol{x}) = g(\boldsymbol{x}) - g(\pi_{i,j}(\boldsymbol{x}))$ and

$$\mathcal{G}^{i,j}_{\text{rwsai}}(n) = \{g(\boldsymbol{x}) : \Delta_{i,j} g(\boldsymbol{x}) \text{ is increasing in } x_j \ge x_i, \text{ for any } x_i\},$$

$$\mathcal{G}^{i,j}_{\text{lwsai}}(n) = \{g(\boldsymbol{x}) : \Delta_{i,j} g(\boldsymbol{x}) \text{ is decreasing in } x_i \le x_j, \text{ for any } x_j\}.$$

Definition 2.3: A random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to be

(i) right tail weakly stochastic arrangement increasing (RWSAI) if $\mathbb{E}[g(\boldsymbol{X})] \ge \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$, for any $g \in \mathcal{G}^{i,j}_{\text{rwsai}}(n)$ and any pair (i,j) such that $1 \le i < j \le n$;
(ii) left tail weakly stochastic arrangement increasing (LWSAI) if $\mathbb{E}[g(\boldsymbol{X})] \ge \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$, for any $g \in \mathcal{G}^{i,j}_{\text{lwsai}}(n)$ and any pair (i,j) such that $1 \le i < j \le n$.

The notions of RWSAI and LWSAI are introduced by Cai and Wei (2014) and have been applied in actuarial science and financial engineering to model the dependence among ordered random risks. interested readers may refer to the outstanding works of Cai and Wei (2014, 2015), You and Li (2015), Zhang et al. (2018). The multivariate Dirichlet distribution, the multivariate F distribution, and the multivariate Pareto distribution of type I have AI joint probability densities whenever the corresponding parameters are arrayed in ascending order.

The following lemmas present some interesting properties of RWSAI and LWSAI, which play a part in the proof of the main results.

Lemma 2.4: *If a random vector $\boldsymbol{X}$ is*

(i) *RWSAI, then $((X_i, X_j)|\boldsymbol{X}_{i,j})$ is RWSAI for any $i, j = 1, 2, \ldots, n$, where $((X_i, X_j)|\boldsymbol{X}_{i,j})$ denotes the conditional bivariate random vector (X_i, X_j) given the values of $\boldsymbol{X}_{i,j} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$;*
(ii) *LWSAI, then $((X_i, X_j)|\boldsymbol{X}_{i,j})$ is LWSAI for any $i, j = 1, 2, \ldots, n$, where $((X_i, X_j)|\boldsymbol{X}_{i,j})$ denotes the conditional bivariate random vector (X_i, X_j) given the values of $\boldsymbol{X}_{i,j} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$.*

Lemma 2.5: *A random vector $\boldsymbol{X}$ is RWSAI[LWSAI] if and only if*

$$\mathbb{E}[g_2(X_1, X_2)] \ge \mathbb{E}[g_1(X_1, X_2)]$$

for all g_1 and g_2 such that

(i) *$g_2(x_1, x_2) - g_1(x_1, x_2)$ is increasing [decreasing] in $x_j \ge x_i [x_i \le x_j]$, for any $x_i[x_j]$;*
(ii) *$g_2(x_1, x_2) + g_2(x_2, x_1) \ge g_1(x_1, x_2) + g_1(x_2, x_1)$, for $x_j \ge x_i$.*

3. Main results

We first discuss the orderings among the coordinates of the optimal allocation policy when the stochastic returns are RWSAI and accompanied with descending threshold values.

Theorem 3.1: *Suppose that $\boldsymbol{X}$ is RWSAI and $\boldsymbol{l} \in \mathcal{D}^n_+$. Then, for any $\boldsymbol{w} \in \mathcal{D}^n_+$, and any permutation $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ of $\{1, 2, \ldots, n\}$, we have*

$$\sum_{i=1}^{n} w_i X_i I(X_i > l_i) \le_{\text{icx}} \sum_{i=1}^{n} w_{\pi_i} X_i I(X_i > l_i) \le_{\text{icx}} \sum_{i=1}^{n} w_{n-i+1} X_i I(X_i > l_i).$$

Proof: Note that any permutation $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ can be obtained by a series of pairwise interchange of permutation $(1, 2, \ldots, n)$, which is needed to consider the case that only one pairwise interchanges. Without loss generality, we give the proof of $\boldsymbol{\pi} = (2, 1, 3, \ldots, n)$. Note that

$$\sum_{i=1}^{n} w_i X_i I(X_i > l_i) = w_1 X_1 I(X_1 > l_1) + w_2 X_2 I(X_2 > l_2) + \sum_{i=3}^{n} w_i X_i I(X_i > l_i),$$
$$\sum_{i=1}^{n} w_{\pi_i} X_i I(X_i > l_i) = w_2 X_1 I(X_1 > l_1) + w_1 X_2 I(X_2 > l_2) + \sum_{i=3}^{n} w_i X_i I(X_i > l_i).$$

Let $a = \sum_{i=3}^{n} w_i X_i I(X_i > l_i)$ for any $\boldsymbol{X}_{(1,2)} = \boldsymbol{x}_{(1,2)}$. Then, for any increasing and convex function $\phi : \mathbb{R} \to \mathbb{R}$, we define

$$g_2(x_1, x_2) = \phi(w_2 x_1 I(x_1 > l_1) + w_1 x_2 I(x_2 > l_2) + a), \text{ and}$$
$$g_1(x_1, x_2) = \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a).$$

From Lemma 2.5, for any $x_1 \leq x_2$, note that the function

$$\begin{aligned}
& g_2(x_1, x_2) + g_2(x_2, x_1) \\
&= \phi(w_2 x_1 I(x_1 > l_1) + w_1 x_2 I(x_2 > l_2) + a) + \phi(w_2 x_2 I(x_2 > l_2) + w_1 x_1 I(x_1 > l_1) + a) \\
&\geq \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a) + \phi(w_1 x_2 I(x_2 > l_2) + w_2 x_1 I(x_1 > l_1) + a) \\
&= g_1(x_1, x_2) + g_1(x_2, x_1),
\end{aligned}$$

which is trivial based on the fact that $g_2(x_1, x_2) = g_1(x_2, x_1)$ and $g_2(x_2, x_1) = g_1(x_1, x_2)$. Thus, we next only prove the function

$$\begin{aligned}
& g_2(x_1, x_2) - g_1(x_1, x_2) \\
&= \phi(w_2 x_1 I(x_1 > l_1) + w_1 x_2 I(x_2 > l_2) + a) - \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a)
\end{aligned}$$

is increasing in $x_2 \geq x_1$ for any x_1. For any fixed x_1, define

$$\Lambda_1(x_2) = (w_2 - w_1) x_1 I(x_1 > l_1) + (w_1 - w_2) x_2 I(x_2 > l_2). \tag{2}$$

Notice that

$$\Lambda_1(x_2) \geq (w_2 - w_1) x_1 I(x_1 > l_1) + (w_1 - w_2) x_1 I(x_1 > l_1) = [(w_2 - w_1) + (w_1 - w_2)] x_1 I(x_1 > l_1) = 0,$$

where the inequality follows from the fact that $xI(x > l)$ is an increasing function in x and $x_2 I(x_2 > l_2) \geq x_1 I(x_1 > l_1)$, for any $l_1 \geq l_2$. Therefore, $\Lambda_1(x_2)$ is non-negative and increasing in x_2, for $w_1 \geq w_2$ and any x_1. For any $x_1 \leq y_1 \leq x_2$, note that the convexity of ϕ implies

$$\begin{aligned}
& \phi(w_1 x_1 I(x_1 > l_1) + w_2 y_1 I(y_1 > l_1) + \Lambda_1(y_1) + a) \\
& \quad - \phi(w_1 x_1 I(x_1 > l_1) + w_2 y_1 I(y_1 > l_1) + a) \\
& \leq \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + \Lambda_1(y_1) + a) \\
& \quad - \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a).
\end{aligned} \tag{3}$$

Furthermore, from $0 \leq \Lambda_1(y_1) \leq \Lambda_1(x_2)$ and the increasing property of ϕ, it follows that

$$\begin{aligned}
& \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + \Lambda_1(y_1) + a) \\
& \quad - \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a) \\
& \leq \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + \Lambda_1(x_2) + a) \\
& \quad - \phi(w_1 x_1 I(x_1 > l_1) + w_2 x_2 I(x_2 > l_2) + a).
\end{aligned} \tag{4}$$

According to (3) and (4), we have

$$\begin{aligned}
&\phi(w_1x_1I(x_1>l_1)+w_2y_1I(y_1>l_1)+\Lambda_1(y_1)+a)\\
&\quad-\phi(w_1x_1I(x_1>l_1)+w_2y_1I(y_1>l_1)+a)\\
&\le\phi(w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)+\Lambda_1(x_2)+a)\\
&\quad-\phi(w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)+a).
\end{aligned}\tag{5}$$

By the definition of $\Lambda_1(x_2)$ in (2), for any $y_1\le x_2$, further, we simplify (5) as

$$\begin{aligned}
&\phi(w_2x_1I(x_1>l_1)+w_1y_1I(y_1>l_1)+\Lambda_1(y_1)+a)\\
&\quad-\phi(w_1x_1I(x_1>l_1)+w_2y_1I(y_1>l_1)+a)\\
&\le\phi(w_2x_1I(x_1>l_1)+w_1x_2I(x_2>l_2)+\Lambda_1(x_2)+a)\\
&\quad-\phi(w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)+a),
\end{aligned}$$

which implies that $g_2(x_1,x_2)-g_1(x_1,x_2)$ is increasing in $x_2\ge x_1$ for any x_1. Therefore, it follows that from Lemma 2

$$\begin{aligned}
&w_1X_1I(X_1>l_1)+w_2X_2I(X_2>l_2)+a\\
&\le_{\text{icx}} w_2X_1I(X_1>l_1)+w_1X_2I(X_2>l_2)+a.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(\sum_{i=1}^n w_iX_iI\left(X_i>l_i\right)\right)\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\phi\left(\sum_{i=1}^n w_iX_iI\left(X_i>l_i\right)\right)\Big|\boldsymbol{X}_{(1,2)}\right]\right]\\
&\le\mathbb{E}\left[\mathbb{E}\left[\phi\left(w_2X_1I\left(X_1>l_1\right)+w_1X_2I\left(X_2>l_2\right)+\sum_{i=3}^n w_iX_iI\left(X_i>l_i\right)\right)\Big|\boldsymbol{X}_{(1,2)}\right]\right]\\
&=\mathbb{E}\left[\phi\left(w_2X_1I\left(X_1>l_1\right)+w_1X_2I\left(X_2>l_2\right)+\sum_{i=3}^n w_iX_iI\left(X_i>l_i\right)\right)\right].
\end{aligned}$$

Repeating the argument, the desired result follows. ■

Based on Theorem 3.1, we can conclude that the worst allocation $\tilde{w}$ for a risk-preferent investor should be fulfilled with $\tilde{w}_1\ge\tilde{w}_2\ge\cdots\ge\tilde{w}_n$. The next result proves that more diversity among the allocations taken in $\mathcal{D}_+^n$ leads to smaller aggregate stochastic returns in the sense of the increasing convex ordering.

Theorem 3.2: *Suppose that* $\boldsymbol{X}$ *is RWSAI and* $\boldsymbol{l}\in\mathcal{D}_+^n$. *Then, for* $\boldsymbol{w},\boldsymbol{v}\in\mathcal{D}_+^n$,

$$\boldsymbol{w}\overset{\text{m}}{\succeq}\boldsymbol{v}\Rightarrow\sum_{i=1}^n w_iX_iI(X_i>l_i)\le_{\text{icx}}\sum_{i=1}^n v_iX_iI(X_i>l_i).$$

Proof: For any increasing and convex function $\phi:\mathbb{R}\to\mathbb{R}$, we need to show that

$$\mathbb{E}\left[\phi\left(\sum_{i=1}^n w_iX_iI\left(X_i>l_i\right)\right)\right]\le\mathbb{E}\left[\phi\left(\sum_{i=1}^n w_iX_iI\left(X_i>l_i\right)\right)\right].$$

By the nature of majorization order, the proof can be completed under the setting of $\boldsymbol{w}=(w_1,w_2,w_3,\ldots,w_n)$, $\boldsymbol{v}=(v_1,v_2,w_3,\ldots,w_n)$ and $(w_1,w_2)\overset{\text{m}}{\succeq}(v_1,v_2)$. Using Lemma 3.A.2.b of Marshall et al. (1979), it is enough to prove that

$$\mathbb{E}\left[\phi\left(w_1X_1I\left(X_1>l_1\right)+w_2X_2I\left(X_2>l_2\right)+\sum_{i=3}^n w_iX_iI\left(X_i>l_i\right)\right)\right]$$

$$\leq \mathbb{E}\left[\phi\left(v_1X_1I(X_1 > l_1) + v_2X_2I(X_2 > l_2) + \sum_{i=3}^{n} w_iX_iI(X_i > l_i)\right)\right].$$

Define

$$g_1(x_1, x_2) = \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a), \text{ and}$$
$$g_2(x_1, x_2) = \phi(v_1x_1I(x_1 > l_1) + v_2x_2I(x_2 > l_2) + a).$$

From Lemma 2, for any $x_1 \leq x_2$ and $(w_1, w_2) \overset{\text{m}}{\succeq} (v_1, v_2)$, it is enough to show that

(i) the function

$$\begin{aligned} &g_2(x_1, x_2) - g_1(x_1, x_2) \\ &= \phi(v_1x_1I(x_1 > l_1) + v_2x_2I(x_2 > l_2) + a) - \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a) \end{aligned}$$

is increasing in $x_2 \geq x_1$ for any x_1, and

(ii) the function

$$\begin{aligned} &g_2(x_1, x_2) + g_2(x_2, x_1) \\ &= \phi(v_1x_1I(x_1 > l_1) + v_2x_2I(x_2 > l_2) + a) + \phi(v_1x_2I(x_2 > l_1) + v_2x_1I(x_1 > l_2) + a) \\ &\geq \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a) + \phi(w_1x_2I(x_2 > l_1) + w_2x_1I(x_1 > l_2) + a) \\ &= g_1(x_1, x_2) + g_1(x_2, x_1), \text{ for any } x_1 \leq x_2. \end{aligned}$$

Proof of step (i): Without loss of generality, we assume $l_1 \geq l_2$ and $x_1 \leq x_2$. From the assumption that $(w_1, w_2) \overset{\text{m}}{\succeq} (v_1, v_2)$, $w_1 \geq w_2$ and $v_1 \geq v_2$, we have $w_2 \leq v_2$. Now, denote

$$\Lambda_2(x_2) = (v_1 - w_1)x_1I(x_1 > l_1) + (v_2 - w_2)x_2I(x_2 > l_2). \tag{6}$$

Observe that

$$\Lambda_2(x_2) \geq (v_1 - w_1)x_1I(x_1 > l_1) + (v_2 - w_2)x_1I(x_1 > l_1) = [(v_2 + v_2) - (w_1 + w_2)]x_1I(x_1 > l_1) = 0,$$

where the inequality follows from the fact that $xI(x > l)$ is an increasing function in x and $x_2I(x_2 > l_2) \geq x_1I(x_1 > l_1)$ in accordance with $l_1 \geq l_2$. Therefore, $\Lambda_2(x_2)$ is non-negative and increasing in x_2, for $w_1 \geq w_2$ and any x_1. Besides, for any $x_1 \leq y_1 \leq x_2$, the convexity of ϕ implies that

$$\begin{aligned} &\phi(w_1x_1I(x_1 > l_1) + w_2y_1I(y_1 > l_1) + \Lambda_2(y_1) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2y_1I(y_1 > l_1) + a) \\ &\leq \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + \Lambda_2(y_1) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a). \end{aligned} \tag{7}$$

Besides, from the that $\Lambda_2(y_1) \leq \Lambda_2(x_2)$ and the increasing property of ϕ, it follows that

$$\begin{aligned} &\phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + \Lambda_2(y_1) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_1 > l_2) + a) \\ &\leq \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + \Lambda_2(x_2) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a). \end{aligned} \tag{8}$$

Combining (7) with (8), we have

$$\begin{aligned} &\phi(w_1x_1I(x_1 > l_1) + w_2y_1I(y_1 > l_1) + \Lambda_2(y_1) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2y_1I(y_1 > l_1) + a) \\ &\leq \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + \Lambda_2(x_2) + a) \\ &\quad - \phi(w_1x_1I(x_1 > l_1) + w_2x_2I(x_2 > l_2) + a). \end{aligned} \tag{9}$$

By the definition of $\Lambda_2(\cdot)$, (9) can be rewritten as

$$\begin{aligned}&\phi(v_1x_1I(x_1>l_1)+v_2y_1I(y_1>l_1)+a)\\&\quad-\phi(w_1x_1I(x_1>l_1)+w_2y_1I(y_1>l_1)+a)\\&\leq\phi(v_1x_1I(x_1>l_1)+v_2x_2I(x_2>l_2)+a)\\&\quad-\phi(w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)+a),\end{aligned}$$

which implies $g_2(x_1,x_2)-g_1(x_1,x_2)$ is increasing in $x_2\geq x_1$, for any x_1.

Proof of step (ii): Based on the assumptions, we have

$$\begin{aligned}&v_1x_2I(x_2>l_2)+v_2x_1I(x_1>l_1)-[w_2x_2I(x_2>l_2)+w_1x_1I(x_1>l_1)]\\&=(v_1-w_2)x_2I(x_2>l_2)+(v_2-w_1)x_1I(x_1>l_1)\\&\geq(v_1-w_2)x_2I(x_1>l_1)+(v_2-w_1)x_1I(x_1>l_1)\\&=(v_1+v_2-w_1-w_2)x_2I(x_1>l_1)=0\end{aligned}\tag{10}$$

and

$$\begin{aligned}&v_1x_2I(x_2>l_2)+v_2x_1I(x_1>l_1)-[w_1x_2I(x_2>l_2)+w_2x_1I(x_1>l_1)]\\&\quad-\Big\{w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)-[v_2x_2I(x_2>l_2)+v_1x_1I(x_1>l_1)]\Big\}\\&=v_1[x_1I(x_1>l_1)+x_2I(x_2>l_2)]+v_2[x_1I(x_1>l_1)+x_2I(x_2>l_2)]\\&\quad-\Big\{w_1[x_1I(x_1>l_1)+x_2I(x_2>l_2)]+w_2[x_1I(x_1>l_1)+x_2I(x_2>l_2)]\Big\}\\&=(v_1+v_2-w_1-w_2)[x_1I(x_1>l_1)+x_2I(x_2>l_2)]=0.\end{aligned}\tag{11}$$

According to (10) and (11), we further have

$$\begin{aligned}&\phi(v_1x_1I(x_1>l_1)+v_2x_2I(x_2>l_2)+a)+\phi(v_1x_2I(x_2>l_1)+v_2x_1I(x_1>l_2)+a)\\&\geq\phi(w_1x_1I(x_1>l_1)+w_2x_2I(x_2>l_2)+a)+\phi(w_1x_2I(x_2>l_1)+w_2x_1I(x_1>l_2)+a).\end{aligned}$$

Then, it holds that $g_2(x_1,x_2)+g_2(x_2,x_1)\geq g_1(x_1,x_2)+g_1(x_2,x_1)$, for any $x_1\leq x_2$. In light of Lemma 2.5, we obtain

$$\begin{aligned}&w_1X_1I(X_1>l_1)+w_2X_2I(X_2>l_2)+a\\&\leq_{\text{icx}}v_1X_1I(X_1>l_1)+v_2X_2I(X_2>l_2)+a.\end{aligned}$$

Therefore, by using the double expectation formula, the desired result follows immediately. ■

Since $(w,0,\dots,0)\overset{\text{m}}{\succeq}(w_1,w_2,\dots,w_n)$ under the space $\mathcal{D}_+^n$, the worst allocation policy can be obtained from Theorem 3.2. This is summarized as the following proposition.

Proposition 3.3: *Under the setup of Theorem 3.2, the worst allocation policy for the risk-preferent investor is $\tilde{\boldsymbol{w}}=(w,0,\dots,0)$.*

According to Theorem 3.1, the optimal allocation policy belongs to $\mathcal{I}_+^n$. In other words, if $\boldsymbol{w}^*$ is the optimal asset allocation policy then it must hold that $w_1^*\leq w_2^*\leq\cdots\leq w_n^*$.

The following theorem characterizes the effect of the dispersiveness among the allocations taken from $\mathcal{I}_+^n$ on the aggregate stochastic return.

Theorem 3.4: *Suppose that $\boldsymbol{X}$ is RWSAI and $\boldsymbol{l}\in\mathcal{D}_+^n$. Then, for $\boldsymbol{w},\boldsymbol{v}\in\mathcal{I}_+^n$,*

$$\boldsymbol{w}\overset{\text{m}}{\succeq}\boldsymbol{v}\Rightarrow\sum_{i=1}^n w_iX_iI(X_i>l_i)\geq_{\text{icx}}\sum_{i=1}^n v_iX_iI(X_i>l_i).$$

Proof: Using the same technique as in the proof of Theorem 3.2, the desired result can be proved similarly, which is thus omitted here for brevity. ■

For a risk-preferent investor with the initial wealth w for n risky assets having RWSAI stochastic returns and decreasing threshold values, Theorem 3.2 states that more heterogeneity among the allocations in the inadmissible set $\mathcal{D}_+^n$ results in smaller stochastic returns, while Theorem 3.4 suggests that more heterogeneity among the allocations in the admissible set $\mathcal{I}_+^n$ leads to larger stochastic returns.

The next proposition follows immediately from Theorem 3.4 by using the fact that $(0, 0, \ldots, w) \overset{\text{m}}{\succeq} (w_1, w_2, \ldots, w_n)$.

Proposition 3.5: *Under the setup of Theorem* 3.4*, the optimal asset allocation policy for the risk-preferent investor is* $\boldsymbol{w}^* = (0, 0, \ldots, w)$.

4. Numerical examples

In this section, we illustrate the main theoretical results developed in the previous section by presenting some numerical examples.

Example 4.1: Consider the multivariate Clayton copula with the generator $\psi(t) = (\theta t + 1)^{-1/\theta}$(which is the log-convex if $\theta \geq 0$) and $\bar{F}_1(x) = \exp(-2x), \bar{F}_2(x) = \exp(-0.1x), \theta = 0.8, \boldsymbol{l} = (7, 5)$. It is easy to examine that (X_1, X_2) is RWSAI by Theorem 5.7 of Cai and Wei (2014). It can be checked that all conditions of Theorem 3.1 are satisfied. To illustrate the increasing convex order of Theorem 3.1, let

$$\eta(\boldsymbol{w}) = \int_{\alpha}^{1} F_{\boldsymbol{X},\boldsymbol{w}}^{-1}(x)\mathrm{d}x, \text{ where } \alpha \in [0, 1].$$

Taking $\boldsymbol{w}_1 = (1, 1)$, $\boldsymbol{w}_2 = (1.5, 0.5)$, $\boldsymbol{w}_3 = (1.6, 0.4)$, $\boldsymbol{w}_4 = (1.9, 0.1)$, and $\boldsymbol{\alpha} = (0, 0.01, 0.02, \ldots, 0.99, 1)$ and $\boldsymbol{w}_1 \overset{\text{m}}{\preceq} \boldsymbol{w}_2 \overset{\text{m}}{\preceq} \boldsymbol{w}_3 \overset{\text{m}}{\preceq} \boldsymbol{w}_4$, Figure 1 plots the curves of $\eta(\boldsymbol{w}_i), i = 1, 2, 3, 4.$, from which we can see that Theorem 3.1 is holding. Therefore, the effectiveness of Theorem 3.1 is confirmed.

It is natural to ask whether the condition $l \in \mathcal{D}_+^n$ in Theorem 3.2 could be dropped. Unfortunately, the following example gives a negative answer.

Example 4.2: Consider the multivariate Clayton copula described by the generator $\psi(t) = (\theta t + 1)^{-1/\theta}$, which is log-convex for $\theta \geq 0$. Set $\bar{F}_1(x) = \exp(-\lambda_1 x), \bar{F}_2(x) = \exp(-\lambda_2 x)$, and $\lambda_1 = 0.6, \lambda_2 = 0.5$ and $\theta = 1$. Then, we can check that (X_1, X_2) is RWSAI by Theorem 5.7 of Cai and Wei (2014). Let $\boldsymbol{l} = (8, 6) \in \mathcal{D}_+^2$ and $\phi(x) = x^\beta$ for $\beta \geq 1$, which is increasing and convex in $x \in \mathbb{R}_+$. It can be checked that all conditions of Theorem 3.1 are satisfied. Note that

$$\begin{aligned} g(\boldsymbol{w}) &:= \mathbb{E}[\phi(w_1 I(X_1 > l_1) + w_2 I(X_2 > l_2))] \\ &= \mathbb{E}[(w_1 I(X_1 > l_1) + w_2 I(X_2 > l_2))^\beta]. \end{aligned}$$

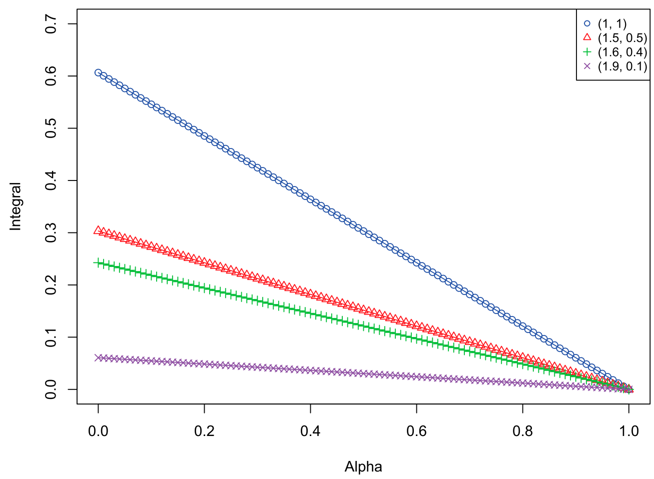

Figure 1. The curves of $\eta(\boldsymbol{w}_i), i = 1, 2, 3, 4$.

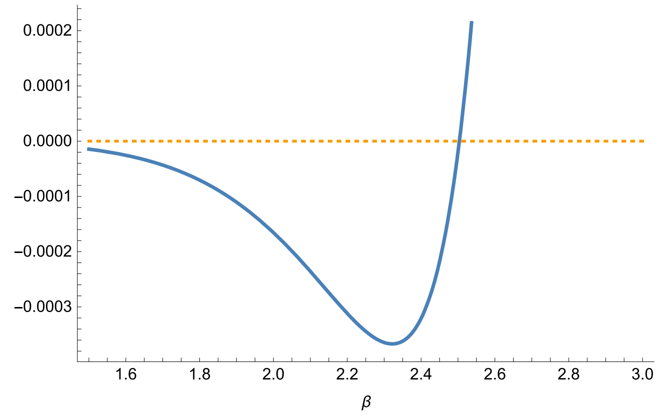

Figure 2. Curves of $g(\boldsymbol{w}_1) - g(\boldsymbol{w}_2)$, for $\beta \in [1.5, 3]$, in Example 4.2.

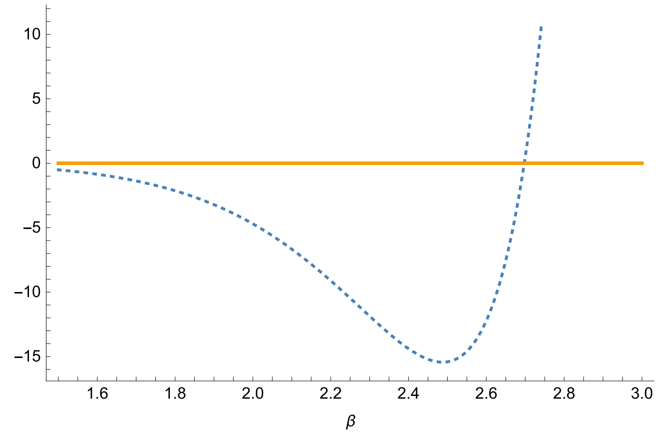

Figure 3. Curves of $g(\boldsymbol{l}_1) - g(\boldsymbol{l}_2)$, for $\beta \in [1.5, 3]$, in Example 4.3.

By applying Corollary 1.6.12 of Denuit et al. (2006), it can be calculated that

$$g(\boldsymbol{w}) = w_1 w_2 \beta(\beta - 1) \int_{l_2}^{\infty} \int_{l_1}^{\infty} (\mathrm{e}^{\lambda_1 x} + \mathrm{e}^{\lambda_2 y} - 1)^{-1} (w_1 x + w_2 y) \mathrm{d}x \mathrm{d}y.$$

Setting $\boldsymbol{l} = (0.1, 20) \notin \mathcal{D}_+^n$ and $(2.3, 2.7) = \boldsymbol{w}_1 \overset{\mathrm{m}}{\preceq} \boldsymbol{w}_2 = (2.2, 2.8)$, from Figure 2 we can see that the curve of $g(\boldsymbol{w}_1) - g(\boldsymbol{w}_2)$ is crossing at line $y = 0$, which implies that the desired Theorem 3.2 is not valid. Therefore, the condition $l \in \mathcal{D}_+^n$ in Theorem 3.2 cannot be relaxed.

Based on Theorem 3.2, one may ask whether the best asset allocation can be obtained under the majorization order of threshold $l \in \mathcal{D}_+^n$. However, the following example provides a negative answer.

Example 4.3: Under the same setup of Example 4.2, take $\boldsymbol{w} = (15, 5) \in \mathcal{D}_+^n$ and $\boldsymbol{l}_1 = (0.8, 0.6), \boldsymbol{l}_2 = (0.9, 0.5)$. Let

$$g(\boldsymbol{l}) = w_1 w_2 \beta(\beta - 1) \int_{l_2}^{\infty} \int_{l_1}^{\infty} (\mathrm{e}^{\lambda_1 x} + \mathrm{e}^{\lambda_2 y} - 1)^{-1} (w_1 x + w_2 y) \mathrm{d}x \mathrm{d}y.$$

Observing that $\boldsymbol{l}_1 \overset{\mathrm{m}}{\preceq} \boldsymbol{l}_2$, unfortunately, it can be seen from Figure 3 that the plot of $g(\boldsymbol{l}_1) - g(\boldsymbol{l}_2)$ is crossing at line $y = 0$, which means that the best asset allocation can be obtained under the majorization order of threshold $l \in \mathcal{D}_+^n$.

5. Conclusion

In the last section, we provide for the theoretical contributions, the potential managerial implications and the future interesting topics of this manuscript.

5.1. Theoretical contributions

In most practical scenarios of insurance engineering and actual science, the investors are always assumed to the risk-averse. However, as discussed in Chen (2003), there are also risk-preference investors in practice market. Therefore, this paper further exploits the optimal asset allocation problems (1) in the context of the stochastic returns under a threshold model. For risk-preference investors, the optimal and the worst allocation policies are given when assets with stochastic returns are right tail weakly stochastic arrangement increasing, respectively.

5.2. Potential managerial insights

For a risk-preference investor, this manuscript analyzes the effect of the different asset allocation policies on the aggregate stochastic return when assets with stochastic returns are right tail weakly stochastic arrangement increasing. The established new methods might provide for very significant managerial implications and decision support for the asset allocation engineers as follows.

(1) The optimal asset allocation policy is $(0, \dots, 0, w)$ when assets with stochastic returns are left tail weakly stochastic arrangement increasing for a risk-preference investor, that is, the optimal allocation policy is to put all the initial wealth on the n-th asset.
(2) The worst asset allocation policy is $(w, 0, \dots, 0)$ when assets with stochastic returns are left tail weakly stochastic arrangement increasing for a risk-preference investor, that is, the optimal allocation policy is to impose all the capital on the first asset.

5.3. Future topics

A potential primary constraint within the existing findings could be attributed to the absence of an asset allocation analysis of the aggregate stochastic return in instances, where there exists statistical interdependence between $\boldsymbol{X}$ and $\boldsymbol{I}$, which presents a more intriguing asset allocation quandary. Nevertheless, due to the intricate nature associated with formulating models for statistically dependent aggregate stochastic returns, these captivating inquiries persist as unresolved, warranting an extended discourse.

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