Statistical Theory and Related Fields

ISSN: 2475-4269 (Print) 2475-4277 (Online) Journal homepage: www.tandfonline.com/journals/tstf20

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To cite this article: Vikas Kumar Sharma, Komal Shekhawat & Princy Kaushik (08 Sep 2025): Lehmann-type family of location-scale *t* distributions with two degrees of freedom, Statistical Theory and Related Fields, DOI: 10.1080/24754269.2025.2540243

To link to this article: https://doi.org/10.1080/24754269.2025.2540243

Published online: 08 Sep 2025.

STATISTICAL THEORY AND RELATED FIELDS
https://doi.org/10.1080/24754269.2025.2540243

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Lehmann-type family of location-scale *t* distributions with two degrees of freedom

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ABSTRACT

This article introduces a three-parameter Lehman-type *t* distribution having 2 degrees of freedom, that is capable of fitting positive and negative skewed data sets. It is shown that the density and hazard functions of the proposed distribution are uni-model. Ordinary moments, entropy measure, ordering, identifiability and order statistics are investigated. Since the quantile function is explicitly defined, quantile-based statistics are also discussed for the proposed distribution. These properties include measures of skewness and kurtosis, L-moments, quantile density and hazard functions, mean residual life function and Parzen's score function. Mechanisms of maximum likelihood, bias correction and matching of percentiles are employed for estimating the unknown parameters of the distribution. Simulation experiments are conducted to compare the performance of these three estimation methods. A real-life data set consisting of strength of glass fibres is fitted to show the adequacy of the proposed distribution over some extensions of the normal and *t* distributions. Parametric regression model is developed along with its parameter estimation using the maximum likelihood approach. Simulation study for the regression model is also presented that endorsed the asymptotic properties of the estimators.

ARTICLE HISTORY
Received 14 June 2023
Accepted 11 July 2025

KEYWORDS
Lehmann-type family; *t* distribution; parametric regression model; quantile modelling; quantile hazard rate; identifiability; moments; entropy; order statistics; stochastic ordering; maximum likelihood estimation; bias correction; percentile estimation

1. Introduction

Student's *t* distribution is a well-known sampling distribution and being widely used to answer various inferential problems. It was pioneered by William Sealy Gosset under the pseudonym 'Student' in Student (1908a, 1908b). Student *t*-distribution is the only distribution among the classical distributions that resembles the Normal distribution in shapes and properties.

Heavy-tailed skewed distributions are of more interest as compared to the symmetric distributions in practice. Interest of researchers has shifted to the skewed distributions with higher degree of kurtosis that are widely applicable in various applied disciplines. Therefore, various skewed families of the normal and *t* distributions have been derived in past by many researchers. de Souza and Tsallis (1997) optimized the generalized entropy with certain constraints yielding student's *t* distribution and *r* distribution.

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Papastathopoulos and Tawn (2013) gave an extension of student's t distribution with negative degrees of freedom and presented its applications for pharmaceutical data. Massing and Ramos (2021) introduced an implementation of mixture of student's t distributions with 3 degrees of freedom on daily and hourly log returns data. Fergusson and Platen (2006) identified that the log-return distribution of a world stock index denominated in different currencies can be characterized by the generalized hyperbolic version of the t distribution with 4 degrees of freedom. Rosco et al. (2011) used sinh–arcsinh transformation to generate skew t distribution. Likelihood inferences are applied for the distribution to analyse the heavy-tailed data of the strength of the fibre glasses. Aas and Haff (2006) introduced generalized hyperbolic skew t distribution with property that one tail has polynomial and the other has exponential behaviour and they explored the applicability of the distribution in modelling financial data. Several skewed families have also been obtained for the normal distribution. See Ma and Genton (2004).

Undoubtedly, the t distribution has phenomenal practical utility. But it has some limitations to use in practice due to its intractable distribution function that leads to the implicit forms of some of its properties. Therefore, many authors have considered t distribution with 2 degrees of freedom (for short t_2) that could show some nice properties in closed form which attracted various applications. It was first reported by Hill (1970) in which author provided a mathematical basis which can be used to handle detailed property of processing systems. Jones (2002) referred this distribution as student's simplest distribution and he explored its various properties and advantages.

Simple characterization of t_2 distribution is discussed by Nevzorov et al. (2003). Regressional properties of order statistics are obtained for the t_2 distribution in Nevzorov (2005). Akhundov et al. (2004) introduced families of distributions which are characterized by the regressional properties of order statistics using the t_2 distribution. Yanev and Ahsanullah (2012) extended the characterization results to the t distribution with more than 2 degrees of freedom. Bai et al. (2014) performed inverse Laplace transformation to obtain an accurate t_2 distribution. For fitting the heavy tailed data, Azzalini (1985) introduced skewed t_2 distribution. Distributional properties and inferences of skewed t distribution are explained by Ahsanullah and Nevzorov (2017). Akhundov and Nevzorov (2012) characterized student's t_3 distribution using the t_2 distribution, and simple regressional properties of order statistics are derived.

The cumulative density function (cdf) of the t_2 distribution is given by

$$F(t) = \frac{1}{2}\left\{1 + \frac{t}{\sqrt{\left(2+t^2\right)}}\right\}, \quad t \in \mathbb{R}. \tag{1}$$

The quantile function (qf) of t_2 distributed random variable (rv) is given by

$$Q(u) = \frac{2u-1}{\sqrt{2u\,(1-u)}}, \quad u \in (0,1). \tag{2}$$

It is beyond doubt that introducing the shape parameter to a base-line distribution provides a better fitting of the real data problems. One way is to define the distribution as αth power of the baseline distribution, i.e., $F_X(x) = (F_Y(x))^{\alpha}, \alpha > 0$, where Y is the baseline rv and the parameter α controls the skewness and flatness. These types of distributions

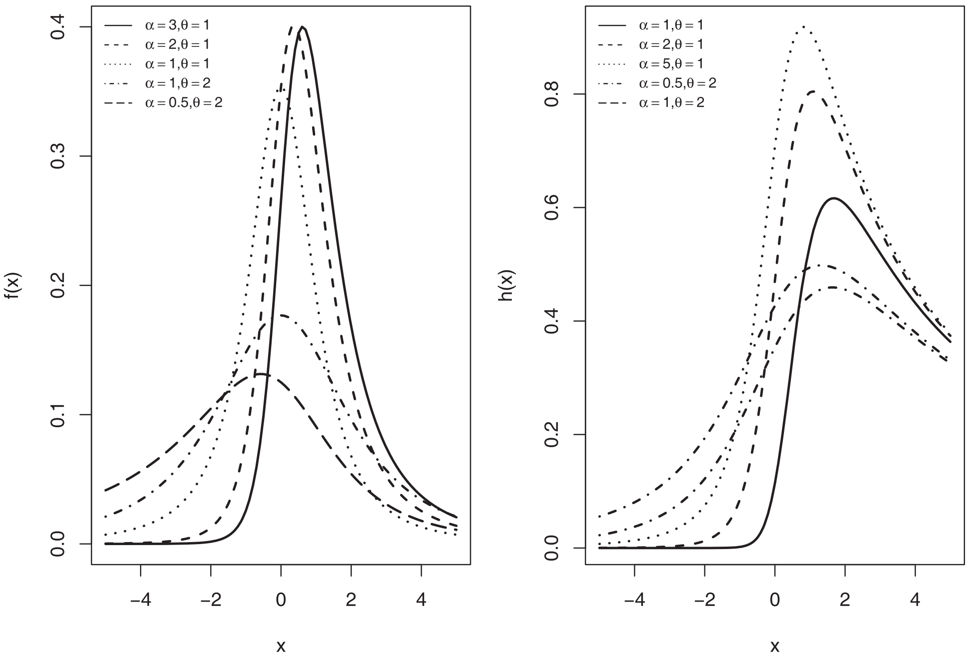

Figure 1. The Lt_2D pdf and hrf plots for different combinations of α and θ when $\mu = 0$.

are known as exponentiated distributions. It was first used by Mudholkar and Srivastava (1993) to propose extended Weibull distribution for fitting bathtub-shaped failure rate data sets. Since then, many authors have contributed towards this field. See Gupta et al. (1998) and Gupta and Kundu (2001). Key idea of such distributions is drawn from Lehmann (1953) in which author suggested to use the distributions of the following forms, $F_X(x) = (F_Y(x))^\alpha$ and $F_X(x) = (1 - F_Y(x))^\alpha$, which are known as Lehmann types I and II, respectively. Sharma et al. (2022) introduced two-parameter exponentiated Teissie*r* distribution using the Lehmann type-I transformation for fitting increasing, decreasing and bathtub hazard functions. The detailed review of the exponentiated distributions can be found in Nadarajah (2006), Nassar and Eissa (2003) and Ristić and Balakrishnan (2012).

From the references cited above, we could observe the following advantages of the Lehman-type distributions. (a) It produces non-monotone (bathtub or upside-down bathtub) hazard rate distribution while the baseline distribution is having monotone hazard rate function. See Sharma et al. (2022). (b) It produces the skewed family of distributions while the baseline distribution is symmetric. See Freeman Modarres (2006). (c) It produces heavy-tailed distributions with varying thickness of tails. See Figure 1 and the references cited above.

Motivated from the above discussion, we, in this article, aim to introduce a new skewed family of t_2 distributions using the Lehman-type I transformation. The proposed distribution would be capable of fitting positive and negative skewed data and upside-down bathtub-shaped hazard rate function (hrf). Besides the useful shapes of the density and hrf, the proposed distribution would have tractable distributional properties such as closed-form expression of hrf and survival function, well-defined mean, median and mode, closed-form qf, etc. Since the qf is mathematically convenient, quantile-based characteristics such as

L-moments, measure of skewness and kurtosis, ordering, quantile density, hrf, and mean residual life function are explicitly defined.

Advantages mentioned above motivate authors to introduce the Lehman-type t_2 distribution ($\mathrm{L}t_2\mathrm{D}$) in statistics literature as a pertinent choice for fitting skewed data over the other skewed families of the normal and t distributions (see Section 9 on real data modelling). The regression model with non-normal response variable is referred to as the parametric regression models that are extensively used in survival data analysis. See Kalbfleisch and Prentice (2002). The $\mathrm{L}t_2\mathrm{D}$ may be an impressive model for modelling regression problems. Linear regression model with the assumption that the response variable follows the $\mathrm{L}t_2\mathrm{D}$ is studied along with maximum likelihood estimation and simulations.

Rest of the paper is structured as follows. In Section 2, we present the genesis of the $\mathrm{L}t_2\mathrm{D}$ and discuss the shapes of the density and hrf. In Section 3, the qf and its related measures are derived. This section also includes the study of the Parzen's score function to investigate quantile hrf and stochastic orderings. Expressions of the ordinary moments and quantiles based L-moments are given in Section 4. In Section 5, order statistics and an entropy measure are explored. Estimation and identifiability of the $\mathrm{L}t_2\mathrm{D}$ are given in Section 6. The $\mathrm{L}t_2\mathrm{D}$ regression model and its parameter estimation are discussed in Section 7. Section 8 consists of the simulation results. The fitting of the strength data is given in Section 9. The last section concludes the entire paper.

2. Lehmann-type t_2 distribution and its shapes

In this section, we consider two-parameter t_2 distribution to introduce its general family of skewed distributions using the Lehmann-type I method as discussed in the earlier section. We also investigate the shapes of the density and hrf. The two-parameter t_2 distribution is defined by the cdf,

$$F(x,\theta,\mu) = \frac{1}{2}\left\{1+\frac{w}{\sqrt{(2+w^2)}}\right\}, \quad w=\frac{x-\mu}{\theta},\ x\in\mathbb{R},\ \theta>0,\ \mu\in\mathbb{R}. \tag{3}$$

If X is an rv that follows t_2 distribution with cdf given by Equation (3), then the rv Y with cdf $F_Y(y) = [F_X(y)]^{\alpha}, \alpha > 0$ is said to follow the $\mathrm{L}t_2\mathrm{D}$. The cdf of the $\mathrm{L}t_2\mathrm{D}$ is given by

$$F(y,\alpha,\mu,\theta) = \frac{1}{2^{\alpha}}\left\{1+\frac{z}{\sqrt{(2+z^2)}}\right\}^{\alpha}, \quad z=\frac{y-\mu}{\theta},\ (y,\mu)\in\mathbb{R},\ (\alpha,\theta)>0. \tag{4}$$

We denote the cdf in (4) by $\mathrm{L}t_2\mathrm{D}(\alpha,\mu,\theta)$. Differentiating the cdf given by (4) with respect to (w.r.t.) y, we obtain probability density function (pdf) as given by

$$f(y,\alpha,\mu,\theta) = \frac{\alpha}{\theta 2^{\alpha-1}\left(2+z^2\right)^{3/2}}\left\{1+\frac{z}{\sqrt{(2+z^2)}}\right\}^{\alpha-1},$$

$$z=\frac{y-\mu}{\theta},\ (y,\mu)\in\mathbb{R},\ (\alpha,\theta)>0. \tag{5}$$

The hrf of the Lt$_2$D is given by

$$h(y, \alpha, \theta, \mu) = \frac{\alpha}{2^{\alpha-1}\theta\left(2+z^2\right)^{3/2}}\left\{1+\frac{z}{\sqrt{(2+z^2)}}\right\}^{\alpha-1} \times\left(1-\frac{1}{2^{\alpha}}\left\{1+\frac{z}{\sqrt{(2+z^2)}}\right\}^{\alpha}\right)^{-1}. \quad (6)$$

Figure 1 shows the various shapes of the pdf and hrf for various parameter values. It is clear from the graph that the skewness and flatness of the Lt$_2$D mainly depend on the shape parameter α and the scale parameter θ. The pdf given in (5) is unimodal with varying degrees of positive and negative skewness and flatness. Figure 1 also reveals that the hrf is upside-down bathtub (uni-modal) shaped for all choices of the parameters. Though the hrf is not analytically tractable, one may use an alternative approach investigated by Glaser (1980) to explore the shapes of the hrf using the quantity $\eta(y) = -\frac{f'(y)}{f(y)}$. We can note here that $\eta(y)$ and $h(y)$ share the same shapes as stated by Glaser (1980).

Proposition 2.1: *The hrf of the Lt$_2$D$(\alpha, 0, 1)$ is an upside down bathtub shaped for all the values of the shape parameter α.*

Proof: For the given pdf (5) of the Lt$_2$D$(\alpha, 0, 1)$, we have

$$\eta(y;\alpha) = -\frac{f'(y;\alpha)}{f(y;\alpha)} = \frac{(\alpha+2)y-(\alpha-1)\sqrt{y^2+2}}{y^2+2}$$

with $\lim_{y\to-\infty}\eta(y;\alpha) = \lim_{y\to-\infty}\left(\frac{2\alpha+1}{y}\right) = 0$ and $\lim_{y\to+\infty}\eta(y;\alpha) = \lim_{y\to+\infty}\left(\frac{3}{y}\right) = 0$. Now, we need to show that there exists a point y_0 such that $\eta(y)$ achieves its maximum at y_0 with $\eta'(y_0) = 0$. We obtain the derivative as follows:

$$\eta'(y;\alpha) = \frac{(\alpha-1)y^3-(\alpha+2)y^2+2\sqrt{y^2+2}(\alpha+2)\sqrt{y^2+2}+2(\alpha-1)y}{\left(y^2+2\right)^{5/2}} = \frac{s(y;\alpha)}{\left(y^2+2\right)^{5/2}}.$$

To obtain the critical points, we need to find the roots of the equation,

$$s(y;\alpha) = y^6(c-1)+y^4(4c+2)+y^2(4c+4)-8 = 0, \quad (7)$$

where $c = \frac{(\alpha-1)^2}{(2+\alpha)^2}$. After some algebraic manipulations, we get the roots of the equation in (7) which are $-\sqrt{\frac{c+\sqrt{c(c+8)}+2}{1-c}}$, $\sqrt{\frac{c+\sqrt{c(c+8)}+2}{1-c}}$, $-\sqrt{\frac{-c+\sqrt{c(c+8)}-2}{c-1}}$, $\sqrt{\frac{-c+\sqrt{c(c+8)}-2}{c-1}}$, $\mathrm{i}\sqrt{2}$ and $\mathrm{i}\sqrt{2}$.

We note here that $\sqrt{c(c+8)} > 2+c$ and $1-c > 0$ and the root that is feasible is

$$y_0 = \sqrt{\frac{c+\sqrt{c(c+8)}+2}{1-c}} = \left[\frac{\alpha^2+2\alpha+3+(\alpha-1)\sqrt{3\alpha^2+10\alpha+11}}{2\alpha+1}\right]^{\frac{\alpha}{2}}.$$

Since $\eta'(y) > 0$ in $(0, y_0)$, $\eta'(y) < 0$ in (y_0, ∞) and $\eta'(y_0) = 0$, we can conclude that the Glaser's quantity is upside-down bathtub shaped. Using the Glaser's lemma, we can state that the hrf is also upside-down bathtub shaped for all values $\alpha > 0$. ■

Proposition 2.2: *Let Y be the Lt$_2$D distributed rv with pdf given by* (5). *The mode of the distribution is given by*

$$Y_{\text{mode}} = \mu + \frac{\sqrt{6}\sqrt{(\alpha-1)^2(2\alpha+1)\theta^2}}{6\alpha+3}.$$

Proof: Proof is straightforward and can be achieved using the derivative test for the log-pdf,

$$\log(f(y)) = \log(\alpha) - \log(\theta) - (\alpha-1)\log(2) - \frac{3}{2}\log(2+z^2)$$
$$+ (\alpha-1)\log\left(1+\frac{z}{\sqrt{2+z^2}}\right),$$

where $z = (y-\mu)/\theta$. For $\alpha = 1$, $Y_{\text{mode}} = \mu$, which is the mode of the t_2 distribution. ■

3. Quantile function and related measures

In this section, we derive qf of the Lt$_2$D and associated measures such as median, inter quartile range, coefficients of skewness and kurtosis, quantile density, quantile hazard and mean residual life function. Since the qf is available in closed from, the above properties can be explicitly defined. The qf is not only helping us to describe the distribution but also provides a basis to the robust estimation. The qf is defined by the following proposition.

Proposition 3.1: *The quantile of the Lt$_2$D with cdf* (4) *is given by*

$$Q(u) = \mu + \frac{\theta\left(2u^{\frac{1}{\alpha}}-1\right)}{\sqrt{2u^{\frac{1}{\alpha}}\left(1-u^{\frac{1}{\alpha}}\right)}}, \quad u \in (0,1),\ \alpha > 0,\ \theta > 0,\ \mu \in \mathbb{R}. \tag{8}$$

Proof: The quantile function is defined by $Q(u) = \min\{y; F(y) \geq u\}, 0 < u < 1$ and can be obtained by solving the probability integral transformation,

$$F_Y(Q(u)) = u, u \in (0,1). \tag{9}$$

Substituting the cdf given in (4) in (9), we have

$$\frac{\left(\frac{Q(u)-\mu}{\theta}\right)^2}{2+\left(\frac{Q(u)-\mu}{\theta}\right)^2} = (2u^{\frac{1}{\alpha}}-1)^2,$$

which in turn implies the result as stated in Proposition 3.1. ■

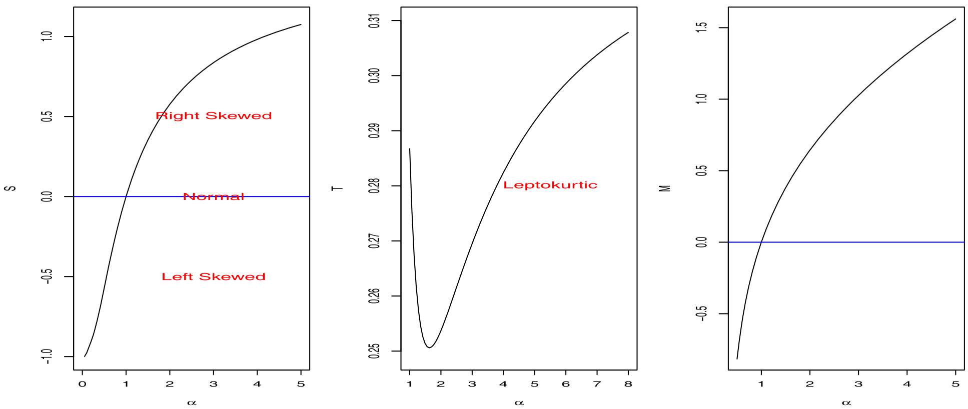

Figure 2. Plots of the coefficients of skewness, kurtosis and median of the Lt_2D with varying shape parameter α with $\mu = 0$ and $\theta = 1$.

The support of the distribution given in (5) is $(Q(0), Q(1)) = (-\infty, \infty)$. Using the quantiles, the median (M) and inter-quantile range (IQR) are respectively given by

$$\mathrm{M} = Q\left(\frac{1}{2}\right) = \frac{2^{-\frac{\alpha+2}{2\alpha}}\left(2^{1/\alpha}-2\right)\theta}{\sqrt{4^{-1/\alpha}\left(2^{1/\alpha}-1\right)}} + \mu,$$

$$\mathrm{IQR} = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) = \frac{\theta}{\sqrt{2}}\left(\frac{2^{\frac{\alpha-2}{\alpha}}3^{1/\alpha}-1}{\sqrt{\left(\frac{9}{16}\right)^{1/\alpha}\left(\left(\frac{4}{3}\right)^{1/\alpha}-1\right)}} - \frac{2^{\frac{\alpha-2}{\alpha}}-1}{\sqrt{2^{-4/\alpha}\left(4^{1/\alpha}-1\right)}}\right).$$

The coefficients of skewness (known as Galton's coefficient, denoted by S) and kurtosis (known as Moor's coefficient, denoted by T) are given by

$$\mathrm{S} = \frac{Q_{0.25} + Q_{0.75} - 2Q_{0.5}}{Q_{0.75} - Q_{0.25}},$$

$$\mathrm{T} = \frac{Q_{0.875} - Q_{0.625} + Q_{0.375} - Q_{0.125}}{Q_{0.75} - Q_{0.25}} - 1.23.$$

In Figure 2, we sketch the curves of S, T and M for increasing sequence of the shape parameter α while $\mu = 0$ and $\theta = 1$ are kept as fixed. From this figure, we can see that the Lt_2D has symmetric ($\mathrm{S} = 0$), left skewed ($\mathrm{S} < 0$), right skewed ($\mathrm{S} > 0$) and leptokurtic ($\mathrm{T} > 0$) shapes of density function. We can state here that the Lt_2D accommodates skewed shapes with higher degree of the peakedness.

We now define two another important functions associated with the qf that are quantile density function (qdf) and mean residual quantile function (mrqf). The qdf is given by

$$q(u) = \frac{\theta u^{\frac{1}{\alpha}-1}}{2\sqrt{2}\alpha\left(-u^{1/\alpha}\left(u^{1/\alpha}-1\right)\right)^{3/2}}, \quad u \in (0,1),\ \alpha > 0,\ \theta > 0,\ \mu \in \mathbb{R}. \qquad (10)$$

Proposition 3.2: *The qdf given in* (10) *is*

(1) *increasing in u if* $u > u_0(\alpha), \forall\, \alpha > 0$;
(2) *decreasing in u if* $u < u_0(\alpha), \forall\, \alpha > 0$,

where $u_0(\alpha) = \left(\frac{2\alpha+1}{4\alpha+2}\right)^{1/\alpha}$.

Proof: The derivative of qdf given in (10) w.r.t. u is

$$q'(u) = \frac{\theta z(\alpha, u)}{g(\alpha, u)}, \tag{11}$$

where

$$z(\alpha, u) = \left(-2\alpha + 2(\alpha+2)u^{1/\alpha} - 1\right),$$
$$g(\alpha, \theta, u) = 4\sqrt{2}\alpha^2 u^2 \left(u^{1/\alpha} - 1\right)^2 \sqrt{-u^{1/\alpha}\left(u^{1/\alpha} - 1\right)}.$$

Since $g(\alpha, u) > 0\, \forall\, \alpha > 0, u \in (0,1), q'(u) = 0$, then $-2\alpha + 2(\alpha+2)u^{1/\alpha} = 1$. That results in the critical point $u_0(\alpha) = \left(\frac{2\alpha+1}{4\alpha+2}\right)^{1/\alpha}$. As $z(\alpha, 0) = -(2\alpha+1) < 0$, $z(\alpha, u_0(\alpha)) = 0$ and $z(\alpha, 1) = 3\ \forall\ \alpha > 0$, $q'(u) > 0$ for $u \in (u_0(\alpha), 1)$ and $q'(u) < 0$ for $u \in (0, u_0(\alpha))$ with $q'(u_0(\alpha)) = 0$. Therefore, the qdf in (10) is increasing (decreasing) in u if $u > (<) \left(\frac{2\alpha+1}{4\alpha+2}\right)^{1/\alpha}$ respectively. Hence, Proposition 3.2 is proved. ■

The qdf may be used for deriving the pdf using the relation, $f(Q(u)) = q(u)^{-1}$. We also get the pdf plot, as given in Figure 1, by plotting the curve between $Q(u)$ and $q(u)^{-1}$. Since this gives the similar shapes as those of Figure 1, we removed it.

The mrqf, $M(u)$, defined as the mean remaining life of individual beyond $100(1-u)\%$ of the distribution, is given by

$$M(u) = \frac{1}{\sqrt{2}(4\alpha^2-1)(1-u)}\left(\frac{4\sqrt{\pi}\theta\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha-1)} - 4\alpha\theta(\alpha-1)\sqrt{u^{1/\alpha+2}\left(1-u^{1/\alpha}\right)}\right.$$
$$\times {}_2F_1\left(1, \alpha+1; \alpha+\frac{3}{2}; u^{1/\alpha}\right)$$
$$\left. + \frac{\theta(2\alpha+1)(2\alpha+2u^{1/\alpha}(-2\alpha+(\alpha-1)u+1)+u-1)}{\sqrt{u^{1/\alpha}\left(1-u^{1/\alpha}\right)}}\right),$$

where ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a)_k (b)_k}{(c)_k z^k}$ is a hypergeometric function.

3.1. Parzen's score function and stochastic orderings

The hazard quantile function (hqf) is given by

$$H(u) = \frac{1}{(1-u)q(u)} = \frac{2\sqrt{2}\alpha u\left(u^{1/\alpha} - 1\right)\sqrt{-u^{1/\alpha}\left(u^{1/\alpha} - 1\right)}}{\theta(u-1)}.$$

Analytical proof of its shapes is not possible at this time. Parzen (1979) has introduced a score function that may be used to identify the shape of the hqf and to establish stochastic orderings. The score function is equivalent to Glaser's lemma given in Glaser (1980). Parzen's score function for the Lt_2D is given by

$$J(u) = \frac{q'(u)}{q^2(u)} = \frac{\sqrt{2u^{1/\alpha}\left(1-u^{1/\alpha}\right)}\left(2u^{1/\alpha}(2+\alpha)-2\alpha-1\right)}{\theta}. \tag{12}$$

Nair et al. (2012) gave the relationship between the hqf and score function as $(1-u)H'(u) = H(u) - J(u)$ that can be used to define the shape of the hqf. Using this relation, we can state that if $H(u)$ is increasing (decreasing) as $H(u) \geq (\leq) J(u)$ for all $u \in (0,1)$ and the change points of non-monotonic $H(u)$ are zeroes of the equation $H(u) - J(u) = 0$.

Proposition 3.3: *The Parzen's score function $J(u)$ given by* (12) *is upside-down bathtub shaped.*

Proof: Note that $\lim_{u\to 0} J(u) = \lim_{u\to 1} J(u) = 0$. We need to prove that there exists a maxima of $J(u)$ for $u \in (0,1)$. We use maxima–minima principle and obtain

$$J'(u) = \frac{u^{\frac{1}{\alpha}-1}\left(-2\alpha+2(5\alpha+7)u^{1/\alpha}-8(\alpha+2)u^{2/\alpha}-1\right)}{\sqrt{2}\alpha\theta\sqrt{-u^{1/\alpha}\left(u^{1/\alpha}-1\right)}}. \tag{13}$$

Equating (13) to 0, we have

$$-2\alpha+2(5\alpha+7)u^{1/\alpha}-8(\alpha+2)u^{2/\alpha}-1=0, \tag{14}$$

as $\frac{u^{\frac{1}{\alpha}-1}}{\sqrt{2}\alpha\theta\sqrt{-u^{1/\alpha}(u^{1/\alpha}-1)}} > 0$. Roots of Equation (14) are

$$u_1 = \left(\frac{-\sqrt{3}\sqrt{3\alpha^2+10\alpha+11}+5\alpha+7}{8(\alpha+2)}\right)^{\alpha},$$
$$u_2 = \left(\frac{\sqrt{3}\sqrt{3\alpha^2+10\alpha+11}+5\alpha+7}{8(\alpha+2)}\right)^{\alpha}.$$

For $\alpha > 0$, root u_2 is feasible and that is the point at which the Parzen's score function has its maxima. Therefore we can state that the Parzen's score function has upside-down bathtub shapes. ■

According to Theorem 2.1 given in Nair et al. (2012), there exists at most one root of $H'(u)$ in the closed interval of the points $0, u_2$ and 1.

Another important application of the score function is to compare quantile distributions in terms of their stochastic order. Let X and Y be two non-negative rvs and $J_Y(u) \geq J_X(u)$. Then X is said to be scholastically greater than Y, i.e., $Y \leq_{\text{st}} X$. The stochastic orders may be defined using some other statistical functions such as quantile, total time on test (TTT) and mrqf. Their implications can be stated as $Y \leq_J X \implies Y \leq_{\text{disp}} X \implies Y \leq_{\text{TTT}} X \implies$

$M_Y(u) \leq M_X(u)$. In classical way, stochastic orders are defined using the cdf, likelihood function, hrf and mean residual life function (mrlf) and their implications are as follows:

$$Y \leq_{\text{likelihood}} X \Longrightarrow Y \underset{\substack{\Downarrow \\ Y \leq_{\text{cdf}} X}}{\leq_{\text{hrf}}} X \Longrightarrow Y \leq_{\text{mrlf}} X.$$

Proposition 3.4: *Let X and Y be two independent random variables following the $Lt_2D(\alpha_1)$ and $Lt_2D(\alpha_2)$, respectively. If $\alpha_1 > \alpha_2$, then $(Y \leq_{\text{likelihood}} X)$, $(Y \leq_{\text{mrlf}} X)$, $(Y \leq_{\text{hrf}} X)$ and $(Y \leq_{\text{cdf}} X)$ for all x.*

Proof: For given $\mu = 0$ and $\theta = 1$, the likelihood ratio for the rvs X and Y is given by

$$\frac{f_X(x,\alpha_1)}{f_Y(x,\alpha_2)} = 2^{\alpha_2-\alpha_1}\frac{\alpha_1}{\alpha_2}\left(1+\frac{x}{\sqrt{x^2+2}}\right)^{\alpha_1-\alpha_2}.$$

Taking logarithmic of likelihood ratio and differentiating it w.r.t. x, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{f_X(x,\alpha_1)}{f_Y(x,\alpha_2)}\right) = (\alpha_1-\alpha_2)\frac{\left(\sqrt{x^2+2}-x\right)}{x^2+2}.$$

As $\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{f_X(x,\alpha_1)}{f_Y(x,\alpha_2)}\right) > 0$ for $\alpha_1 > \alpha_2$, the likelihood ratio is an increasing function of sample point x. Hence, $X \geq_{\text{likelihood}} Y$. That implies $(Y \leq_{\text{mrlf}} X)$, $(Y \leq_{\text{hrf}} X)$ and $(Y \leq_{\text{cdf}} X)$. ■

4. Moments and L-Moments

In this section, we derive the expressions of the ordinary moments obtained as the expectation of the rth $(r = 1, 2, 3, \ldots)$ power of the rv and L-moments as the expectation of linear combination of the order statistics. The moments are employed to explore the characteristics of the distribution that mainly includes mean, variance, skewness and kurtosis. The L-moments are defined using the qf as given by

$$\mathcal{L}_r = \sum_{q=0}^{r-1}(-1)^{r-1-q}\binom{r-1}{q}\binom{r-1+q}{q}\int_0^1 u^q Q(u)\,\mathrm{d}u.$$

The main advantages of L-moments over the conventional moments are that they are robust to sampling variability and outliers, and gave more precise inference results. For more theory on the L-moments, we suggest readers to follow Hosking (1990) and Marshall and Olkin (2007).

Proposition 4.1: *Suppose Y is the Lt_2D rv that is governed by the quantile function given in* (8). *Then the rth L-moment of Y is given by*

$$\mathcal{L}_r = \sum_{k=0}^{r-1}(-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k} \times \left(\frac{\mu}{k+1} + \frac{\sqrt{\frac{\pi}{2}}\alpha\theta(\alpha(k+1)-1)\Gamma\left(\alpha(k+1)-\frac{1}{2}\right)}{\Gamma(\alpha(k+1)+1)}\right) \tag{15}$$

where $\alpha > \frac{1}{2}$.

Proof: Proof is straightforward and removed. ■

Corresponding to $r = 1, 2, 3, 4$, we obtain first four L-moments as given by

$$\mathcal{L}_1 = \frac{\sqrt{\frac{\pi}{2}}\theta\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha - 1)} + \mu,$$

$$\mathcal{L}_2 = \sqrt{\frac{\pi}{2}}\theta\left(\frac{\Gamma\left(2\alpha - \frac{1}{2}\right)}{\Gamma(2\alpha - 1)} - \frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha - 1)}\right),$$

$$\mathcal{L}_3 = \sqrt{\frac{\pi}{2}}\theta\left(\frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha - 1)} + \frac{6\alpha(1 - 2\alpha)\Gamma\left(2\alpha - \frac{1}{2}\right)}{\Gamma(2\alpha + 1)} + \frac{2\Gamma\left(3\alpha - \frac{1}{2}\right)}{\Gamma(3\alpha - 1)}\right),$$

$$\mathcal{L}_4 = \sqrt{\frac{\pi}{2}}\theta\left(-\frac{2\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha)} + \frac{\alpha\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha)} + \frac{6\Gamma\left(2\alpha - \frac{1}{2}\right)}{\Gamma(2\alpha - 1)} + \frac{20\alpha(4\alpha - 1)\Gamma\left(4\alpha - \frac{1}{2}\right)}{\Gamma(4\alpha + 1)} - \frac{10\Gamma\left(3\alpha - \frac{1}{2}\right)}{\Gamma(3\alpha - 1)}\right).$$

The mean of the Lt_2D is $\mathcal{L}_1$ and L-coefficient of variation, $\mathcal{L} - \mathrm{Var}(Y)$, is

$$\mathcal{L} - \mathrm{Var}(Y) = \frac{\Gamma(\alpha - 1)\Gamma(2\alpha - 1/2) - \Gamma(\alpha - 1/2)\Gamma(2\alpha - 1)}{\Gamma(\alpha - 1) - \frac{\mu}{\theta}\sqrt{\frac{2}{\pi}}\Gamma(\alpha - 1)}.$$

We can also obtain the L-coefficient of skewness (τ_3) and L-coefficient of kurtosis (τ_4), respectively, defined by

$$\mathcal{L} - \mathrm{Skew}(Y) = \frac{\mathcal{L}_3}{\mathcal{L}_2} \quad \text{and} \quad \mathcal{L} - \mathrm{Kurt}(Y) = \frac{\mathcal{L}_4}{\mathcal{L}_2}.$$

Proposition 4.2: *Suppose Y is the Lt_2D distributed rv that is governed by the pdf given in* (5). *The rth ordinary moment of Y is given by*

$$E\left[Y^r\right] = \sum_{r=0}^{n}\binom{n}{r}2^{\frac{1}{2}(-2\alpha+r+1)}\alpha\mu^{n-r}\theta^r \times \left(\frac{2^\alpha\Gamma\left(1 - \frac{r}{2}\right){}_2F_1\left(-r, -\alpha; -\frac{r}{2} - \alpha + 1; \frac{1}{2}\right)\Gamma\left(\frac{r}{2} + \alpha\right)}{\Gamma(\alpha + 1)} - \frac{\pi 2^{-\frac{r}{2}}\Gamma(r + 1){}_2F_1\left(\frac{r}{2}, \alpha - \frac{r}{2}; \frac{r}{2} + \alpha + 1; \frac{1}{2}\right)\left(\csc\left(\pi\alpha + \frac{\pi r}{2}\right) + (-1)^r\csc\left(\frac{1}{2}\pi(r - 2\alpha)\right)\right)}{\Gamma\left(\frac{r}{2} - \alpha + 1\right)\Gamma\left(\frac{r}{2} + \alpha + 1\right)}\right), \tag{16}$$

where

$$r \in \begin{cases}(2\alpha, 2), & \text{if } \alpha \in (0, 1),\\ (2, 2\alpha), & \text{if } \alpha \in (1, \infty).\end{cases}$$

Proof: Proof is straightforward and removed. ■

The mean of the Lt_2D is given by

$$E[Y] = \mu + \frac{\sqrt{\frac{\pi}{2}}\theta\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha - 1)}, \quad \alpha > 1,$$

which is the same as the first L-moment.

5. Order statistics and entropy measure

Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ be the ordered sample of size n from the Lt_2D. In this section, we derive the distribution of the rth order statistic $Y_{(r)}, 0 \leq r \leq n$ with its expectation.

Proposition 5.1: *Let Y be the Lt_2D distributed rv with pdf given in* (5). *The pdf of the rth order statistic $Y_{(r)}$ is*

$$\begin{aligned} f_{Y_{(r)}}(y) &= \frac{n!}{(j-1)!(n-j)!} \frac{\alpha}{2^{\alpha-1}\left(2+z^2\right)^{3/2}} \\ &\times \left\{1 + \frac{z}{\sqrt{(2+z^2)}}\right\}^{\alpha-1} \left[\frac{1}{2}\left(1 + \frac{z}{\sqrt{(2+z^2)}}\right)\right]^{\alpha(j-1)} \\ &\times \left[1 - \frac{1}{2^\alpha}\left(1 + \frac{z}{\sqrt{(2+z^2)}}\right)^{\alpha}\right]^{n-j}, \quad z = \frac{y-\mu}{\theta} \in \mathbb{R}, \ \alpha > 0, \ \theta > 0, \end{aligned} \quad (17)$$

and its expectation is

$$\begin{aligned} E[Y_{(r)}] &= \sum_{t=0}^{n-j} \sqrt{\pi}\alpha(-1)^{n-j}2^{\alpha+\alpha(-j)+\alpha(j(-t)+j+nt)-\frac{1}{2}}(\alpha(j(-t)+j+nt)-1) \\ &\times \frac{\Gamma\left((-tj+j+nt)\alpha - \frac{1}{2}\right)}{\Gamma((-tj+j+nt)\alpha+1)}, \quad \alpha(j(-t)+j+nt) > \frac{1}{2}. \end{aligned} \quad (18)$$

Proof: Proof is straightforward and removed. ■

The means of the minimum and maximum order statistics are given by

$$\begin{aligned} E\left[Y_{(1)}\right] &= \sum_{t=0}^{n-1} \frac{\sqrt{\pi}\alpha(-1)^{n-1}2^{\alpha t(n-1)-\frac{1}{2}}(\alpha(n(t-1)+1)-1)}{\Gamma((t(n-1)+1)\alpha+1)} \\ &\times \Gamma\left((-t+1+nt)\alpha - \frac{1}{2}\right) \end{aligned}$$

$$E\left[Y_{(n)}\right] = \frac{\sqrt{\pi}\alpha 2^{\alpha-\frac{1}{2}}(\alpha n-1)\Gamma\left(n\alpha - \frac{1}{2}\right)}{\Gamma(n\alpha+1)},$$

respectively.

Entropy measure is useful in collecting information about the uncertainty of the random experiments. We derive a measure of randomness called Renyi entropy (Rényi, 1961) that generalizes the Shannon entropy. The Renyi entropy for the distribution of the rv Y with pdf $f(y)$ is given by

$$\mathcal{RI}_\gamma = \frac{1}{1-\gamma} \log\left[\int_{-\infty}^{\infty} f^\gamma\,(y)\,\mathrm{d}y\right],$$

where $\gamma > 0$ and $\gamma \neq 1$. The Renyi entropy reduces to the Shannon entropy as γ approaches to 0.

Proposition 5.2: *Let Y be the Lt_2D distributed rv with pdf given in* (5). *The Renyi entropy is given by*

$$\begin{aligned}\mathcal{RI}_\gamma &= 2\log\left(2\alpha\gamma - \frac{13\gamma}{2} - \frac{1}{2}\right) + \log\left(\Gamma\left(\frac{5}{2} - \frac{3\gamma}{2}\right)\right) + \log\left(\Gamma\left(2\alpha\gamma - 5\gamma - 2\right)\right)\\ &\quad - \log\left(\Gamma\left(2\alpha\gamma - \frac{13\gamma}{2} + \frac{1}{2}\right)\right) - \log\left(1-\gamma\right),\\ &\quad (\gamma) < \frac{5}{3} \wedge ((5-2\alpha)\gamma) < -2 \wedge \gamma \neq 1,\end{aligned}$$

where $\Gamma(n) = \int_0^\infty \mathrm{e}^{-y} y^{n-1}\,\mathrm{d}y$ *denotes the gamma function.*

Proof: Proof is straightforward and removed. ■

6. Identifiability and estimation

This section provides insight on how to perform the estimation of Lt_2D parameters using the complete sample of observations. We discuss two different methods of estimation based on the maximum likelihoods and matching the theoretical and empirical percentiles. Both the methods are well known in estimation theory and can be effectively used for practical problems. We also suggest to compute bias-corrected maximum likelihood estimates (BMLEs). Identifiability issue is also discussed that may ensure the uniqueness of the estimates.

6.1. Identifiability

Identifiability is a required condition for a model so that the precise inferences can be made. A distribution is said to be identifiable in parameters if the two members of family are equal, i.e., $f_1(x;\Theta_1) = f_2(x;\Theta_2)$, then $\Theta_1 = \Theta_2$ for all x. Theorem 1 of Basu and Ghosh (1980) for the identifiable distribution states that 'the density ratio of $\frac{f_1(x;\Theta_1)}{f_2(x;\Theta_2)}$ of two distinct members of the family defined on real line either converges to 0 or diverges to ∞', as $x \to -\infty$. For the Lt_2D family, we have

$$\lim_{x\to-\infty} \frac{f_1(x;\alpha_1)}{f_2(x;\alpha_2)} = \begin{cases} 0, & \text{if } \alpha_1 > \alpha_2,\\ \infty, & \text{if } \alpha_1 < \alpha_2,\\ 1, & \text{if } \alpha_1 = \alpha_2.\end{cases}$$

According to Theorem 1 of Basu and Ghosh (1980), the Lt_2D is identifiable in shape parameter, α.

6.2. When μ is known

We investigate the estimation of the parameters α and θ when $\mu = 0$ is given. Suppose $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is a random sample of size n from the Lt_2D defined by the cdf (4). Assuming that the random observations are independently distributed, the log-likelihood function is obtained as

$$\ell(\alpha,\theta \mid \mathbf{y}) = n\ln(\alpha) - n\ln(\theta) - n(\alpha-1)\ln(2)$$
$$-\frac{3}{2}\sum_{i=1}^{n}\ln\left(2+\left(\frac{y_i}{\theta}\right)^2\right) + (\alpha-1)\sum_{i=1}^{n}\ln\left(1+\frac{y_i}{\theta\sqrt{2+\left(\frac{y_i}{\theta}\right)^2}}\right). \quad (19)$$

The log-likelihood equations corresponding to (19) are given by

$$\frac{n}{\alpha} - n\ln(2) + \sum_{i=1}^{n}\ln\delta(y_i) = 0, \quad (20)$$

$$-\frac{n}{\theta} - \frac{3}{2}\sum_{i=1}^{n}\frac{\phi'_\theta(y_i)}{\phi(y_i)} + (\alpha-1)\sum_{i=1}^{n}\frac{\delta'_\theta(y_i)}{\delta(y_i)} = 0, \quad (21)$$

where $\phi(y_i) = \left(2+\left(\frac{y_i}{\theta}\right)^2\right)$, $\phi'_\theta(y_i) = -2\frac{y_i^2}{\theta^3}$, $\delta(y_i) = \left(1+\frac{y_i}{\theta\sqrt{2+\left(\frac{y_i}{\theta}\right)^2}}\right)$ and $\delta'_\theta(y_i) = -\left(\frac{y_i}{\theta}\right)$ $\left\{\frac{1}{\theta\sqrt{\phi(y_i)}} + \frac{\phi'_\theta(y_i)}{2\phi(y_i)^{3/2}}\right\}$.

The maximum likelihood estimates (MLEs) of the parameters α and θ, respectively, denoted by $\hat{\alpha}_m$ and $\hat{\theta}_m$, can be obtained by solving Equations (20) and (21). Once $\hat{\theta}_m$ is obtained, $\hat{\alpha}_m$ can be uniquely determined using Equation (20) as given by

$$\hat{\alpha}_m(\theta) = n\left(n\ln(2) - \sum_{i=1}^{n}\ln\hat{\delta}(y_i)\right)^{-1}, \quad (22)$$

where

$$\hat{\delta}(y_i) = \delta(y_i) = \left(1+\frac{y_i}{\hat{\theta}}\left(2+\left(\frac{y_i}{\hat{\theta}}\right)^2\right)^{-1/2}\right).$$

However, $\hat{\theta}_m$ can be determined as the solution of Equation (21) for given $\hat{\alpha}_m(\theta)$ shown in Equation (22). Since the closed-form expression of $\hat{\theta}_m$ is not possible, an iterative method is used for numerical computation of the estimate. To estimate $\hat{\theta}_m$, we have to solve the following non-linear equation:

$$\phi(\theta) = \frac{-n}{\theta} + \frac{3}{2}\sum_{i=1}^{n}\frac{2y_i^2}{\theta^3\left(2+\left(\frac{y_i}{\theta}\right)^2\right)} + (\hat{\alpha}_m - 1)\sum_{i=1}^{n}\frac{y_i\left(y_i - \theta\sqrt{2+\frac{y_i^2}{\theta^2}}\right)}{\theta(y_i^2+2\theta^2)} = 0. \quad (23)$$

Finding the existence and uniqueness of nonlinear equation solutions can be difficult and depends on a number of variables. So, in this section, we will give the equation defined in (23)

Table 1. The MLE $\hat{\theta}_m$ for various combinations of n and α.

	Sample size (n)			
$\hat{\alpha}_m$	30	60	100	150
0.5	1.700	1.582	1.411	1.489
1.0	2.390	2.719	2.008	2.287
2.0	7.755	4.784	4.789	6.072
4.0	5.354	5.701	5.973	6.346

a visual representation that's simpler to comprehend. Potential problems like having many solutions or none at all are identified. Table 1 demonstrates that solutions of θ exist for different n and α combinations, further demonstrating the uniqueness and existence of $\hat{\theta}$. Hence, from Table 1, we can conclude that solution of θ does exist and is unique.

The $100(1-\gamma)\%$ two-sided asymptotic confidence intervals (ACIs) for the unknown parameters α and θ are given by

$$\hat{\alpha}_m \mp z_{\gamma/2}\sqrt{\hat{\kappa}^{11}}, \quad \hat{\theta}_m \mp z_{\gamma/2}\sqrt{\hat{\kappa}^{22}},$$

where $z_{\gamma/2}$ is obtained from the probability $P[Z > z_{\gamma/2}] = \gamma/2$ with Z as the standard normal rv and $(\hat{\kappa}^{11}, \hat{\kappa}^{22})$ are the diagonal elements of the inverse Fisher's information matrix estimated using the MLEs $\hat{\alpha}_m$ and $\hat{\theta}_m$.

The Fisher's information matrix is given by

$$I(\alpha,\theta) = \begin{bmatrix} -E\left[\dfrac{\partial^2 \log \mathcal{L}}{\partial \alpha^2}\right] & -E\left[\dfrac{\partial^2 \log \mathcal{L}}{\partial \theta \partial \alpha}\right] \\ -E\left[\dfrac{\partial^2 \log \mathcal{L}}{\partial \alpha \partial \theta}\right] & -E\left[\dfrac{\partial^2 \log \mathcal{L}}{\partial \theta^2}\right] \end{bmatrix}$$

where

$$E\left[\frac{\partial^2 \log \mathcal{L}}{\partial \alpha^2}\right] = -\frac{n}{\alpha^2},$$

$$E\left[\frac{\partial^2 \log \mathcal{L}}{\partial \theta \partial \alpha}\right] = \sum_{i=1}^{n} E\left[\frac{y_i\left(y_i - \theta\sqrt{2 + \frac{y_i^2}{\theta^2}}\right)}{\theta(y_i^2 + 2\theta^2)}\right],$$

$$E\left[\frac{\partial^2 \log \mathcal{L}}{\partial \theta^2}\right] = \sum_{i=1}^{n} E\left[\frac{4\theta^4 n + y_i^4(-\alpha + n - 2) + 2\theta^2 y_i^2(2n - 3(\alpha+2))}{\theta^2\left(2\theta^2 + y_i^2\right)^2}\right.$$
$$\left. + \frac{4(\alpha-1)\theta^3 y_i\sqrt{\frac{y_i^2}{\theta^2} + 2} + (\alpha-1)\theta y_i^3\sqrt{\frac{y_i^2}{\theta^2} + 2}}{\theta^2\left(2\theta^2 + y_i^2\right)^2}\right].$$

As the closed expressions are not possible for Fisher's information matrix. Using R software, we can easily compute the Hessian matrix which is approximate to Fisher's information matrix. Notwithstanding that the MLE is the most popular method for estimating the unknown parameters, it is not necessarily unbiased. The fitness of the distribution may undoubtedly be affected by the bias. So for reducing the bias, we adopt a corrective approach to derive the MLE with possible minimum mean squared error (MSE). Bartlett (1953) was first to introduce a bias correction formula for one parameter case. It was then extended by Cox and Snell (1968) for multi-parameter models. In recent past, many authors used this approach to propose the bias corrected MLE for some probability distributions. Following Cox and Snell (1968), the general estimator of the bias of the parameter $\tau_s, s = 1, 2, \ldots, p$ is given by

$$\mathbb{B}(\hat{\tau}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \kappa^{si}\kappa^{jl}\left[\frac{1}{2}\kappa_{ijl} + \kappa_{ij,l}\right] + O(n^{-2}), \quad s = 1, 2, \ldots, p, \tag{24}$$

where κ^{ij} is the (i, j)th element of the inverse of Fisher's information matrix,

$$\kappa_{ij} = E\left[\frac{\partial^2 \ell}{\partial\lambda_i \partial\lambda_j}\right], \quad \kappa_{ijl} = E\left[\frac{\partial^3 \ell}{\partial\lambda_i \partial\lambda_j \partial\lambda_l}\right],$$
$$\kappa_{ij,l} = E\left[\left(\frac{\partial^2 \ell}{\partial\lambda_i \partial\lambda_j}\right)\left(\frac{\partial \ell}{\partial\lambda_l}\right)\right], \quad \kappa_{ij}^{l} = \frac{\partial \kappa_{ij}}{\partial \lambda_l}, \quad i, j, l = 1, 2, \ldots, p,$$

and ℓ denotes the log-likelihood function.

In the case considered here, we have $p = 2$, $s = 1$, 2, $\tau_1 = \alpha, \tau_2 = \theta$ and biases of the MLEs are

$$\mathbb{B}\left(\hat{\alpha}\right) = \frac{1}{2}\left(\kappa^{11}\right)^2 \kappa_{111} + \frac{1}{2}\kappa^{12}\kappa^{22}\kappa_{222} + \frac{3}{2}\kappa^{11}\kappa^{12}\kappa_{112} + \frac{1}{2}\kappa^{11}\kappa^{22}\kappa_{122} + \left(\kappa^{12}\right)^2 \kappa_{122},$$
$$\mathbb{B}(\hat{\theta}) = \frac{1}{2}\kappa^{12}\kappa^{11}\kappa_{111} + \frac{1}{2}\left(\kappa^{22}\right)^2 \kappa_{222} + \frac{1}{2}\kappa_{112}\left[2\left(\kappa^{12}\right)^2 + \kappa^{11}\kappa^{22}\right] + \frac{3}{2}\kappa_{122}\,\kappa^{12}\kappa^{22}.$$

After thorough algebraic manipulation, we obtained the following expressions for the Lt_2D:

$$\kappa_{111} = \frac{\partial^3 \ell}{\partial\alpha\partial\alpha\partial\alpha} = \frac{2n}{\alpha^3}, \quad \kappa_{112} = \frac{\partial^3 \ell}{\partial\alpha\partial\alpha\partial\theta} = 0,$$
$$\kappa_{122} = \frac{\partial^3 \ell}{\partial\alpha\partial\theta\partial\theta} = \sum_{i=1}^{n}\left[\frac{\delta''_\theta}{\delta} - \left(\frac{\delta'_\theta}{\delta}\right)^2\right],$$
$$\kappa_{222} = \frac{\partial^3 \ell}{\partial\theta\partial\theta\partial\theta} = \frac{-2n}{\theta^3} - \frac{3}{2}\sum_{i=1}^{n}\left[\frac{\phi'''_\theta}{\phi} - \frac{\phi''_\theta\phi'_\theta}{\phi^2} - \frac{2\phi'_\theta\phi''_\theta}{\phi^2} + 2\left(\frac{\phi'_\theta}{\phi}\right)^3\right]$$
$$+ (\alpha - 1)\sum_{i=1}^{n}\left[\frac{\delta'''_\theta}{\delta} - \frac{\delta''_\theta\delta'_\theta}{\delta^2} - \frac{2\delta''_\theta\delta'_\theta}{\delta^2} + 2\left(\frac{\delta'_\theta}{\delta}\right)^3\right],$$

where

$$\phi'_\theta = \frac{-2y_i^2}{\phi^3}, \quad \phi''_\theta = \frac{6y_i^3}{\theta^4}, \quad \phi'''_\theta = \frac{-24y_i^3}{\theta^5},$$

$$\delta'_\theta = -y_i\left[\frac{1}{\theta^2\sqrt{\phi}} - \frac{\phi'_\theta}{2\theta\phi^{3/2}}\right],$$

$$\delta''_\theta = y_i\left[\frac{2}{\theta^3\sqrt{\phi}} + \frac{\phi'_\theta}{2\phi^{3/2}\theta^2} + \frac{\phi''_\theta}{2\theta\phi^{3/2}} - \frac{3(\phi'_\theta)^2}{4\theta\phi^{5/2}} - \frac{\phi'_\theta}{2\theta^2\phi^{3/2}}\right],$$

$$\delta'''_\theta = \frac{-6}{\theta^4\sqrt{\phi}} - \frac{\phi'_\theta}{\phi^{3/2}\theta^3} + \frac{\phi'''_\theta}{2\theta\phi^{3/2}} - \frac{9\phi'_\theta\phi''_\theta}{4\theta\phi^{5/2}} + \frac{15(\phi'_\theta)^2}{8\theta\phi^{7/2}} + \frac{3(\phi'_\theta)^2}{4\theta^2\phi^{5/2}} - \frac{\phi''_\theta}{2\theta^2\phi^{3/2}}.$$

All the terms appearing above are computed using the MLEs of parameters. Then BMLEs of α and θ are obtained as $\hat{\alpha}_b = \hat{\alpha}_m - \mathbb{B}(\hat{\alpha}_m)$ and $\hat{\theta}_b = \hat{\theta}_m - \mathbb{B}(\hat{\theta}_m)$, respectively.

6.2.1. Method of percentiles

The pth percentile of the distribution is an observation that divides the whole area into the ratio of $100(p)\%$ and $100(1-p)\%$. Let $(y_1, y_2, \ldots, y_n)$ be a random sample drawn from the quantile function $Q(u; \Theta)$ given in (8), where Θ is a vector of parameters. A key idea of getting percentile estimates is to equate the theoretical and sample percentiles as the assumed distribution should possess the characteristics of the data if the assumption of the distribution is correct. Using the order statistics $(y_{(1:n)}, y_{(2:n)}, \ldots, y_{(n:n)})$ corresponding to random sample $(y_1, y_2, \ldots, y_n)$, the pth percentile is defined by

$$\zeta\left(p_i\right) = \begin{cases} Y_{([np]+1:n)}, & \text{if } np \in \mathbb{Z}, \\ Y_{(np:n)}, & \text{if } np \notin \mathbb{Z}, \end{cases} \tag{25}$$

where $[a]$ is the largest integral. The percentile estimates (PEs) $(\hat{\alpha}_p, \hat{\theta}_p)$ of the parameters (α, θ) can be obtained by solving the following non-linear quantile equations:

$$\zeta\left(p_i\right) = Q\left(p_i\right), \quad i = 1, 2.$$

The percentile points $p_i, i = 1, 2$ need to be decided by the experimenter. It has been observed that the distribution sometimes lacks to explain the tails of the frequency curve. We may consider the percentiles from the tails. Here, we suggest to use $(p_1 = 0.1, p_2 = 0.9)$ or $(p_1 = 1/4, p_2 = 3/4)$. The PEs $\hat{\alpha}_p$ and $\hat{\theta}_p$ of α and θ, respectively, can be computed by solving the following non-linear quantile equations:

$$\zeta\left(p_1\right) = \frac{\theta\left(2p_1^{\frac{1}{\alpha}} - 1\right)}{\sqrt{2p_1^{\frac{1}{\alpha}}\left(1 - p_1^{\frac{1}{\alpha}}\right)}}, \tag{26}$$

$$\zeta\left(p_2\right) = \frac{\theta\left(2p_2^{\frac{1}{\alpha}} - 1\right)}{\sqrt{2p_2^{\frac{1}{\alpha}}\left(1 - p_2^{\frac{1}{\alpha}}\right)}}. \tag{27}$$

Subtracting (26) from (27), the PE $\hat{\theta}_p$ given $\hat{\alpha}_p$ is computed by solving the following relation:

$$\hat{\theta}_p = \frac{\zeta(p_2) - \zeta(p_1)}{g(p_1, p_2, \hat{\alpha}_p)}, \tag{28}$$

where $g(p_1, p_2, \alpha) = \frac{(2p_2^{\frac{1}{\alpha}} - 1)}{\sqrt{2p_2^{\frac{1}{\alpha}}(1 - p_2^{\frac{1}{\alpha}})}} - \frac{(2p_1^{\frac{1}{\alpha}} - 1)}{\sqrt{2p_1^{\frac{1}{\alpha}}(1 - p_1^{\frac{1}{\alpha}})}}$ and $\hat{\alpha}_p$ can be obtained as the solution of the following non-linear equation:

$$\zeta(p_1)\sqrt{2p_1^{\frac{1}{\hat{\alpha}_p}}\left(1 - p_1^{\frac{1}{\hat{\alpha}_p}}\right)} - \frac{\zeta(p_2) - \zeta(p_1)}{g(p_1, p_2, \hat{\alpha}_p)}\left(2p_1^{\frac{1}{\hat{\alpha}_p}} - 1\right) = 0. \tag{29}$$

Equation (29) can be solved numerically.

6.3. When μ is unknown

Suppose $(y_1, y_2, \ldots, y_n)$ is a random sample of size n from the $\mathrm{Lt_2D}(\alpha, \theta, \mu)$ with pdf given by (5). The log-likelihood function of $\Theta = (\alpha, \theta, \mu)$ is given by

$$\ell(\Theta \mid \mathbf{y}) = n\ln(\alpha) - n\ln(\theta) - n(\alpha - 1)\ln(2)$$
$$-\frac{3}{2}\sum_{i=1}^{n}\ln\left(2 + \left(\frac{y_i - \mu}{\theta}\right)^2\right) + (\alpha - 1)\sum_{i=1}^{n}\ln\left(1 + \frac{y_i - \mu}{\theta\sqrt{2 + \left(\frac{y_i - \mu}{\theta}\right)^2}}\right). \tag{30}$$

The following log-likelihood equations are obtained from the log-likelihood function (30),

$$\frac{n}{\alpha} - n\ln(2) + \sum_{i=1}^{n}\ln\delta(y_i) = 0, \tag{31}$$

$$-\frac{n}{\theta} - \frac{3}{2}\sum_{i=1}^{n}\frac{\phi'_\theta(y_i)}{\phi(y_i)} + (\alpha - 1)\sum_{i=1}^{n}\frac{\delta'_\theta(y_i)}{\delta(y_i)} = 0, \tag{32}$$

$$-\frac{3}{2}\sum_{i=1}^{n}\frac{\phi'_\mu(y_i)}{\phi(y_i)} + (\alpha - 1)\sum_{i=1}^{n}\frac{\delta'_\mu(y_i)}{\delta(y_i)} = 0, \tag{33}$$

where

$$\phi(y_i) = \left(2 + \left(\frac{y_i - \mu}{\theta}\right)^2\right), \quad \phi'_\mu(y_i) = -2\left(\frac{y_i - \mu}{\theta}\right), \quad \phi'_\theta(y_i) = -2\frac{(y_i - \mu)^2}{\theta^3},$$

$$\delta(y_i) = \left(1 + \frac{y_i - \mu}{\theta\sqrt{2 + \left(\frac{y_i - \mu}{\theta}\right)^2}}\right), \quad \delta'_\theta(y_i) = -\left(\frac{y_i - \mu}{\theta}\right)\left\{\frac{1}{\theta\sqrt{\phi(y_i)}} + \frac{\phi'_\theta(y_i)}{2\phi(y_i)^{3/2}}\right\},$$

$$\delta'_\mu(y_i) = -\frac{1}{\theta}\left(\frac{1}{\sqrt{\phi(y_i)}} + \frac{(y_i - \mu)\phi'_\mu(y_i)}{2\phi(y_i)^{3/2}}\right).$$

Equations (31), (32) and (33) can be simultaneously solved to get the MLEs $\hat{\alpha}_m$, $\hat{\mu}_m$ and $\hat{\theta}_m$. From Equation (31), the MLE $\hat{\alpha}_m$ can be uniquely determined using the following relation:

$$\hat{\alpha}_m = n\left(n\ln(2) - \sum_{i=1}^{n} \ln\hat{\delta}(y_i)\right)^{-1}, \tag{34}$$

where $\hat{\delta}(y_i) = \delta(y_i) = \left(1 + \frac{y_i - \hat{\mu}_m}{\hat{\theta}_m}\left(2 + \left(\frac{y_i - \hat{\mu}_m}{\hat{\theta}_m}\right)^2\right)^{-1/2}\right)$.

However, the MLEs $\hat{\mu}_m$ and $\hat{\theta}_m$ can be determined as the simultaneous solution of Equations (32) and (33) for given $\hat{\alpha}_m$ defined in (34). Since the closed-form expressions of $\hat{\theta}_m$ and $\hat{\mu}_m$ are not possible to derive, an iterative method is used for numerical computation of the estimates.

The ACIs and BMLEs for the parameters can be determined using the procedure discussed in the previous section. We omit the discussion from here.

To estimate the parameters (α, θ, μ) using the percentiles, we need to equate three theoretical and sample percentiles. The PEs $\hat{\alpha}_p$, $\hat{\theta}_p$ and $\hat{\mu}_p$ can be computed by solving the following non-linear quantile equations:

$$\zeta\left(p_i\right) = Q\left(p_i\right), \quad i = 1, 2, 3. \tag{35}$$

Simplifying Equation (35), the $\hat{\alpha}_p$ is obtained by solving the equation,

$$\frac{\zeta\left(p_3\right) - \zeta\left(p_2\right)}{\zeta\left(p_2\right) - \zeta\left(p_1\right)} - \frac{h(p_2, p_3, \alpha)}{g(p_1, p_2, \alpha)} = 0, \tag{36}$$

where

$$h(p_2, p_3, \alpha) = \frac{\left(2p_3^{\frac{1}{\alpha}} - 1\right)}{\sqrt{2p_3^{\frac{1}{\alpha}}\left(1 - p_3^{\frac{1}{\alpha}}\right)}} - \frac{\left(2p_2^{\frac{1}{\alpha}} - 1\right)}{\sqrt{2p_2^{\frac{1}{\alpha}}\left(1 - p_2^{\frac{1}{\alpha}}\right)}}$$

and $g(p_1, p_2, \alpha)$ is defined in the previous section.

For given $\hat{\alpha}_p$, the PE $\hat{\theta}_p$ is

$$\hat{\theta}_p = \frac{\zeta\left(p_2\right) - \zeta\left(p_1\right)}{g(p_1, p_2, \hat{\alpha}_p)}. \tag{37}$$

Using $\hat{\alpha}_p$ and $\hat{\theta}_p$, the PE $\hat{\mu}_p$ is given by

$$\hat{\mu}_p = \zeta\left(p_1\right) - \hat{\theta}_p \frac{\left(2p_1^{\frac{1}{\hat{\alpha}_p}} - 1\right)}{\sqrt{2p_1^{\frac{1}{\hat{\alpha}_p}}\left(1 - p_1^{\frac{1}{\hat{\alpha}_p}}\right)}}. \tag{38}$$

Equations (36), (37) and (38) can be solved numerically.

7. Accelerated failure time regression and estimation

From the previous sections, we could see the impressive features of the Lt_2D including the useful skewed shapes, uni-modal hrf, explicitly defined statistical measures and convenient estimation procedures. We may curiously think of studying the use of the Lt_2D in parametric regression analysis. There are various situations where the assumption of normal random error does not meet the requirement. In such cases, it is sought to use non-normal distribution for the random error component. Great efforts have been made for the development of the regression models with non-normal response and error distributions. See Kalbfleisch and Prentice (2002, Section 2.3) and Liu (2012, Chapter 4). There are various ways to incorporate the co-factors into parametric regression model. We may consider either of the following functions: hrf, mean parameter, shape and scale parameters and linear predictor.

In this paper, we consider the linear model having only one predictor variable,

$$y = \beta_0 + \beta_1 x + \sigma\epsilon, \tag{39}$$

where y is the response variable defined on $(-\infty, \infty)$, β_0 is the intercept, β_1 is the regression coefficient, x is the independent/predictor variable, σ is the scale parameter and ϵ is the random disturbance that is assumed to follow the Lt_2D(α) defined on $(-\infty, \infty)$.

The model defined by Equation (39) is called accelerated failure time regression model. The reliability/survival function for the ith response is

$$\begin{aligned} S_{Y_i}\left(y \mid x\right) &= \Pr\left(\epsilon_i \geq \frac{y - \beta_0 - \beta_1 x_i}{\sigma}\right), \\ &= S_0\left(\frac{y - \beta_0 - \beta_1 x_i}{\sigma}\right), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{40}$$

where $S_0(.)$ is the survival function associated with Lt_2D(α).

The corresponding hrf is given by

$$h_{Y_i}\left(y \mid x\right) = \frac{1}{\sigma} h_0\left(\frac{y - \beta_0 - \beta_1 x_i}{\sigma}\right), \quad i = 1, 2, \ldots, n, \tag{41}$$

where $h_0(.)$ is the survival function associated with Lt_2D(α).

The pdf for the ith response is readily obtained as

$$f_{Y_i}\left(y \mid x, \Theta\right) = \frac{1}{\sigma} h_0\left(\frac{y - \beta_0 - \beta_1 x_i}{\sigma}\right) S_0\left(\frac{y - \beta_0 - \beta_1 x_i}{\sigma}\right), \quad i = 1, 2, \ldots, n, \tag{42}$$

where $-\infty < y < \infty$ and $\Theta = \{\alpha, \beta_0, \beta_1, \sigma\}$.

The pdf can be easily extended for more than one predictors by replacing the linear predictor, $\beta_0 + \beta_1 x_i$ in (42) by $X'\beta = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$ for p variates.

For the Lt_2D regression model defined by (42), the MLEs of Θ can be obtained by maximizing the log-likelihood function,

$$\log L\left(\Theta \mid \underset{-}{y}, \underset{-}{x}\right) = -n \log \sigma + \sum_{i=1}^{n} \log h_0\left(z_i\right) + \sum_{i=1}^{n} \log S_0\left(z_i\right), \tag{43}$$

where $z_i = \frac{y_i - \beta_0 - \beta_1 x_i}{\sigma}$. The MLEs $\hat{\alpha}_m$, $\hat{\beta}_{0m}$, $\hat{\beta}_{1m}$ and $\hat{\sigma}_m$ can be determined by solving the following log-likelihood equations:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \log L\left(\Theta \mid \underset{-}{y}, \underset{-}{x}\right) = \sum_{i=1}^{n} \frac{h_0^{\alpha}(z_i)}{h_0(z_i)} + \sum_{i=1}^{n} \frac{S_0^{\alpha}(z_i)}{S_0(z_i)} = 0, \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}\beta_0} \log L\left(\Theta \mid \underset{-}{y}, \underset{-}{x}\right) = \sum_{i=1}^{n} \frac{h_0^{\beta_0}(z_i)}{h_0(z_i)} + \sum_{i=1}^{n} \frac{S_0^{\beta_0}(z_i)}{S_0(z_i)} = 0, \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}\beta_1} \log L\left(\Theta \mid \underset{-}{y}, \underset{-}{x}\right) = \sum_{i=1}^{n} \frac{x_i h_0^{\beta_1}(z_i)}{h_0(z_i)} + \sum_{i=1}^{n} \frac{x_i S_0^{\beta_1}(z_i)}{S_0(z_i)} = 0, \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \log L\left(\Theta \mid \underset{-}{y}, \underset{-}{x}\right) = n + \sum_{i=1}^{n} \frac{z_i h_0^{\sigma}(z_i)}{h_0(z_i)} + \sum_{i=1}^{n} \frac{z_i S_0^{\sigma}(z_i)}{S_0(z_i)} = 0. \tag{47}$$

Equations (44), (45), (46) and (47) can be solved numerically using a suitable iterative method as discussed in Section 6.2.1. We may also obtain ACIs for the parameters involved in (42) using the asymptotic property of the MLE.

8. Simulations

In this section, we carry out Monte Carlo simulations for the estimation of the parameters of the Lt_2D discussed in Section 7 and the Lt_2D regression model given in Section 8. In the first case, we consider location parameter to be known and estimation of the scale and shape parameters is performed. The simulation experiments may be easily extended for three parameters case as it is done for the regression model with four parameters. In the latter case, we conduct simulation experiments for the estimation of the parameters α, β_0, β_1 and σ while the independent variable is taken to be uniformly distributed in $(0, 10)$. The comparison is made on the basis of bias (B), MSE and absolute bias (AB) of the estimators. Following combinations are considered to execute the simulation experiments: $(\alpha = 1, \theta = 0.5)$, $(\alpha = 1, \theta = 1)$, $(\alpha = 2, \theta = 1)$, $(\alpha = 1.5, \theta = 1)$ and $(\alpha = 2, \theta = 2)$ for varying sample sizes $n = 20, 40, 60, 80$.

We simulated 10000 random samples from the Lt_2D using the quantile function

$$x_i = \mu + \frac{\theta\left(2u^{1/\alpha} - 1\right)}{\sqrt{2u^{1/\alpha}\left(1 - u^{1/\alpha}\right)}},$$

where $u_i \sim \mathrm{Uniform}(0, 1)$, $i = 1, 2, \ldots, n$. Then we compute bias, MSE and AB for MLEs, BMLEs and PEs. The MSE, bias and AB are computed as follows:

$$\mathrm{MSE}(\tau) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\tau - \hat{\tau}_i\right)^2, \quad B(\tau) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{\tau}_i - \tau\right),$$

$$\mathrm{AB}(\lambda) = \frac{1}{10000} \sum_{i=1}^{10000} |\tau_i - \hat{\tau}_i|,$$

where $\hat{\tau}$ is either of the MLE, BMLE and PE of τ. For the BMLEs, the bias can be extracted directly using Equation (24).

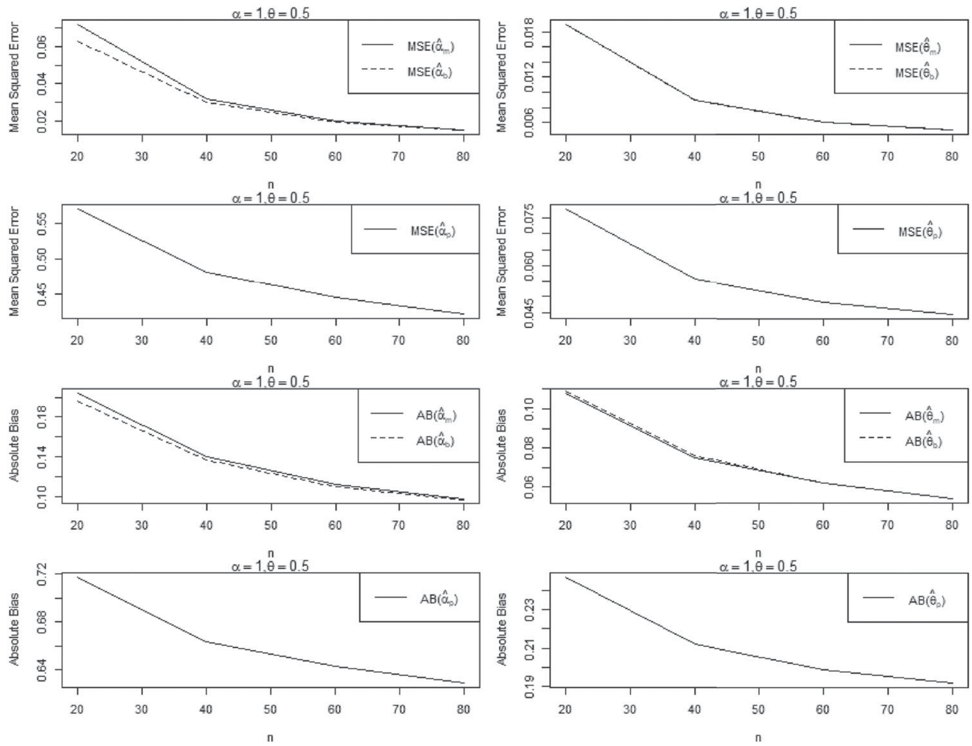

Figure 3. The MSE and absolute bias of $(\hat{\alpha}_m, \hat{\theta}_m)$, $(\hat{\alpha}_b, \hat{\theta}_b)$ and $(\hat{\alpha}_p, \hat{\theta}_p)$ for simulated sample with $\alpha = 1$ and $\theta = 0.5$.

The results are shown in Figures 3–7. From these figures, we observe that the MSEs of BMLEs of both the parameters are smaller than that of the corresponding MLEs and PEs. Bias of $\hat{\alpha}_b$ also appears substantially smaller than the biases of $\hat{\alpha}_m$ and $\hat{\alpha}_p$ for all the combinations of the parameters and for every n. We also note here that ABs of $\hat{\alpha}_b$ and $\hat{\theta}_b$ are smaller than the ABs of $(\hat{\alpha}_m, \hat{\alpha}_p)$ and $(\hat{\theta}_m, \hat{\theta}_p)$, respectively. We also found from the simulations that increasing sample size results in decreasing the bias, MSE and AB of all the estimators in all the cases of the n. From the above study, we can state here that the BMLEs are better than the MLEs and PEs for both parameters.

We now conduct simulation experiments to assess the long-run performance of the MLEs of Lt_2D regression parameters. We take varying sample sizes $n = 20, 40, 60, 80, 100$ and different combinations of the parameter values $\alpha = 0.5, 1, 2$, $\beta_0 = 0, 2$, $\beta_1 = 1, 2$ and $\sigma = 1$. We generate n uniform random variates ($X \sim \text{Uniform}(0, 10)$) as a co-variate and n random disturbances distributed as $\epsilon \sim \text{L}t_2\text{D}(\alpha)$. The model given in Equation (39) is used to simulate the response variates for the given values of the parameters. The MLEs are computed using the simulated samples with varying sample sizes. The average MLEs, biases and MSEs of the MLEs are reported on the basis of 10,000 iterations. Simulation results are shown in Table 2. Following are the conclusions from the simulation.

- We observe that β_1 is showing smaller bias on almost all the sample sizes as compared to the other parameters.

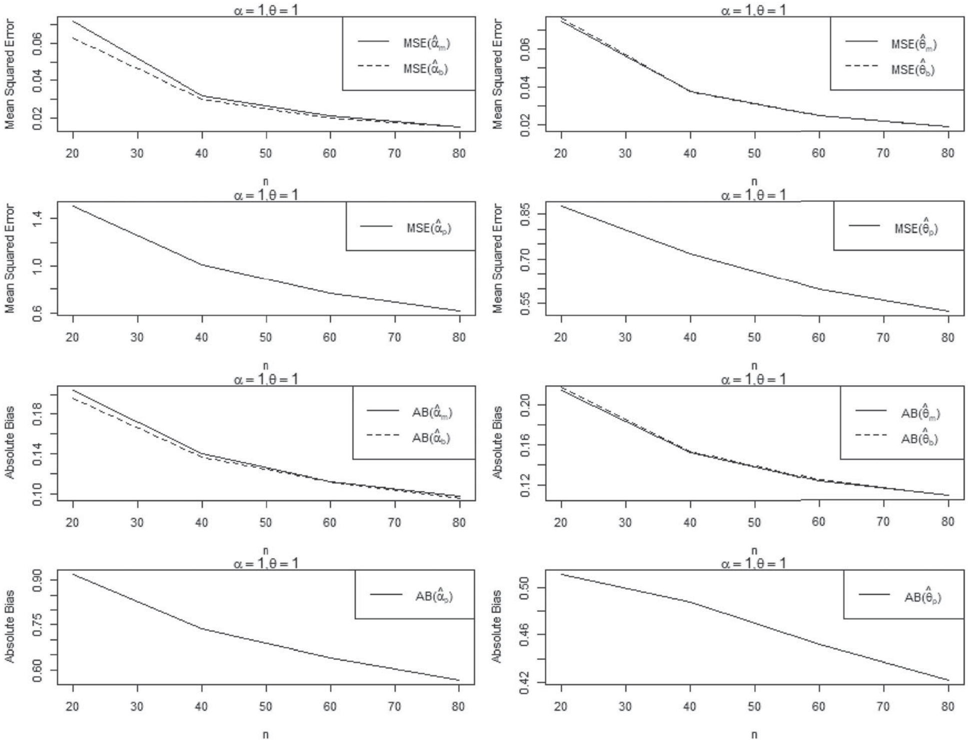

Figure 4. The MSE and absolute bias of $(\hat{\alpha}_m, \hat{\theta}_m)$, $(\hat{\alpha}_b, \hat{\theta}_b)$ and $(\hat{\alpha}_p, \hat{\theta}_p)$ for simulated sample with $\alpha = 1$ and $\theta = 1$.

- We also note that the MSE of MLEs of all Lt_2D regression parameters decreases as the sample size increases. This shows that the MLEs are consistent.
- The average estimate and bias are also decreasing as the sample size is increasing.

9. Strength data modelling

This section proposes the use of the methodologies, presented in the previous sections, for modelling strength data obtained from a life-test of 1.5 cm glass fibres. The life test was conducted at UK National Physical Laboratory. Obubu et al. (2019) used this data set in their study of fitting Gompertz length-biased exponential distribution. It is apparent from the statistics given in Table 3 that the strength data is left tailed and has higher peak than the normal curve since the Pearson's coefficient of skewness $\beta_1 < 0$ and kurtosis $\beta_2 > 3$. The Lt_2D may be a pertinent model for fitting this data set because its characteristics are matching with that of the strength data. To prove the efficacy of the Lt_2D for real life data applications, we compare the goodness-of-fit test results with five well-known flexible extensions of the normal and t distributions. The pdfs are defined as follows:

(1) Topp–Leone normal distribution (TLND) by Sharma (2018)

$$f(x) = 2\alpha\phi(x)\left[1 - \Phi(x)\right]\Phi(x)^{\alpha-1}\left[2 - \Phi(x)\right]^{\alpha-1}, \quad -\infty < x < \infty,\ \alpha > 0.$$

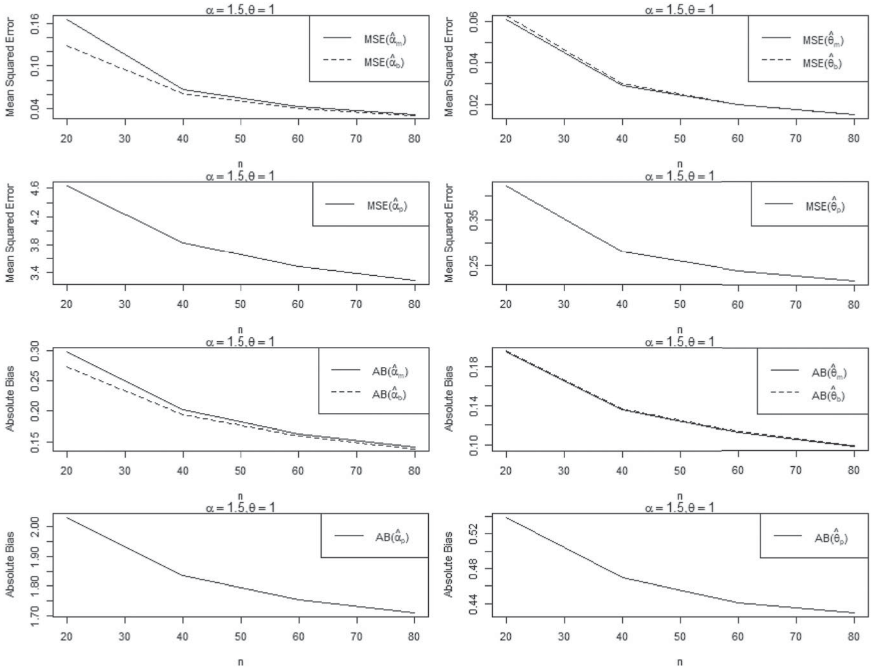

Figure 5. The MSE and absolute bias of $(\hat{\alpha}_m, \hat{\theta}_m)$, $(\hat{\alpha}_b, \hat{\theta}_b)$ and $(\hat{\alpha}_p, \hat{\theta}_p)$ for simulated sample with $\alpha = 1.5$ and $\theta = 1$.

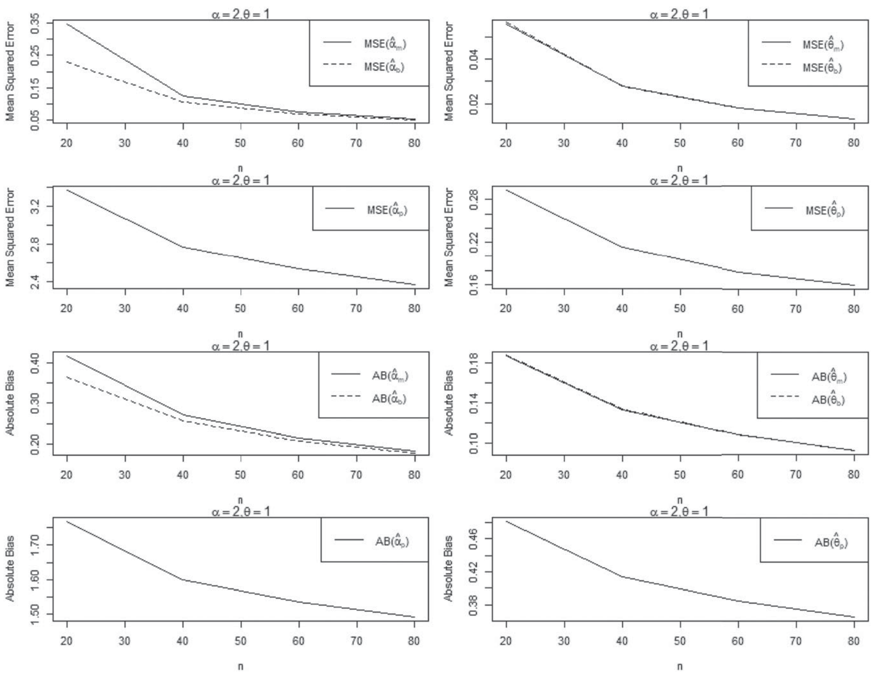

Figure 6. The MSE and absolute bias of $(\hat{\alpha}_m, \hat{\theta}_m)$, $(\hat{\alpha}_b, \hat{\theta}_b)$ and $(\hat{\alpha}_p, \hat{\theta}_p)$ for simulated sample with $\alpha = 2$ and $\theta = 1$.

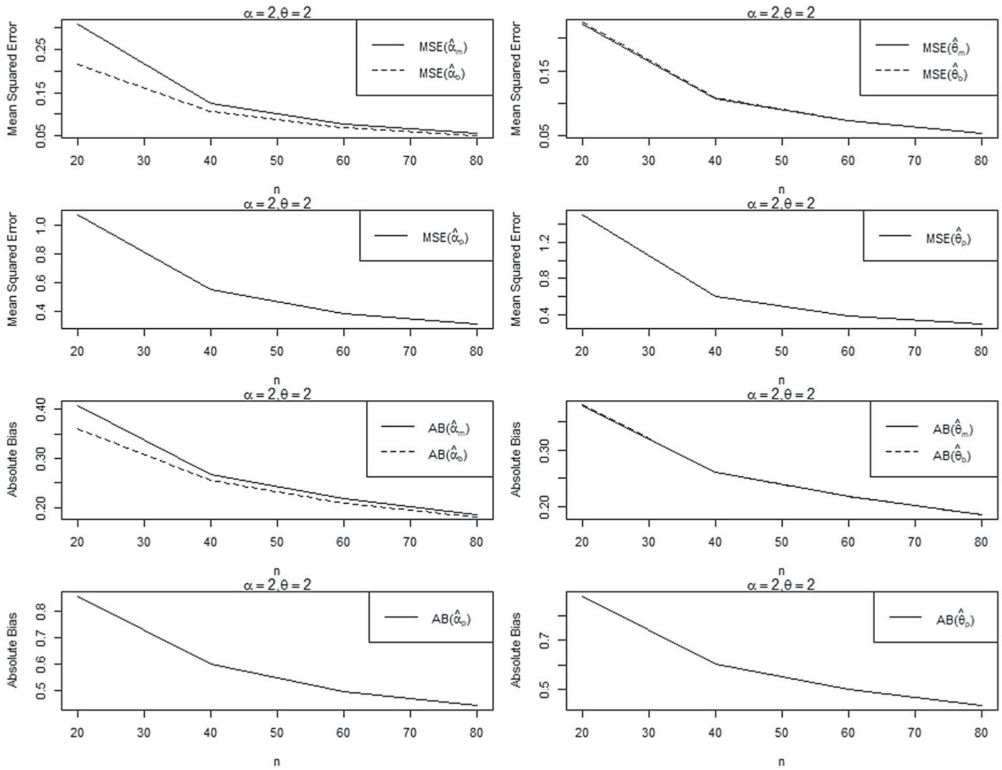

Figure 7. The MSE and absolute bias of $(\hat{\alpha}_m, \hat{\theta}_m)$, $(\hat{\alpha}_b, \hat{\theta}_b)$ and $(\hat{\alpha}_p, \hat{\theta}_p)$ for simulated sample with $\alpha = 2$ and $\theta = 2$.

(2) Skew normal distribution (SND) by Azzalini (1985)

$$f(x) = 2\phi(x)\Phi(\alpha x), \quad -\infty < x < \infty,\ \alpha > 0.$$

(3) Power normal distribution (PND) by Gupta and Gupta (2008)

$$f(x) = \alpha\phi(x)\,[\Phi(x)]^{\alpha-1}, \quad -\infty < x < \infty,\ \alpha > 0.$$

(4) Beta-t distribution (BtD) by Basalamah et al. (2018)

$$f(x) = \frac{1}{\mathcal{B}(\alpha,\beta)} f_t(x)F_t(x)^{\alpha-1}(1-F_t(x))^{\beta-1}, \quad -\infty < x < \infty,\ \alpha > 0,\ \beta > 0.$$

(5) Kumaraswamy skew-t distribution (KStD) by Said et al. (2018)

$$f(x) = \alpha\beta f_t(x)F_t(x)^{\alpha-1}(1-F(x)^{\alpha})^{\beta-1}, \quad -\infty < x < \infty,\ \alpha > 0,\ \beta > 0.$$

In the above pdfs, $\phi(x)$ $(\Phi(x))$ denotes the pdf(cdf) of the normal distribution with mean μ and variance $\sigma^2 > 0$ and $f_t(x)$ $(F_t(x))$ denotes the pdf(cdf) of the t distribution with ν degrees of freedom.

The appropriacy of the distributions is assessed and compared using the information criterion such as Akaike information criterion ($\text{AIC} = -2\ln(\ell) + 2p$) and Bayesian information

Table 2. Average MLE, MSE and bias for the Lt_2D regression parameters using simulated samples.

		$(\alpha = 1, \beta_0 = 0, \beta_1 = 1, \sigma = 1)$				$(\alpha = 1, \beta_0 = 2, \beta_1 = 2, \sigma = 1)$			
		α	β_0	β_1	σ	α	β_0	β_1	σ
$n = 20$	Mean	1.285	2.285	3.285	4.285	1.287	1.754	2.000	0.882
	Bias	0.285	−0.245	0.000	−0.118	0.287	−0.246	0.000	−0.118
	MSE	1.031	1.046	0.015	0.123	1.043	1.050	0.015	0.123
$n = 40$	Mean	1.134	−0.091	1.001	0.966	1.134	1.909	2.001	0.966
	Bias	0.134	−0.091	0.001	−0.034	0.134	−0.091	0.001	−0.034
	MSE	0.279	0.324	0.006	0.046	0.279	0.324	0.006	0.046
$n = 60$	Mean	1.071	−0.053	1.000	0.984	1.071	1.947	2.000	0.984
	Bias	0.071	−0.053	0.000	−0.016	0.071	−0.053	0.000	−0.016
	MSE	0.100	0.187	0.004	0.030	0.100	0.187	0.004	0.030
$n = 80$	Mean	1.049	−0.040	1.001	0.987	1.049	1.960	2.001	0.987
	Bias	0.049	−0.040	0.001	−0.013	0.049	−0.040	0.001	−0.013
	MSE	0.059	0.133	0.003	0.022	0.059	0.133	0.003	0.022
$n = 100$	Mean	1.037	−0.026	1.000	0.989	1.037	1.974	2.000	0.989
	Bias	0.037	−0.026	0.000	−0.011	0.037	−0.026	0.000	−0.011
	MSE	0.042	0.102	0.002	0.018	0.042	0.102	0.002	0.018
		$(\alpha = 2, \beta_0 = 0, \beta_1 = 2, \sigma = 1)$				$(\alpha = 0.5, \beta_0 = 0, \beta_1 = 1, \sigma = 1)$			
		α	β_0	β_1	σ	α	β_0	β_1	σ
$n = 20$	Mean	2.428	−0.370	1.999	0.690	0.5614	−0.1281	0.9987	0.9297
	Bias	0.428	−0.370	−0.001	−0.310	0.0614	−0.1281	−0.0013	−0.0703
	MSE	2.983	1.134	0.010	0.229	0.1146	1.0801	0.0262	0.1552
$n = 40$	Mean	2.517	−0.178	2.001	0.892	0.5236	−0.0639	1.0009	0.9743
	Bias	0.517	−0.178	0.001	−0.108	0.0236	−0.0639	0.0009	−0.0257
	MSE	2.392	0.464	0.004	0.064	0.0200	0.3823	0.0101	0.0693
$n = 60$	Mean	2.404	−0.109	2.000	0.950	0.5126	−0.0359	1.0003	0.9856
	Bias	0.404	−0.109	0.000	−0.050	0.0126	−0.0359	0.0003	−0.0144
	MSE	1.500	0.257	0.003	0.027	0.0101	0.2333	0.0063	0.0447
$n = 80$	Mean	2.299	−0.076	2.001	0.966	0.5092	−0.0287	1.0005	0.9879
	Bias	0.299	−0.076	0.001	−0.034	0.0092	−0.0287	0.0005	−0.0121
	MSE	0.955	0.172	0.002	0.018	0.0071	0.1695	0.0045	0.0328
$n = 100$	Mean	2.223	−0.052	2.000	0.975	0.5079	−0.0195	0.9998	0.9904
	Bias	0.223	−0.052	0.000	−0.025	0.0079	−0.0195	−0.0002	−0.0096
	MSE	0.637	0.125	0.001	0.013	0.0055	0.1314	0.0036	0.0260

Table 3. Descriptive statistics of strength data.

Statistic	Strength
Minimum	0.55
Maximum	2.24
Mean	1.507
Median	1.59
SD	0.324
Q1	1.375
Q3	1.685
Skewness	−0.899
Kurtosis	3.924

criterion (BIC $= -2\ln(\ell) + 2\log(n)$), where $\ln(\ell)$ is the value of log-likelihood function, p is the number of estimated parameters. The AIC and BIC are used for ranking the fitting of the models based on their likelihoods. Smaller values of these statistics correspond to the better fitted model. Kolmogorov– Smirnov (KS) statistic with its p-value is also computed for all the distributions. The KS statistic is employed to test the hypothesis that a distribution fits the data significantly at given level of significance. If p-value of the KS statistic is greater than 0.025, it suggests that the distribution fits the data significantly at 5% level of significance. The MLE with their standard errors (S.E.), negative log-likelihood (NLL) and KS statistic

Table 4. The MLEs, negative log likelihood(NLL), KS statistic along with p-value for the distributions based on strength data.

Model		MLEs			S.E.		NLL	KS	p-value
Lt_2D	0.665	1.64	0.139	0.13	0.034	0.028	12.985	0.072	0.899
TLND	0.264	1.924	0.222	0.189	0.093	0.069	15.057	0.157	0.09
SND	0.017	1.502	0.322	2.281	0.587	0.03	17.912	0.181	0.032
PND	0.103	1.941	0.137	0.105	0.131	0.057	15.636	0.16	0.079
BtD	31.021	2.662	67.596	5.848	0.506	103.405	21.004	0.205	0.01
KStD	24.551	3.69	66.305	3.464	1.493	246.507	18.79	0.19	0.021

Table 5. AIC and BIC of the distributions for the strength data.

Model	AIC	BIC
Lt_2D	31.969	38.399
TLND	36.114	42.543
SND	41.824	48.253
PND	37.272	43.701
BtD	48.009	54.438
KStD	43.58	50.009

Table 6. MLE, BMLE and PE for Lt_2D based on strength data.

	MLE	Bias	BMLE	PE	SE	95% CI
α	0.6650	0.0244	0.6406	2.212	0.1300	(0.410, 0.920)
μ	1.6404	−0.0046	1.6450	0.856	0.0344	(1.573, 1.708)
θ	0.1393	0.0000	0.1393	0.195	0.0276	(0.085, 0.193)

along with the corresponding p-value are exhibited in Table 4. The AIC and BIC are shown in Table 5.

We note from Tables 4 and 5 that the AIC, BIC and KS statistic have smaller values for the Lt_2D as compared to the extensions of the normal and t distributions. From above fitting results, we can state that the Lt_2D gives a better fitting than all the other distributions used for comparison. From Figure 8, we can also see that the Lt_2D captures well both tails and peakedness of the frequency distribution of the strength data. We also compute the estimates of the parameters using the percentiles and bias corrected MLE methods. We take $u_1 = 0.25, u_2 = 0.5$ and $u_3 = 0.95$, and obtain the PEs of α, θ and μ that are presented in Table 6. Using the MLE, BMLE and PE, the estimated quantiles and reliability are plotted in Figures 9(a) and (b), respectively. We can see that the BMLE provides the estimates closer to the empirical estimates. The PE is far away from the reality.

Besides the uni-variate data fitting, we wish to illustrate the regression model for strength data set. Since co-variates are not available for the strength data, we simulate a co-variate from uniform population in (0,10) and estimate the parameters and reliability for the strength data. Table 7 shows the MLE, SE Bias, BMLE, PE and 95% confidence intervals for the Lt_2D regression model. Conditional reliability is estimated using the MLE for the regression model for the given different values of x (see Figure 10). We can note that the median strength increases as x increases. The estimated reliability is closely fitted with the empirical one.

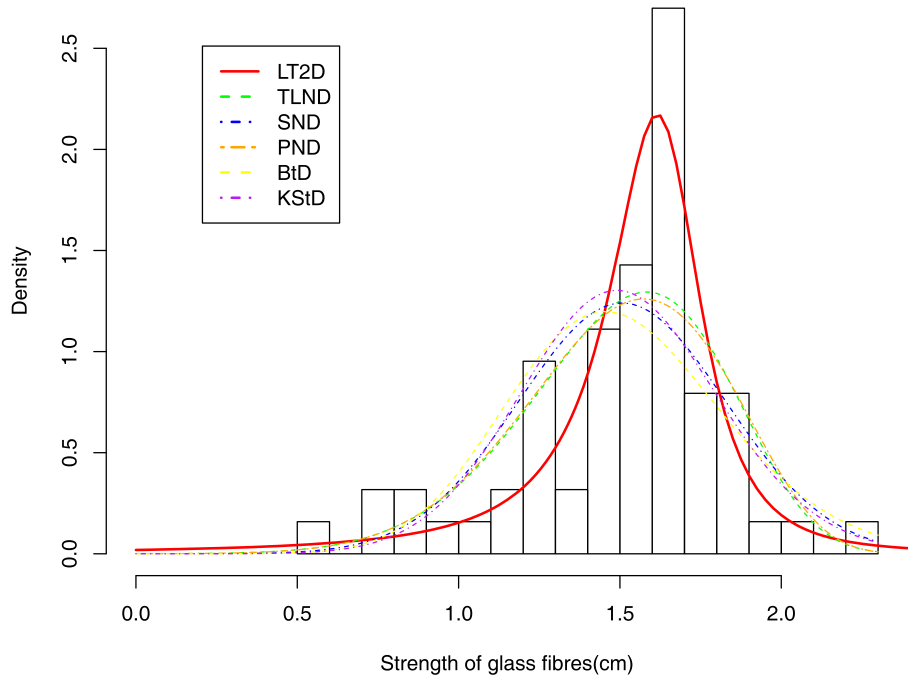

Figure 8. Fitted density plots of the distributions for strength data.

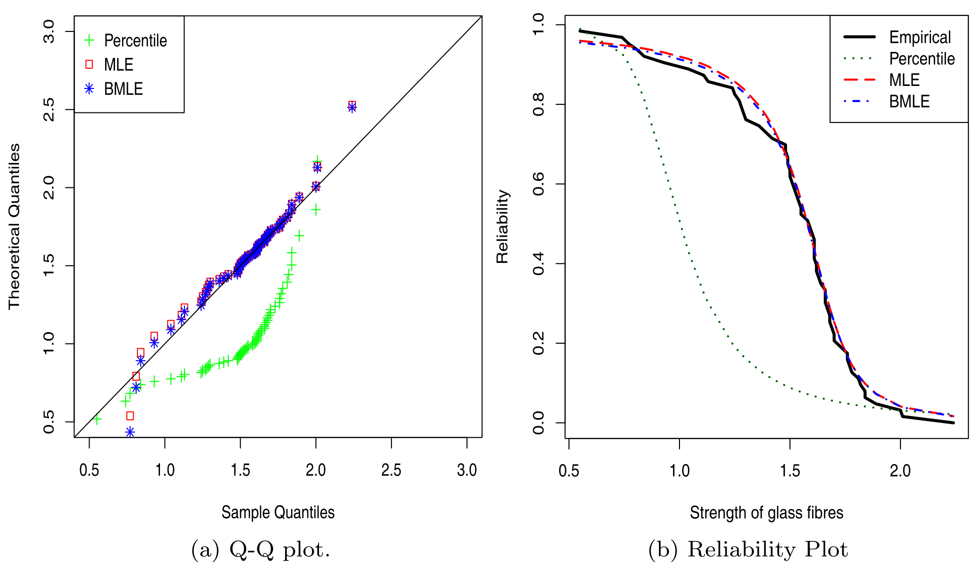

Figure 9. Q–Q and reliability plots for the strength data using Lt_2D: (a) Q–Q plot and (b) reliability plot.

10. Conclusion

In this article, we proposed three-parameter Lehmann-type I *t* distribution with 2 degrees of freedom, which accomodates upside-down bathtub-shaped hrf. We also derived the statistical properties such as shape of the density and hrf, moments, order statistics and associated

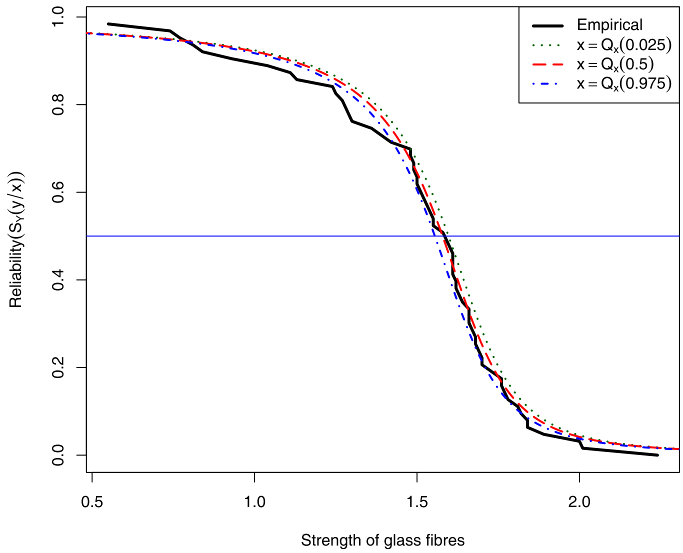

Figure 10. Reliability plots for strength of glass fibre data using the Lt_2D regression model.

Table 7. MLE, SE and 95% CI for Lt_2D regression parameters for strength data.

	MLE	SE	95% CI
α	0.561	0.102	(0.361, 0.760)
β_0	0.511	0.038	(0.436, 0.586)
β_1	0.000	0.006	(−0.012, 0.012)
σ	0.078	0.015	(0.047, 0.108)

measures, entropy, quantile function, density quantile function, quantile hazard and mean residual quantile function. Measures of dispersion using the moments and L-moments are derived. We also study the skewness and kurtosis of the distribution based on quantile function. It is observed that Lt_2D accommodates the shapes with varying degrees of skewness and higher levels of the peakedness.

The Lt_2D holds the identifiability condition in the skewness parameter. The unknown parameters of the distribution are estimated using the method of MLE, BMLE and PE. Their performances were compared through the simulation experiments. It was observed that the BMLE outperforms the MLE and PE.

We fit the strength data using the Lt_2D and compare with five another extensions of the normal and t distributions. It was found that the Lt_2D is enough flexible to model negative skewed and high peaked data.

The distribution is also utilized to develop parametric regression model to study the significance of co-variate on glass fibre strength data. Simulation experiments are performed for the MLE of the Lt_2D regression model. So, by summing up it can be stated that Lt_2D can be effectively used for non-monotone shaped hazard rate data sets.

Acknowledgments

The authors sincerely thank the Editor-in-Chief, Professor Yincai Tang, for considering our manuscript for possible publication in this esteemed journal and for efficiently managing the entire review process. We also express our gratitude to the two anonymous referees for their valuable comments and constructive suggestions that helped improve the quality of this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

Dr. Sharma received the financial support from Science and Engineering Research Board, Department of Science & Technology, Government of India, under the scheme Early Career Research Award (file no. ECR/2017/002416). Dr. Sharma also acknowledges Banaras Hindu University, Varanasi, India, for providing financial support as seed grant under the Institute of Eminence Scheme (Scheme no. Dev. 6031).

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