Statistical Theory and Related Fields

ISSN: 2475-4269 (Print) 2475-4277 (Online) Journal homepage: www.tandfonline.com/journals/tstf20

# Remaining useful life prediction based on exponential dispersion process with random drifts

Pingping Wang & Yincai Tang

STATISTICAL THEORY AND RELATED FIELDS
https://doi.org/10.1080/24754269.2025.2555043

Taylor & Francis
Taylor & Francis Group



# Remaining useful life prediction based on exponential dispersion process with random drifts

Pingping Wang [a] and Yincai Tang [b]

[a]Department of Statistics, Nanjing University of Finance and Economics, Nanjing, People's Republic of China;
[b]KLATASDS-MOE, School of Statistics, East China Normal University, Shanghai, People's Republic of China

**ABSTRACT**
Remaining Useful Life (RUL) is one of the most important indicators to detect a component failure. RUL can be predicted by historical data by adopting a model-based method. The stochastic process models have become the most popular way to model degradation data for high-quality products, such as the Wiener process, gamma process and inverse Gaussian process. However, this leads to poor reliability assessment if the model is misspecified. Application of the Tweedie exponential dispersion (TED) process, including the above-mentioned classical stochastic processes as special cases, transforms the model selection problem into a parameter estimation problem dexterously. In this paper, we propose a TED process with random drifts for degradation data and a TED process with random drifts and covariates for accelerated degradation data. A hierarchical Bayesian method is adopted to estimate the parameters of the proposed models. We also derive the failure-time distribution and the remaining useful life distribution for the proposed models. The simulation study shows that the proposed model outperforms the wrongly specified models. Two illustrative examples demonstrate the performance of the proposed TED process with random drifts and the TED process with random drifts and covariates.

**ARTICLE HISTORY**
Received 1 April 2025
Accepted 22 August 2025

**KEYWORDS**
Bayesian method; degradation data; accelerated degradation test; exponential dispersion process; random drifts; covariates

## 1. Introduction

To assess the reliability of the newly designed products, manufacturers often design the degradation test (DT) or the accelerated degradation test (ADT) to shorten the testing time by loading higher stress than normal use condition, such as a combination of random vibration, higher temperature, voltage, humidity and pressure. To hasten the degradation, the ADTs are used under severe stress to quickly obtain the degradation information. The obtained DT or ADT data facilitates the reliability inference for highly reliable products. For an overview of research in this area, see Limon et al. (2017) and Meeker et al. (1998) for details. In DT or ADT, the failure of a product is often determined by one or more quality characteristics (QC). The degradation of this QC accumulates over time and induces a failure when it exceeds a predefined threshold. This threshold combines the product degradation and reliability (Ye et al., 2014).

**CONTACT** Yincai Tang yctang@stat.ecnu.edu.cn KLATASDS-MOE, School of Statistics, East China Normal University, Shanghai, People's Republic of China

There are two classes of models for DT and ADT data: the general path models (dos Santos & Colosimo, 2015; Lu & Meeker, 1993; Shat, 2024) and stochastic process models (Bae et al., 2007; Lim & Yum, 2011; Meeker et al., 1998; Shi & Meeker, 2011). Ye and Xie (2015) have given an excellent review on this area. The Wiener process, gamma process and inverse Gaussian (IG) process are three popular stochastic models to analyse degradation data. Although they have received intensive applications in degradation data analysis, it is hard to suggest an appropriate stochastic model for a specific dataset. Recently, Tseng and Lee (2016) proposed the exponential-dispersion (ED) process to fit the degradation paths, which includes the Wiener process, gamma process and IG process as special cases. They mainly focused on the optimum allocation problem in ADT. Zhou and Xu (2019) developed the statistical inference for the parameter estimation method for TED process. Chen et al. (2019) used the TED process to predict the remaining useful life (RUL) of products and design ADTs. Luo et al. (2022) derived the system reliability, in which the degradation path for each component is assumed to follow an ED process.

However, there exists unit-to-unit variability in the degradation paths, because of the operation environment, initial variation in raw materials and manufacturing processes or product heterogeneity. There is a significant body of literature studying the heterogeneity by introducing the random effects into the degradation model. Wang (2010) gave rise to the maximum likelihood inference on a class of Wiener processes with random effects for degradation data. Lawless and Crowder (2004) incorporated the covariates and random effects to gamma process. Wang (2008) investigated a semi-parametric pseudo-likelihood inference for nonhomogeneous gamma process with random effects for degradation data. Peng (2015) proposed a degradation model based on an inverse normal-gamma mixture of the IG process. Fang et al. (2022) studied a novel multivariate degradation model based on the inverse Gaussian process. As for the ED degradation process, Duan and Wang (2018) further generalized ED with random dispersion parameters and covariates, in which the EM algorithm is developed to obtain the maximum likelihood estimation and Birnbaum–Saunder distribution is used to approximate the life-time distributions. However, the dispersion parameters only affect the variance of the increment instead of the mean. Chen et al. (2021) proposed the TED process with both random drifts and dispersions and adopted the variational inference for parameter inference. The TED is a generalized stochastic process that includes the Wiener, gamma, and IG processes as its special cases. The TED process has independent increments, which are characterized by the TED distribution in Jørgensen (1997).

From the results of these studies, we can find that the TED process is more flexible than other processes since it incorporates the classical stochastic processes as special cases, by which the model selection problem turns into the parameter estimation problem dexterously. We also find that incorporating random effects into the TED models can achieve better performance in degradation modelling. For TED process with random effects, most statistical inferences for parameter estimation are based on the EM algorithm or variational inference. However, these methods are usually of great complexity and cannot provide the estimation results for random effect parameters. In this paper, we propose a TED model with random drifts for degradation data and a TED model with both random drifts and covariates for accelerated degradation data, respectively. The hierarchical Bayesian method is adopted to estimate the parameters, which can bring about the shrinkage of the random effects in that they share a common distribution and provide the posterior estimation of the random effects.

The rest of this paper is organized as follows. In Section 2. we introduce some basic properties of the TED process and its approximation method. Sections 3 and 4 give rise to the

TED process with random drifts and TED process with both random drifts and covariates along with their corresponding posterior inference algorithms and the distributions of failure time and remaining useful life (RUL). In Section 5, a simulation study is conducted to show the performance and superiority of the proposed models under mis-specification situations. In Section 6, the GaAs laser data and stress relaxation data examples are used to illustrate the validity of the proposed models. Finally, we conclude our findings in Section 7.

## 2. ED process and its approximation

### 2.1. ED process

Let $Y(t)$ be the degradation characteristics at measurement time $t$ and it can be modelled as an ED process, which has the following properties: (1) $Y(0) = 0$; (2) $\{Y(t) \mid t \geq 0\}$ has stationary and independent increments, i.e., $\Delta Y = Y(t + \Delta t) - Y(t)$ follows ED distribution $\mathrm{ED}(\mu \Delta t, \lambda)$ with probability density function (PDF)

$$f(\Delta y \mid \mu, \lambda, \Delta t) = c(\Delta y \mid \lambda, \Delta t) \exp\{\lambda[\Delta y\theta - \Delta t\kappa(\theta)]\},$$

where $c(\cdot)$ is the canonical function and $\kappa(\cdot)$ is the cumulant function, satisfying $\mu = \kappa'(\theta)$. $\mu$ is the drift parameter and $\lambda$ is the dispersion parameter. According to the moment generating function (MGF) for $\Delta Y$, we can obtain the expectation and variance. The MGF is derived as

$$\begin{aligned} M_{\Delta Y}(s) = E\left[\mathrm{e}^{s\Delta y}\right] &= \int \mathrm{e}^{s\Delta y} c(\Delta y \mid \lambda, \Delta t) \exp\{\lambda[\Delta y\theta - \Delta t\kappa(\theta)]\} \mathrm{d}\Delta y \\ &= \exp\left\{\lambda \Delta t\left[\kappa\left(\theta + \frac{s}{\lambda}\right) - \kappa(\theta)\right]\right\}. \end{aligned} \tag{1}$$

Then the expectation can be derived as $E(\Delta y \mid \mu, \lambda, \Delta t) = \frac{\mathrm{d}M_{\Delta Y}(s)}{\mathrm{d}s}|_{s=0} = \Delta t\kappa'(\theta) = \mu \Delta t$; the variance can be derived as $\mathrm{Var}(\Delta y \mid \mu, \lambda, \Delta t) = E(\Delta y^2 \mid \mu, \lambda, \Delta t) - [E(\Delta y \mid \mu, \lambda, \Delta t)]^2 = \frac{\mathrm{d}^2 M_{\Delta Y}(s)}{\mathrm{d}^2 s}|_{s=0} - \Delta t^2\kappa'(\theta)^2 = \Delta t^2\kappa'(\theta)^2 + \frac{\Delta t}{\lambda}\kappa''(\theta) - \Delta t^2\kappa'(\theta)^2 = \frac{\Delta t}{\lambda}\kappa''(\theta) = \frac{\Delta t V(\mu)}{\lambda}$, where $V(\mu) = \kappa''(\theta)$ is called the variance function.

An important family of ED process is proposed by Jørgensen (1997) with $V(\mu) = \mu^p, p \in (-\infty, 0] \cup [1, \infty)$. This family of ED process is called the Tweedie ED (TED) process. The commonly used stochastic processes are special cases of TED process, such as the Wiener process ($p = 0$), the gamma process ($p = 2$), the inverse Gaussian process ($p = 3$), the Poisson process ($p = 1$) and the compound Poisson process ($1 < p < 2$). Besides the Wiener process, the parameter $\mu$ controls both the expectation and partially the variation in the data. The distribution, expectation and variance of the increment for different processes are given in Table 1.

### 2.2. Saddlepoint approximation

Although the TED process can be used as a general stochastic process to model the degradation data, a significant challenge is that the canonical function $c(\cdot)$ and the cumulant function $k(\cdot)$ have no closed-form expressions. However, Jørgensen (1986, 1997) adopted the saddlepoint approximation (Daniels, 1954) method to approximate the PDF of the Tweedie ED distribution. The approximation results are given in the following lemma.

**Table 1.** The distribution, expectation and variance of the increment for different processes.

| Process | Distribution of the increment | $p$ | Expectation | Variance |
|---|---|---|---|---|
| ED process | $\mathrm{ED}(\mu\Delta t,\lambda), V(\mu)=\mu^p$ | – | $\mu\Delta t$ | $\frac{\mu^p\Delta t}{\lambda}$ |
| Wiener process | $\mathcal{N}(\mu\Delta t, t/\lambda)$ | 0 | $\mu\Delta t$ | $\frac{\Delta t}{\lambda}$ |
| gamma process | $\mathrm{Gamma}(\lambda\Delta t, \mu/\lambda)$ | 2 | $\mu\Delta t$ | $\frac{\mu^2\Delta t}{\lambda}$ |
| inverse Gaussian process | $\mathcal{IG}(\mu\Delta t, \lambda\Delta t^2)$ | 3 | $\mu\Delta t$ | $\frac{\mu^3\Delta t}{\lambda}$ |
| Poisson process | $\mathrm{Poisson}(\mu\Delta t), \lambda=1$ | 1 | $\mu\Delta t$ | $\mu\Delta t$ |

**Lemma 2.1.** *The PDF of the degradation increment of the TED process can be approximated using saddlepoint approximation method as follows:*

$$f(\Delta y\,|\,\mu,\lambda,p,\Delta t)\cong\sqrt{\frac{\lambda}{2\pi\Delta t^{1-p}y^p}}\exp\left[-\frac{\lambda\Delta t}{2}d\left(\frac{\Delta y}{\Delta t};\mu,p\right)\right],$$

*where*

$$d\left(\frac{\Delta y}{\Delta t};\mu,p\right)=\begin{cases}\left(\frac{\Delta y}{\Delta t}-\mu\right)^2, & p=0,\\ 2\left[\frac{\Delta y}{\Delta t}\log\left(\frac{\Delta y}{\mu\Delta t}\right)-\frac{\Delta y}{\Delta t}+\mu\right], & p=1,\\ 2\left[\log\left(\frac{\mu\Delta t}{\Delta y}\right)+\frac{\Delta y}{\mu\Delta t}-1\right], & p=2,\\ 2\left[\frac{\left(\frac{\Delta y}{\Delta t}\right)^{2-p}}{(1-p)(2-p)}-\frac{\frac{\Delta y}{\Delta t}\mu^{1-p}}{1-p}+\frac{\mu^{2-p}}{2-p}\right], & p\neq 0,1\text{ or }2.\end{cases}\tag{2}$$

The accuracy of the approximation has been discussed in some literature: (1) for $p=0,3$, the approximated results are the exact distributions of the TED model; (2) for $p\neq 0,3$, the approximated results are accurate (Duan & Wang, 2018; Dunn & Smyth, 2008). As shown in Table 1, for $p=1$ and $p=2$, the TED process reduces to the Poisson and gamma processes.

## 3. TED process with random drifts

### 3.1. Model

Denote the degradation observations as $Y_{i,1},\ldots,Y_{i,m_i}$ at times $0<t_{i,1}<\cdots<t_{i,m_i}$, where $i=1,\ldots,n$ and $m_i$ is the number of inspection measurements for the $i$th item. Considering the unit-to-unit variability, we assume that there are different degradation drift parameters for each path. Based on this specification, the degradation increment $\Delta Y_{i,j}\equiv Y_{i,j+1}-Y_{i,j}$ on the time interval $\Delta t_{i,j}(\equiv t_{i,j+1}-t_{i,j})$ follows $\mathrm{ED}(\mu_i,\lambda,p)$ with approximated PDF

$$f_{\Delta Y_{i,j}}(\Delta y_{i,j}\,|\,\mu_i,\lambda,p,\Delta t_{i,j})=\sqrt{\frac{\lambda}{2\pi\Delta t_{i,j}^{1-p}\Delta y_{i,j}^{p}}}\exp\left[-\frac{\lambda\Delta t_{i,j}}{2}d\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}};\mu_i,p\right)\right],\tag{3}$$

and $d(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}; \mu_i, p)$ is

$$d\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}; \mu_i, p\right) = \begin{cases} \left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}} - \mu_i\right)^2, & p = 0, \\ 2\left[\frac{\Delta y_{i,j}}{\Delta t_{i,j}} \log\left(\frac{\Delta y_{i,j}}{\mu_i \Delta t_{i,j}}\right) - \frac{\Delta y_{i,j}}{\Delta t_{i,j}} + \mu_i\right], & p = 1, \\ 2\left[\log\left(\frac{\mu_i \Delta t_{i,j}}{\Delta y_{i,j}}\right) + \frac{\Delta y_{i,j}}{\mu_i \Delta t_{i,j}} - 1\right], & p = 2, \\ 2\left[\frac{\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}\right)^{2-p}}{(1-p)(2-p)} - \frac{\frac{\Delta y_{i,j}}{\Delta t_{i,j}} \mu_i^{1-p}}{1-p} + \frac{\mu_i^{2-p}}{2-p}\right], & p \neq 0, 1 \text{ or } 2. \end{cases} \tag{4}$$

For real data analysis, we can discuss the results for these four scenarios by model selection methods. Thus, without loss of generality, we use the fourth scenario ($p \neq 0, 1$ or 2) for statistical inference. Based on the derivation above, the joint density of random increment of $\mathbf{\Delta} Y \equiv (\mathbf{\Delta} Y_1, \ldots, \mathbf{\Delta} Y_n)^\top$ with $\mathbf{\Delta} Y_i \equiv (\Delta Y_{i,1}, \ldots, \Delta Y_{i,m_i})^\top$ is

$$f_{\mathbf{\Delta} Y}(\mathbf{\Delta} y \mid \boldsymbol{\mu}, \lambda, p) = \prod_{i=1}^{n} \prod_{j=1}^{m_i - 1} f_{\Delta Y_{i,j}}(\Delta y_{i,j} \mid \mu_i, \lambda, p, \Delta t_{i,j}), \tag{5}$$

where $\mathbf{\Delta} y \equiv (\mathbf{\Delta} y_1, \ldots, \mathbf{\Delta} y_n)^\top$, $\mathbf{\Delta} y_i \equiv (\Delta y_{i,1}, \ldots, \Delta y_{i,m_i})^\top$ and $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_n)^\top$.

### 3.2. Prior specification

The hierarchical Bayesian approach can flexibly and effectively estimate the model parameters with random effects, which specifies a common distribution for the unit-specific random parameters. The similarities of the data bring about the correlations of drift parameters for each items. To be specific, the prior distributions are specified as follows.

(1) The drift parameters $\mu_i$ are assumed to follow a normal distribution with mean $\eta$ and variance $\sigma^2$; that is, $\pi(\mu_i) \sim \mathcal{N}(\eta, \sigma^2)$, where $\mathcal{N}(\cdot)$ denotes a normal distribution.
(2) For $(\eta, \sigma^2)^\top$, we consider the normal-inverse gamma distributions (Bernardo & Smith, 1994) $\mathcal{NIG}(\lambda_\mu, \eta_\mu, \nu_\mu, \xi_\mu)$ with hyperparameter vector (0.0001, 10, 0.0001, 0.0001), which is low informative prior.
(3) Assign a gamma distribution Gamma$(\alpha, \beta)$ with $\alpha = \beta = 0.0001$ to $\lambda$.
(4) The normal distribution is assigned to $p \sim \mathcal{N}(\gamma, \delta^2)$.

### 3.3. Posterior distributions and Gibbs algorithm

After specifying the model and prior distributions, the posterior distributions can be derived according to the Bayes' theorem. Let $\mathbf{\Theta} \equiv (\boldsymbol{\mu}, \eta, \sigma^2, \lambda, p)^\top$ denote the set of the parameter vector in the hierarchical Bayesian ED model. The resulting joint posterior distribution of $\mathbf{\Theta}$

is

$$\begin{aligned}\pi(\mathbf{\Theta}\,|\,y) &\propto \pi(\eta,\sigma^2\,|\,\lambda_\mu,\eta_\mu,\nu_\mu,\xi_\mu)\pi(\lambda\,|\,\alpha,\beta)\pi(p\,|\,\gamma,\delta^2)\pi(\boldsymbol{\mu}\,|\,\eta,\sigma^2)f_{\mathbf{\Delta}Y}(\mathbf{\Delta}y\,|\,\boldsymbol{\mu},\eta,\sigma^2,\lambda,p)\\ &\propto (\sigma^2)^{-(1/2+\nu_\mu+1)}\exp\left\{-\frac{2\xi_\mu+\lambda_\mu(\eta-\eta_\mu)^2}{2\sigma^2}\right\}\lambda^{\alpha-1}\exp(-\beta\lambda)\\ &\quad\times\exp\left\{-\frac{(p-\gamma)^2}{2\delta^2}\right\}\prod_{i=1}^{n}\frac{1}{\sigma}\exp\left\{-\frac{(\mu_i-\eta)^2}{2\sigma^2}\right\}\prod_{j=1}^{m_i-1}\sqrt{\frac{\lambda}{2\pi\,\Delta t_{i,j}^{1-p}\,\Delta y_{i,j}^{p}}}\\ &\quad\times\exp\left[-\frac{\lambda\Delta t_{i,j}}{2}d\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}};\mu_i,p\right)\right].\end{aligned}\tag{6}$$

The Metropolis Hasting within Gibbs sampler (MHGS) shown in Algorithm 1 can be implemented to generate posterior samples of parameters with their full conditional posterior distributions given in Appendix A. Specifically, the parameters in $\mathbf{\Theta}$ except $p$ and $\mu_i$ have full conditional distributions of closed-form, and thus they can be updated directly. For $p$ and $\mu_i$, $i=1,\ldots,n$, the MH algorithm can be used to obtain their posterior samples.

### 3.4. *Failure-time distribution and RUL distribution*

The failure-time $T$ is defined as the first hitting time at which the degradation path passes the predefined threshold $D_f$, i.e., $T=\{t\,|\,Y(t)\geq D_f, Y(0)<D_f\}$. Hong and Ye (2017) have obtained that the failure-time distribution for a process with drift $\mu$ and volatility $\sigma$ can be approximated by a Birnbaum–Saunders (BS) distribution (Balakrishnan & Kundu, 2019) $\mathcal{BS}(\frac{\sigma}{\sqrt{\mu D_f}},\frac{D_f}{\mu})$. The failure-time distribution and RUL distribution for our proposed TED process with random drifts are given in the following two theorems.

**Theorem 3.1.** *If the degradation path $Y_i(t)$ follows the TED process defined in Section 3.1, then the cumulative distribution function (CDF) of the failure-time $\mathcal{E}[T_i]=\frac{\mu_i^{p-2}}{2\lambda}+\frac{D_f}{\mu_i}$*

***Proof:*** According to the results obtained by Hong and Ye (2017), the ED degradation process $\{Y_i(t)\,|\,t\geq 0\}$ has drift parameter $\mu_i$ and volatility parameter $\vartheta=\frac{\mu_i^{\frac{p}{2}}}{\lambda^{\frac{1}{2}}}$. The lifetime $T_i$ can be approached by the BS distribution with CDF

$$F_{T_i}(t)\cong\Phi\left(\frac{\mu_i\sqrt{t}}{\vartheta}-\frac{D_f}{\sqrt{t}\vartheta}\right).$$

By replacing $\vartheta=\frac{\mu_i^{\frac{p}{2}}}{\lambda^{\frac{1}{2}}}$, we obtain

$$F_{T_i}(t)\cong\Phi\left[\frac{\lambda^{\frac{1}{2}}}{\mu_i^{\frac{p}{2}}}\left(\mu_i\sqrt{t}-\frac{D_f}{t}\right)\right].$$

**Algorithm 1:** MHGS algorithm for TED process with random drifts.

**input :** The number of MHGS iteration times $T$, proposal parameter $s_\theta$, MH algorithm iteration times $N$, and the initial value $\boldsymbol{\Theta}^{(0)} \leftarrow (\boldsymbol{\mu}^{(0)}, \eta^{(0)}, \sigma^{2(0)}, \lambda^{(0)}, p^{(0)})$.

**output:** The posterior sample of $\boldsymbol{\Theta}$.

**for** $t \leftarrow 1$ **to** $T$ **do**

Draw $(\eta^{(t)}, \sigma^{2(t)})$ from $\mathcal{NIG}(\lambda_\mu'^{(t-1)}, \eta_\mu'^{(t-1)}, \nu_\mu'^{(t-1)}, \xi_\mu'^{(t-1)})$, where $\lambda_\mu'^{(t-1)} = \lambda_\mu + n$, $\eta_\mu'^{(t-1)} = (\lambda_\mu \eta_\mu + \sum_{i=1}^n \mu_i^{(t-1)})/(\lambda_\mu + n)$, $\nu_\mu'^{(t-1)} = n/2 + \nu_\mu$, and $\xi_\mu'^{(t-1)} = \xi_\mu + \lambda_\mu \eta_\mu^2/2 + \sum_{i=1}^n \mu_i^{(t-1)2}/2 - (\lambda_\mu \eta_\mu + \sum_{i=1}^n \mu_i^{(t-1)})^2/(2(\lambda_\mu + n))$ ;

Draw $\lambda^{(t)}$ from Gamma$(\alpha + \frac{\sum_{i=1}^n m_i - n}{2}, \beta + \sum_{i=1}^n \sum_{j=1}^{m_i - 1} \frac{\Delta t_{i,j}}{2} d(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}; \mu_i^{(t-1)}, p^{(t-1)}))$;

$p^{\text{cur}} \leftarrow p^{(t-1)}$ **for** $g \leftarrow 1$ **to** $N$ **do**

Proposed $p'$ from $\mathcal{N}(p^{\text{cur}}, s_\theta^p)$;

Calculatelog $\alpha = \log \frac{\pi(p' | \boldsymbol{\mu}^{(t-1)}, \eta^{(t)}, \sigma^{2(t)}, \lambda^{(t)}, \boldsymbol{\Delta y})}{\pi(p^{\text{cur}} | \boldsymbol{\mu}^{(t-1)}, \eta^{(t)}, \sigma^{2(t)}, \lambda^{(t)}, \boldsymbol{\Delta y})}$;

Draw $u$ from $U(0,1)$;

**if** $\log u < \log \alpha$ **then** $p^{\text{cur}} \leftarrow p'$;

**end**

$p^{(t)} \leftarrow p^{\text{cur}}$ **for** $i \leftarrow 1$ **to** $n$ **do**

$\mu_i^{\text{cur}} \leftarrow \mu_i^{(t-1)}$ **for** $g \leftarrow 1$ **to** $N$ **do**

Proposed $\mu_i' \sim N(\mu_i^{\text{cur}}, s_\theta^\mu)$;

Calculatelog $\alpha =$

$\log \frac{\pi(\mu_i' | \mu_1^{(t)}, \ldots, \mu_{i-1}^{(t)}, \mu_{i+1}^{(t-1)}, \ldots, \mu_n^{(t-1)}, \eta^{(t)}, \sigma^{2(t)}), \lambda^{(t)}, p^{(t)}, \boldsymbol{\Delta y})}{\pi(\mu_i^{\text{cur}} | \mu_1^{(t)}, \ldots, \mu_{i-1}^{(t)}, \mu_{i+1}^{(t-1)}, \ldots, \mu_n^{(t-1)} \eta, \sigma^{2(t)}, \lambda^{(t)}, p^{(t)}, \boldsymbol{\Delta y})}$;

Draw $u$ from $U(0,1)$;

**if** $\log u < \log \alpha$ **then** $\mu_i^{\text{cur}} \leftarrow \mu_i'$;

**end**

$\mu_i^{(t)} \leftarrow \mu_i^{\text{cur}}$

**end**

$\boldsymbol{\Theta}^{(t)} \leftarrow (\boldsymbol{\mu}^{(t)}, \eta^{(t)}, \sigma^{2(t)}, \lambda^{(t)}, p^{(t)})^\top$.

**end**

For the two-parameter $\mathcal{BS}(\varsigma, \psi)$ (Balakrishnan & Kundu, 2019), the CDF can be written as

$$\Phi \left[ \frac{1}{\varsigma} \left\{ \left( \frac{t}{\psi} \right)^{1/2} - \left( \frac{\psi}{t} \right)^{1/2} \right\} \right].$$

Thus, we have

$$\begin{cases} \dfrac{1}{\varsigma\,\psi^{1/2}} = \dfrac{\lambda^{1/2}}{\mu_i^{p/2-1}}, \\ \dfrac{\psi^{1/2}}{\varsigma} = \dfrac{\lambda^{1/2} D_f}{\mu_i^{p/2}}. \end{cases} \tag{7}$$

By solving the equations above, we have

$$\varsigma = \frac{\mu_i^{\frac{p-1}{2}}}{\sqrt{\lambda D_f}}, \psi = \frac{D_f}{\mu_i}. \tag{8}$$

Then we obtain $T \sim \mathcal{BS}\left(\frac{\mu_i^{\frac{p-1}{2}}}{\sqrt{\lambda D_f}}, \frac{D_f}{\mu_i}\right)$. According to the results given by Balakrishnan and Kundu (2019), the expectation of the $\mathcal{BS}(\varsigma, \psi)$ is

$$\mathcal{E}(T_i) = \frac{\psi}{2}(\varsigma^2 + 2) = \frac{D_f}{2\mu_i}\left(\frac{\mu_i^{p-1}}{\lambda D_f} + 2\right) = \frac{\mu_i^{p-2}}{2\lambda} + \frac{D_f}{\mu_i}.$$

■

**Theorem 3.2.** *Define the RUL, $R_{it}$, of the ith unit at time t as* $R_{it} = \inf\{r_{it} > 0 \mid Y_i(t + r_{it}) \geq D_f, Y_i(t) < D_f\}$, *and then* $R_{it} \sim \mathcal{BS}\left(\frac{\mu_i^{\frac{p-1}{2}}}{\sqrt{\lambda(D_f - Y_i(t))}}; \frac{D_f - Y_i(t)}{\mu_i}\right)$ *and the mean residual life (MRL) function for the ith unit* $\mathcal{E}[R_{it}] = \frac{\mu_i^{p-2}}{2\lambda} + \frac{D_f - Y_i(t)}{\mu_i}$.

The proof of Theorem 3.2 can be easily obtained analogy to Theorem 3.1.

## 4. TED process with random drifts and covariates

### 4.1. Model

In this section, we consider the accelerated ED degradation model with random drifts and covariates. Assume that there are $K$ stress levels, $S_k$

Considering the unit-to-unit variability, we assume that there is different degradation drift parameter for each path. Based on this specification, the degradation increments $\Delta Y_{k,i,j} \equiv Y_{k,i,j+1} - Y_{k,i,j}$ on the time interval $\Delta t_{k,i,j} (\equiv t_{k,i,j+1} - t_{k,i,j})$ follow ED$(\mu_{ki}, \lambda, p)$. Then the PDF of $\Delta Y_{k,i,j}$ can be written as follows:

$$f_{\Delta Y_{k,i,j}}(\Delta y_{k,i,j} \mid \mu_{ki}, \lambda, p, \Delta t_{k,i,j}) = \sqrt{\frac{\lambda}{2\pi\,\Delta t_{k,i,j}^{1-p}\,\Delta y_{k,i,j}^{p}}}\,\exp\left[-\frac{\lambda\,\Delta t_{k,i,j}}{2} d\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}; \mu_{ki}, p\right)\right], \tag{9}$$

and $d(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}; \mu_{ki}, p)$ is

$$d\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}; \mu_{ki}, p\right) = \begin{cases} \left(\dfrac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}} - \mu_{ki}\right)^2, & p = 0, \\ 2\left[\dfrac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}} \log\left(\dfrac{\Delta y_{k,i,j}}{\mu \Delta t_{k,i,j}}\right) - \dfrac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}} + \mu_{ki}\right], & p = 1, \\ 2\left[\log\left(\dfrac{\mu_{ki}\Delta t_{k,i,j}}{\Delta y_{k,i,j}}\right) + \dfrac{\Delta y_{k,i,j}}{\mu_{ki}\Delta t_{k,i,j}} - 1\right], & p = 2, \\ 2\left[\dfrac{\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\right)^{2-p}}{(1-p)(2-p)} - \dfrac{\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\mu_{ki}^{1-p}}{1-p} + \dfrac{\mu_{ki}^{2-p}}{2-p}\right], & p \neq 0, 1 \text{ or } 2. \end{cases} \tag{10}$$

Based on the derivation above, the joint density of random increment of $\mathbf{\Delta} Y \equiv (\mathbf{\Delta} Y_1 \ldots, \mathbf{\Delta} Y_K)^\top$ for $\mathbf{\Delta} Y_k \equiv (\Delta Y_{k,1,1}, \ldots, \Delta Y_{k,1,m_k}, \ldots, \Delta Y_{k,n_k,1}, \ldots, \Delta Y_{k,n_k,m_k})^\top$ is

$$f_{\mathbf{\Delta} Y}(\mathbf{\Delta} y \mid \boldsymbol{\mu}, \lambda, p) = \prod_{k=1}^{K}\prod_{i=1}^{n_k}\prod_{j=1}^{m_k} f_{\Delta Y_{k,i,j}}(\Delta y_{k,i,j} \mid \mu_{ki}, \lambda, p, \Delta t_{k,i,j}), \tag{11}$$

for $\mathbf{\Delta} y \equiv (\mathbf{\Delta} y_1, \ldots, \mathbf{\Delta} y_K)^\top$, where $\mathbf{\Delta} y_k \equiv (\Delta y_{k,1,1}, \ldots, \Delta y_{k,1,m_k}, \ldots, \Delta y_{k,n_k,1}, \ldots, \Delta y_{k,n_k,m_k})^\top$ and $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K)$, $\boldsymbol{\mu}_k = (\mu_{k1}, \ldots, \mu_{kn_k})$.

### 4.2. Prior specification

Analogous to the prior specification for ED models with random drifts, we also assume that the drift parameters are random and follow the same distribution for each path under the same stress level. To be specific, the prior distributions are specified as follows.

(1) The drifts $\mu_{ki}$ are assumed to follow a normal distribution with mean $\mu_0 \exp(\beta\Phi(S_k))$ and variance $\sigma_k^2$; that is, $\pi(\mu_{ki}) \sim \mathcal{N}(\mu_0 \exp(\beta\Phi(S_k)), \sigma_k^2)$, $k = 1, \ldots, K$. $\sigma_k^2$ is assumed to follow inverse gamma distribution $\mathcal{IG}(\kappa, \eta)$, where $\kappa = 2.0001$, $\eta = 1$.
(2) Assign a gamma distribution Gamma$(\alpha, \xi)$ with $\alpha = \xi = 0.0001$ to $\lambda$.
(3) The normal distribution is assigned to $p \sim \mathcal{N}(\gamma, \delta^2)$.
(4) $\mu_0$ is assumed to follow $\mathcal{N}(\vartheta_\mu, \phi_\mu^2)$, and $\beta$ is assumed to follow $\mathcal{N}(\vartheta_\beta, \phi_\beta^2)$ with $\vartheta_\mu = \vartheta_\beta = 0, \phi_\mu = \phi_\beta = 100$.

### 4.3. Posterior distribution

After specifying the model and prior distributions, the posterior distributions can be derived according to the Bayes' theorem. Let $\mathbf{\Theta} \equiv (\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \lambda, p, \mu_0, \beta)^\top$ denote the set of the parameter vector in the hierarchical Bayesian TED process with random drifts and covariates, where $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K)^\top$, $\boldsymbol{\mu}_k = (\mu_{k1}, \ldots, \mu_{k,n_k})^\top$ and $\boldsymbol{\sigma}_k^2 = (\sigma_1^2, \ldots, \sigma_K^2)^\top$. The resulting joint posterior distribution of $\boldsymbol{\Theta}$ is

$$\pi(\boldsymbol{\Theta} \mid y) \propto \pi(\mu_0)\pi(\beta)\pi(\lambda \mid \alpha, \xi)\pi(p \mid \gamma, \delta^2)\pi(\boldsymbol{\sigma}^2 \mid \kappa, \eta)\pi(\boldsymbol{\mu} \mid \mu_0, \boldsymbol{\sigma}^2) f_{\Delta Y}(\boldsymbol{\Delta y} \mid \boldsymbol{\mu}, \boldsymbol{\eta}, \boldsymbol{\sigma}^2, \lambda, p)$$
$$\propto \exp\left\{-\frac{(\mu_0 - \vartheta_\mu)^2}{2\phi_\mu^2}\right\} \exp\left\{-\frac{(\beta - \vartheta_\beta)^2}{2\phi_\beta^2}\right\} \lambda^{\alpha-1} \exp(-\xi\lambda) \exp\left\{-\frac{(p-\gamma)^2}{2\delta^2}\right\}$$
$$\times \prod_{k=1}^{K}\left[(\sigma_k^2)^{-(\kappa+1)} \exp\left\{-\frac{\eta}{\sigma_k^2}\right\}\left\{\prod_{i=1}^{n_k}\left(\frac{1}{\sigma_k}\exp\left[-\frac{\{\mu_{ki} - \mu_0 \exp(\beta\phi(S_k))\}^2}{2\sigma_k^2}\right]\right.\right.\right.$$
$$\left.\left.\left.\times \prod_{j=1}^{m_k-1}\sqrt{\frac{\lambda}{\Delta t_{k,i,j}^{1-p}\Delta y_{k,i,j}^{p}}} \exp\left[-\frac{\lambda \Delta t_{k,i,j}}{2} d\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}; \mu_{ki}, p\right)\right]\right)\right\}\right]. \tag{12}$$

The MHGS shown in Algorithm 2 can be implemented to generate posterior samples of parameters with their full conditional posterior distributions given in Appendix B.

**Table 2.** True values of the parameters and their corresponding AB, MSE and CP by using four models under three scenarios in the simulation study.

| | | | Wiener | | | gamma | | | IG | | | TED | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Scenario | Parameter | true | AB | MSE | CP | AB | MSE | CP | AB | MSE | CP | AB | MSE | CP |
| I (Wiener) | $\mu_1$ | 1.720 | 0.089 | 0.012 | 0.940 | 0.101 | 0.016 | 0.990 | 0.141 | 0.026 | 1.000 | 0.089 | 0.013 | 0.940 |
| | $\mu_2$ | 2.082 | 0.073 | 0.009 | 0.970 | 0.063 | 0.007 | 1.000 | 0.054 | 0.005 | 1.000 | 0.073 | 0.009 | 0.970 |
| | $\mu_3$ | 1.626 | 0.098 | 0.015 | 0.890 | 0.120 | 0.022 | 0.970 | 0.169 | 0.037 | 0.990 | 0.100 | 0.016 | 0.890 |
| | $\mu_4$ | 2.713 | 0.102 | 0.016 | 0.930 | 0.189 | 0.042 | 0.990 | 0.292 | 0.089 | 1.000 | 0.106 | 0.017 | 0.940 |
| | $\mu_5$ | 2.147 | 0.080 | 0.010 | 0.970 | 0.066 | 0.007 | 1.000 | 0.053 | 0.004 | 1.000 | 0.079 | 0.009 | 0.980 |
| | $\mu_6$ | 1.633 | 0.101 | 0.015 | 0.900 | 0.128 | 0.022 | 0.990 | 0.179 | 0.039 | 1.000 | 0.103 | 0.015 | 0.920 |
| | $\mu_7$ | 2.218 | 0.077 | 0.009 | 0.970 | 0.067 | 0.007 | 1.000 | 0.059 | 0.005 | 1.000 | 0.076 | 0.009 | 0.990 |
| | $\mu_8$ | 2.330 | 0.100 | 0.016 | 0.900 | 0.095 | 0.014 | 0.990 | 0.104 | 0.015 | 1.000 | 0.099 | 0.016 | 0.920 |
| | $\lambda$ | 16.000 | 0.561 | 0.459 | 1.000 | 5.949 | 35.720 | 0.000 | 6.835 | 47.524 | 0.010 | 0.807 | 0.833 | 1.000 |
| II (gamma) | $\mu_1$ | 1.720 | 0.164 | 0.042 | 0.920 | 0.171 | 0.042 | 0.950 | 0.200 | 0.054 | 0.980 | 0.202 | 0.057 | 0.920 |
| | $\mu_2$ | 2.082 | 0.178 | 0.047 | 0.970 | 0.160 | 0.038 | 0.990 | 0.125 | 0.023 | 1.000 | 0.156 | 0.037 | 0.990 |
| | $\mu_3$ | 1.626 | 0.200 | 0.064 | 0.870 | 0.206 | 0.065 | 0.880 | 0.244 | 0.082 | 0.950 | 0.244 | 0.087 | 0.840 |
| | $\mu_4$ | 2.713 | 0.305 | 0.137 | 0.730 | 0.283 | 0.115 | 0.890 | 0.351 | 0.147 | 0.950 | 0.322 | 0.142 | 0.800 |
| | $\mu_5$ | 2.147 | 0.180 | 0.051 | 0.940 | 0.156 | 0.039 | 0.990 | 0.118 | 0.022 | 1.000 | 0.151 | 0.038 | 0.970 |
| | $\mu_6$ | 1.633 | 0.201 | 0.062 | 0.930 | 0.208 | 0.063 | 0.940 | 0.257 | 0.084 | 0.960 | 0.249 | 0.085 | 0.880 |
| | $\mu_7$ | 2.218 | 0.194 | 0.061 | 0.860 | 0.169 | 0.047 | 0.980 | 0.133 | 0.029 | 1.000 | 0.163 | 0.044 | 0.970 |
| | $\mu_8$ | 2.330 | 0.233 | 0.084 | 0.820 | 0.200 | 0.062 | 0.950 | 0.172 | 0.043 | 1.000 | 0.200 | 0.062 | 0.950 |
| | $\lambda$ | 10.000 | 6.603 | 43.876 | 0.000 | 0.504 | 0.427 | 0.990 | 1.528 | 3.169 | 0.690 | 0.712 | 0.731 | 0.970 |
| III (IG) | $\mu_1$ | 1.720 | 0.208 | 0.076 | 0.960 | 0.228 | 0.083 | 0.930 | 0.230 | 0.082 | 0.950 | 0.254 | 0.101 | 0.890 |
| | $\mu_2$ | 2.082 | 0.220 | 0.075 | 0.970 | 0.223 | 0.079 | 0.970 | 0.179 | 0.052 | 0.990 | 0.204 | 0.066 | 0.980 |
| | $\mu_3$ | 1.626 | 0.234 | 0.085 | 0.930 | 0.237 | 0.086 | 0.870 | 0.262 | 0.099 | 0.920 | 0.289 | 0.123 | 0.830 |
| | $\mu_4$ | 2.713 | 0.426 | 0.245 | 0.700 | 0.359 | 0.176 | 0.850 | 0.358 | 0.166 | 0.920 | 0.392 | 0.205 | 0.870 |
| | $\mu_5$ | 2.147 | 0.214 | 0.081 | 0.920 | 0.213 | 0.079 | 0.960 | 0.167 | 0.049 | 0.990 | 0.189 | 0.065 | 0.970 |
| | $\mu_6$ | 1.633 | 0.207 | 0.069 | 0.930 | 0.214 | 0.069 | 0.920 | 0.232 | 0.079 | 0.940 | 0.267 | 0.104 | 0.900 |
| | $\mu_7$ | 2.218 | 0.294 | 0.127 | 0.840 | 0.295 | 0.125 | 0.870 | 0.234 | 0.079 | 1.000 | 0.243 | 0.089 | 0.970 |
| | $\mu_8$ | 2.330 | 0.296 | 0.132 | 0.830 | 0.274 | 0.117 | 0.880 | 0.222 | 0.077 | 0.990 | 0.252 | 0.095 | 0.940 |
| | $\lambda$ | 10.000 | 8.194 | 67.425 | 0.000 | 1.956 | 4.175 | 0.210 | 0.583 | 0.515 | 1.000 | 0.738 | 0.757 | 0.970 |

Specifically, the parameters $\beta$, $p$ and $\mu_{ki}$ in $\mathbf{\Theta}$ have no closed-form full conditional distributions, and thus they can be updated directly. To obtain the posterior samples of $\beta$, $p$ and $\mu_{ki}$, $k = 1, \ldots, K, i = 1, \ldots, n_k$, the MH algorithm can be used.

---

**Algorithm 2:** MHGS algorithm for TED process with both random drifts and covariates.

---

**input** : The number of MHGS iteration times $T$, proposal parameter $s_\theta$, MH algorithm iteration times $N$, and the initial value $\mathbf{\Theta}^{(0)} \leftarrow (\boldsymbol{\mu}^{(0)}, \boldsymbol{\sigma}^{2(0)}, \lambda^{(0)}, p^{(0)}, \mu_0^{(0)}, \beta^{(0)})$.

**output** : The posterior sample of $\mathbf{\Theta}$.

**for** $t \leftarrow 1$ **to** $T$ **do**

  Draw $\mu_0^{(t)}$ from $\mathcal{N}(\frac{B^{(t-1)}}{A^{(t-1)}}, \frac{1}{A^{(t-1)}})$, $A^{(t-1)} = \frac{1}{\phi_\mu^2} + \sum_{k=1}^{K} \sum_{i=1}^{n_k} \frac{\exp(2\beta^{(t-1)}\phi(S_k))}{\sigma_k^{2(t-1)}}$,

  $B^{(t-1)} = \frac{\vartheta_\mu}{\phi_\mu^2} + \sum_{k=1}^{K} \sum_{i=1}^{n_k} \frac{\exp(\beta^{(t-1)}\phi(S_k))\mu_{ki}^{(t-1)}}{\sigma_k^{2(t-1)}}$ ;

  **for** $k \leftarrow 1$ **to** $n$ **do**

    Draw $\sigma_k^{2(t)}$ from $\mathcal{IG}(\kappa + \frac{n_k}{2}, \eta^{(t-1)} + \frac{1}{2} \sum_{i=1}^{n_k} \{\mu_{ki}^{(t-1)} - \mu_0^{(t)} \exp(\beta^{(t-1)}\phi(S_k))\}^2)$;

  **end**

  Draw $\lambda^{(t)}$ from

  Gamma$(\alpha + \frac{\sum_{k=1}^{K} n_k(m_k-1)}{2}, \xi + \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_k-1} \frac{\Delta t_{k,i,j}}{2} d(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}; \mu_{ki}^{(t-1)}, p^{(t-1)}))$;

  $\beta^{\text{cur}} \leftarrow \beta^{(t-1)}$;

  **for** $g \leftarrow 1$ **to** $N$ **do**

    Proposed $\beta'$ from $\mathcal{N}(\beta^{\text{cur}}, s_\theta^\beta)$;

    Calculatelog $\alpha = \log \frac{\pi(\beta' | \boldsymbol{\mu}^{(t-1)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, p^{(t-1)}, \mu_0^{(t)}, \boldsymbol{\Delta} y)}{\pi(\beta^{\text{cur}} | \boldsymbol{\mu}^{(t-1)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, p^{(t-1)}, \mu_0^{(t)}, \boldsymbol{\Delta} y)}$;

    Draw $u$ from $U(0, 1)$;

    **if** $\log u < \log \alpha$ **then** $\beta^{\text{cur}} \leftarrow \beta'$;

  **end**

  $\beta^{(t)} \leftarrow \beta^{\text{cur}} p^{\text{cur}} \leftarrow p^{(t-1)}$;

  **for** $g \leftarrow 1$ **to** $N$ **do**

    Proposed $p'$ from $\mathcal{N}(p^{\text{cur}}, s_\theta^p)$;

    Calculatelog $\alpha = \log \frac{\pi(p' | \boldsymbol{\mu}^{(t-1)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, \mu_0^{(t)}, \beta^{(t)}, \boldsymbol{\Delta} y)}{\pi(p^{\text{cur}} | \boldsymbol{\mu}^{(t-1)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, \mu_0^{(t)}, \beta^{(t)}, \boldsymbol{\Delta} y)}$;

    Draw $u$ from $U(0, 1)$;

    **if** $\log u < \log \alpha$ **then** $p^{\text{cur}} \leftarrow p'$;

  **end**

  $p^{(t)} \leftarrow p^{\text{cur}}$;

  **for** $k \leftarrow 1$ **to** $K$ **do**

    **for** $i \leftarrow 1$ **to** $n_k$ **do**

      $\mu_{ki}^{\text{cur}} \leftarrow \mu_{ki}^{(t-1)}$**for** $g \leftarrow 1$ **to** $N$ **do**

        Proposed $\mu'_{ki} \sim N(\mu_{ki}^{\text{cur}}, s_\theta^\mu)$;

        Calculatelog $\alpha = \log \frac{\pi(\mu'_{ki} | \mu_{11}^{(t)}, \ldots, \mu_{ki-1}^{(t)}, \mu_{ki+1}^{(t-1)}, \ldots, \mu_{Kn_K}^{(t)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, p^{(t)}, \mu_0^{(t)}, \beta^{(t)}, \boldsymbol{\Delta} y)}{\pi(\mu_{ki}^{\text{cur}} | \mu_{11}^{(t)}, \ldots, \mu_{ki-1}^{(t)}, \mu_{ki+1}^{(t-1)}, \ldots, \mu_{Kn_K}^{(t)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, p^{(t)}, \mu_0^{(t)}, \beta^{(t)}, \boldsymbol{\Delta} y)}$;

        Draw $u$ from $U(0, 1)$;

        **if** $\log u < \log \alpha$ **then** $\mu_{ki}^{\text{cur}} \leftarrow \mu'_{ki}$;

      **end**

      $\mu_{ki}^{(t)} \leftarrow \mu_{ki}^{\text{cur}}$

    **end**

  **end**

  $\mathbf{\Theta}^{(t)} \leftarrow (\boldsymbol{\mu}^{(t)}, \boldsymbol{\sigma}^{2(t)}, \lambda^{(t)}, p^{(t)}, \mu_0^{(t)}, \beta^{(t)})^\top$.

**end**

---

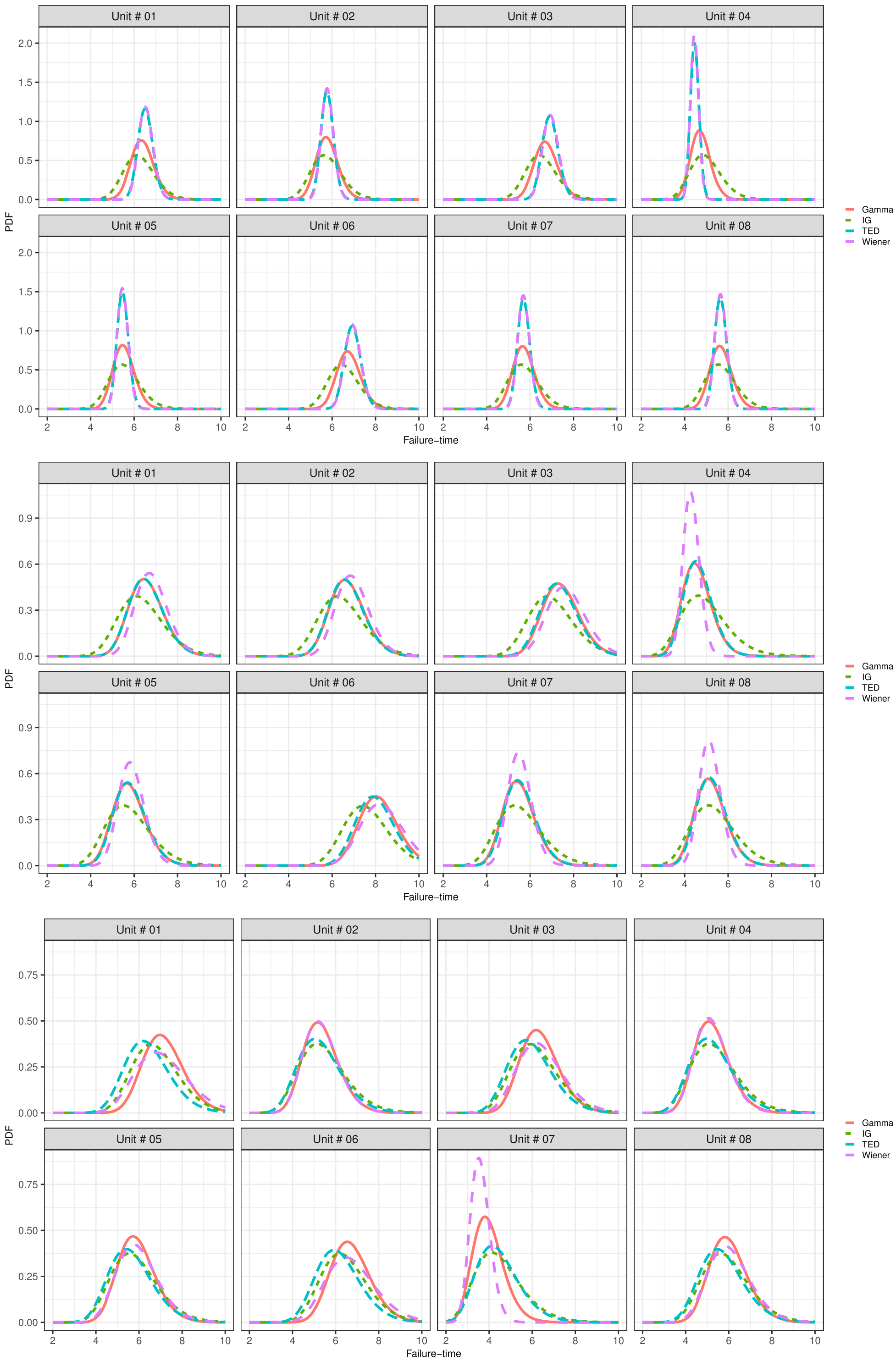


**Figure 1.** PDFs of failure-time based on Wiener process, gamma process, IG process and TED process model and BS distribution: Upper panel is for scenario I; Middle panel is for scenario II; Lower panel is for scenario III in the simulation study.

### 4.4. Failure-time distribution and RUL distribution

Similar to Theorems 3.1 and 3.2, we can obtain the failure-time distribution and RUL distribution for a TED process $\{Y_0(t) \mid t \geq 0\}$ under usage stress. Its corresponding degradation paths under accelerated stress are assumed to follow TED process with random drifts and covariates.

**Theorem 4.1.** *The failure-time* $T = \inf\{t \mid Y_0(t) \geq D_f\}$ *for a TED process* $\{Y_0(t) \mid t \geq 0\}$ *under usage stress with its corresponding degradation paths under accelerated stress following TED process with random drifts and covariates defined in Section 4.1, follows* $\mathcal{BS}\left(\frac{\mu_0^{\frac{p-1}{2}}}{\sqrt{\lambda D_f}}; \frac{D_f}{\mu_0}\right)$, *and the MTTF is* $\mathcal{E}[T] = \frac{\mu_0^{p-2}}{2\lambda} + \frac{D_f}{\mu_0}$.

**Theorem 4.2.** *For a TED process* $\{Y_0(t) \mid t \geq 0\}$ *under usage stress, corresponding degradation paths follow TED process with random drifts and covariates under accelerated stress defined in Section 4.1. The RUL,* $R_t$*, at time* $t$ *is defined as* $R_t = \inf\{r_t > 0 \mid Y_0(t + r_t) \geq D_f, Y_0(t) < D_f\}$*, and then* $R_t \sim \mathcal{BS}(\frac{\mu_0^{\frac{p-1}{2}}}{\sqrt{\lambda D_f}}; \frac{D_f - Y(t)}{\mu_0})$ *and the MRL function is* $\mathcal{E}[R_t] = \frac{\mu_0^{p-2}}{2\lambda} + \frac{D_f - Y_0(t)}{\mu_0}$.

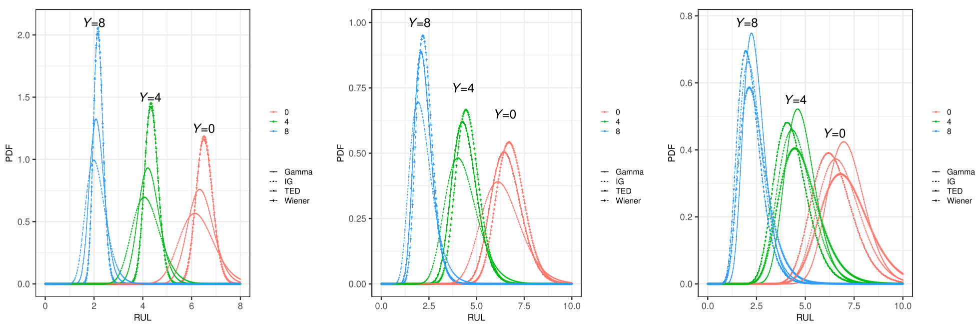


**Figure 2.** PDFs of RUL based on Wiener process, gamma process, IG process and TED process when the current degradations are 0, 4 and 8, respectively: Left panel is for scenario I; Middle panel is for scenario II; Right panel is for scenario III in the simulation study.

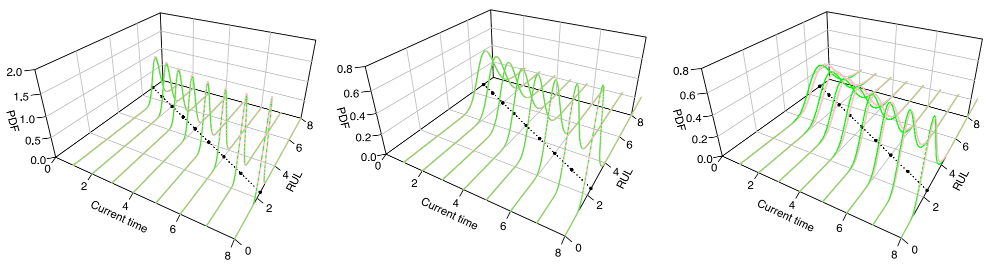


**Figure 3.** The pink solid line and green dashed line are the PDFs of the RUL distribution derived based on the estimated parameters obtained from true model and TED process, respectively. The black dashed line is the MRLs based on the true parameters. Left panel is for scenario I, middle panel is for scenario II and right panel is for scenario III in the simulation study.

The proofs of Theorems 4.1 and 4.2 are analogous to the proof of Theorem 3.1 and are therefore omitted.

**Table 3.** The MTTFs for three scenarios by using four models in the simulation study.

| Scenario | Model | Unit #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 |
|---|---|---|---|---|---|---|---|---|---|
| I(Wiener) | Wiener | 6.54 | 5.79 | 6.95 | 4.46 | 5.48 | 6.96 | 5.70 | 5.66 |
| | gamma | 6.39 | 5.78 | 6.74 | 4.74 | 5.53 | 6.77 | 5.71 | 5.68 |
| | IG | 6.25 | 5.76 | 6.54 | 5.00 | 5.58 | 6.55 | 5.73 | 5.68 |
| | TED | 6.53 | 5.78 | 6.93 | 4.47 | 5.48 | 6.95 | 5.71 | 5.66 |
| II(gamma) | Wiener | 6.82 | 6.97 | 7.70 | 4.31 | 5.89 | 8.35 | 5.54 | 5.17 |
| | gamma | 6.59 | 6.73 | 7.45 | 4.59 | 5.81 | 8.17 | 5.52 | 5.23 |
| | IG | 6.37 | 6.49 | 7.02 | 4.96 | 5.79 | 7.62 | 5.58 | 5.39 |
| | TED | 6.58 | 6.70 | 7.37 | 4.64 | 5.84 | 8.02 | 5.56 | 5.27 |
| III(IG) | Wiener | 7.12 | 5.36 | 6.44 | 5.24 | 5.97 | 6.77 | 3.60 | 6.08 |
| | gamma | 7.16 | 5.36 | 6.37 | 5.25 | 5.91 | 6.73 | 3.99 | 6.01 |
| | IG | 6.82 | 5.46 | 6.19 | 5.40 | 5.85 | 6.47 | 4.52 | 5.93 |
| | TED | 6.44 | 5.33 | 5.97 | 5.27 | 5.68 | 6.18 | 4.47 | 5.73 |

**Table 4.** The MRLs for three scenarios by using true model and TEDP model in the simulation study.

| | Model | Wiener | | | gamma | | | IG | | | TED | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Scenario | $Y(t)$ | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 |
| I(Wiener) | # 1 | 6.54 | 4.36 | 2.18 | 6.39 | 4.27 | 2.15 | 6.25 | 4.18 | 2.11 | 6.53 | 4.36 | 2.18 |
| | # 2 | 5.79 | 3.86 | 1.93 | 5.78 | 3.86 | 1.94 | 5.76 | 3.86 | 1.95 | 5.78 | 3.85 | 1.93 |
| | # 3 | 6.95 | 4.64 | 2.32 | 6.75 | 4.50 | 2.26 | 6.54 | 4.37 | 2.21 | 6.93 | 4.62 | 2.32 |
| | # 4 | 4.46 | 2.97 | 1.49 | 4.74 | 3.17 | 1.59 | 5.00 | 3.35 | 1.70 | 4.46 | 2.98 | 1.49 |
| | # 5 | 5.48 | 3.65 | 1.83 | 5.53 | 3.69 | 1.86 | 5.58 | 3.73 | 1.89 | 5.48 | 3.65 | 1.83 |
| | # 6 | 6.96 | 4.64 | 2.33 | 6.78 | 4.52 | 2.27 | 6.55 | 4.38 | 2.21 | 6.95 | 4.64 | 2.32 |
| | # 7 | 5.71 | 3.81 | 1.91 | 5.71 | 3.82 | 1.92 | 5.73 | 3.83 | 1.94 | 5.71 | 3.81 | 1.91 |
| | # 8 | 5.66 | 3.78 | 1.89 | 5.68 | 3.79 | 1.91 | 5.68 | 3.80 | 1.92 | 5.66 | 3.77 | 1.89 |
| | Model | Wiener | | | gamma | | | IG | | | TED | | |
| | $Y(t)$ | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 |
| II (gamma) | # 1 | 6.82 | 4.56 | 2.30 | 6.59 | 4.41 | 2.23 | 6.37 | 4.27 | 2.18 | 6.58 | 4.40 | 2.23 |
| | # 2 | 6.97 | 4.66 | 2.35 | 6.73 | 4.50 | 2.28 | 6.49 | 4.35 | 2.22 | 6.70 | 4.49 | 2.27 |
| | # 3 | 7.71 | 5.15 | 2.60 | 7.45 | 4.98 | 2.52 | 7.02 | 4.71 | 2.39 | 7.37 | 4.93 | 2.49 |
| | # 4 | 4.31 | 2.88 | 1.45 | 4.59 | 3.08 | 1.56 | 4.96 | 3.34 | 1.73 | 4.64 | 3.11 | 1.58 |
| | # 5 | 5.89 | 3.94 | 1.98 | 5.81 | 3.89 | 1.97 | 5.79 | 3.90 | 1.99 | 5.84 | 3.91 | 1.98 |
| | # 6 | 8.35 | 5.59 | 2.83 | 8.17 | 5.46 | 2.76 | 7.62 | 5.11 | 2.59 | 8.02 | 5.37 | 2.71 |
| | # 7 | 5.54 | 3.70 | 1.86 | 5.52 | 3.69 | 1.87 | 5.58 | 3.76 | 1.93 | 5.56 | 3.72 | 1.88 |
| | # 8 | 5.17 | 3.46 | 1.74 | 5.23 | 3.50 | 1.77 | 5.39 | 3.63 | 1.87 | 5.27 | 3.53 | 1.79 |
| | Model | Wiener | | | gamma | | | IG | | | TED | | |
| | $Y(t)$ | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 | 0 | 4 | 8 |
| III (IG) | # 1 | 7.12 | 4.79 | 2.45 | 7.16 | 4.79 | 2.43 | 6.82 | 4.57 | 2.33 | 6.44 | 4.32 | 2.20 |
| | # 2 | 5.36 | 3.59 | 1.83 | 5.36 | 3.59 | 1.83 | 5.46 | 3.68 | 1.89 | 5.33 | 3.58 | 1.84 |
| | # 3 | 6.43 | 4.32 | 2.21 | 6.37 | 4.27 | 2.17 | 6.19 | 4.16 | 2.13 | 5.97 | 4.01 | 2.05 |
| | # 4 | 5.24 | 3.51 | 1.78 | 5.25 | 3.52 | 1.79 | 5.40 | 3.64 | 1.87 | 5.27 | 3.54 | 1.82 |
| | # 5 | 5.97 | 4.01 | 2.04 | 5.91 | 3.96 | 2.01 | 5.85 | 3.94 | 2.02 | 5.68 | 3.82 | 1.96 |
| | # 6 | 6.77 | 4.55 | 2.32 | 6.73 | 4.51 | 2.29 | 6.46 | 4.34 | 2.22 | 6.18 | 4.15 | 2.12 |
| | # 7 | 3.60 | 2.41 | 1.22 | 3.99 | 2.68 | 1.37 | 4.51 | 3.06 | 1.59 | 4.47 | 3.02 | 1.56 |
| | # 8 | 6.08 | 4.08 | 2.08 | 6.01 | 4.03 | 2.04 | 5.93 | 3.99 | 2.04 | 5.73 | 3.85 | 1.97 |

## 5. Simulation study

In this section, we carry out a simulation study to demonstrate the effectiveness of the TED process with random drifts. The simulated degradation data is generated from the TED process. The number of units is $n = 8$ and the number of inspection time points is $n = 21$. The inspection time starts from 0 and ends to 5 with the identical inspection time interval. The setting-ups of parameter vector $(\eta, \sigma^2, \lambda, p)$ under three scenarios are, respectively, I (2, 0.2, 16, 0); II (2, 0.2, 10, 2); III (2, 0.2, 10, 3), which correspond to the Wiener process with random drifts, gamma process with random drifts, and IG process with random drifts. For each scenario, we adopt four models, the Wiener process with random drifts, the gamma process with random drifts, the IG process with random drifts, and the proposed TED process with random drifts to fit the simulated degradation data. The purpose of this simulation study is to show the superiority of the proposed model in parameter estimation, reliability analysis compared with other classical stochastic process with random drifts if the model is mis-specified.

### 5.1. Parameter estimation

Wiener process, gamma process, IG process with random drifts are special cases of TED process with random drifts, and we keep the same prior specification for each parameters in these models with TED process, excluding the $p$ parameters as it is a known constant. For each scenario, we use the right model, TED process with random drifts, and two wrong models to fit the simulated degradation data. According to Algorithm 1, we use the MHGS

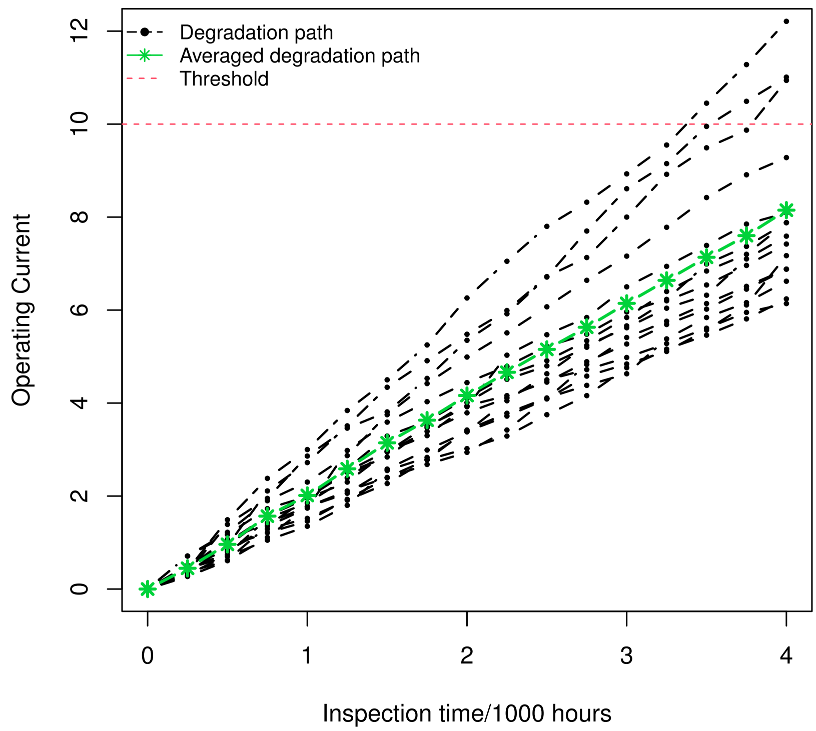


**Figure 4.** The degradation paths for the GaAs laser data along with the averaged mean trend.

algorithm to obtain the posterior sample. Actually, we use the *R2Openbug* package in R to call the *Openbugs* software to achieve this algorithm. We first run 5000 iterations as the burn-in period and check the convergency of the Markov chains for each parameter by using the trace plots, ergodic mean plots and autocorrelation plots. Then we further run 10,000 iterations to obtain the posterior samples. The absolute bias (AB), mean square error (MSE) and coverage probability (CP) for the drift parameters and dispersion parameter under each scenario are computed, which are listed in Table 2. We can see that the right model always gives rise to the best estimation results for three scenarios and TED process has almost equally well results with the right model. However, two wrong models lead to worse results than the right model

**Table 5.** The posterior estimation of the drift parameters $\mu_i$, disperison $\lambda$ and $p$ along with the log-likelihood and AIC for TED process along with Wiener process, gamma process and IG process for the GaAs data.

| | GaAs | | Wiener | | | gamma | |
|---|---|---|---|---|---|---|---|
| Parameter | *mean* | 2.5% | 97.5% | *mean* | 2.5% | 97.5% | |
| | #1 | 2.640 | 2.321 | 2.959 | 2.602 | 2.245 | 3.010 |
| | #2 | 2.282 | 1.968 | 2.599 | 2.288 | 1.966 | 2.652 |
| | #3 | 1.762 | 1.444 | 2.078 | 1.775 | 1.510 | 2.080 |
| | #4 | 1.603 | 1.288 | 1.920 | 1.601 | 1.356 | 1.885 |
| | #5 | 1.917 | 1.604 | 2.229 | 1.934 | 1.653 | 2.256 |
| $\mu$ | #6 | 2.656 | 2.331 | 2.980 | 2.616 | 2.260 | 3.028 |
| | #7 | 1.826 | 1.513 | 2.138 | 1.840 | 1.569 | 2.146 |
| | #8 | 1.624 | 1.305 | 1.943 | 1.626 | 1.378 | 1.910 |
| | #9 | 1.979 | 1.667 | 2.291 | 1.997 | 1.708 | 2.326 |
| | #10 | 2.915 | 2.583 | 3.245 | 2.824 | 2.436 | 3.269 |
| | #11 | 1.880 | 1.565 | 2.192 | 1.897 | 1.618 | 2.216 |
| | #12 | 1.979 | 1.664 | 2.294 | 1.998 | 1.710 | 2.326 |
| | #13 | 2.024 | 1.712 | 2.337 | 2.043 | 1.750 | 2.377 |
| | #14 | 1.763 | 1.448 | 2.076 | 1.774 | 1.509 | 2.076 |
| | #15 | 1.707 | 1.389 | 2.022 | 1.714 | 1.457 | 2.009 |
| $\lambda$ | | 8.645 | 7.120 | 10.300 | 35.160 | 29.360 | 41.480 |
| $p$ | | | 0.000 | | | 2.000 | |
| log likelihood | | | 90.365 | | | 112.331 | |
| AIC | | | −148.731 | | | −192.662 | |
| DIC | | | −161.839 | | | −204.979 | |

| | | | IG | | | TED | |
|---|---|---|---|---|---|---|---|
| Parameter | GaAs | *mean* | 2.5% | 97.5% | *mean* | 2.5% | 97.5% |
| | #1 | 2.527 | 2.128 | 3.032 | 2.602 | 2.245 | 3.020 |
| | #2 | 2.274 | 1.912 | 2.720 | 2.291 | 1.976 | 2.653 |
| | #3 | 1.806 | 1.513 | 2.170 | 1.777 | 1.521 | 2.075 |
| | #4 | 1.633 | 1.369 | 1.965 | 1.603 | 1.366 | 1.877 |
| | #5 | 1.959 | 1.644 | 2.348 | 1.938 | 1.662 | 2.258 |
| $\mu$ | #6 | 2.533 | 2.135 | 3.036 | 2.613 | 2.257 | 3.027 |
| | #7 | 1.870 | 1.566 | 2.240 | 1.844 | 1.578 | 2.153 |
| | #8 | 1.656 | 1.387 | 1.992 | 1.627 | 1.387 | 1.906 |
| | #9 | 2.019 | 1.692 | 2.416 | 2.001 | 1.716 | 2.330 |
| | #10 | 2.688 | 2.267 | 3.232 | 2.818 | 2.426 | 3.274 |
| | #11 | 1.924 | 1.616 | 2.306 | 1.901 | 1.627 | 2.213 |
| | #12 | 2.019 | 1.693 | 2.422 | 2.001 | 1.721 | 2.328 |
| | #13 | 2.060 | 1.728 | 2.469 | 2.047 | 1.760 | 2.379 |
| | #14 | 1.806 | 1.515 | 2.167 | 1.778 | 1.519 | 2.079 |
| | #15 | 1.748 | 1.466 | 2.100 | 1.717 | 1.466 | 2.011 |
| $\lambda$ | | 48.830 | 43.160 | 54.850 | 40.880 | 35.190 | 47.070 |
| $p$ | | | 3.000 | | 2.158 | 1.838 | 2.489 |
| log likelihood | | | 103.086 | | | 115.520 | |
| AIC | | | −174.171 | | | −197.040 | |
| DIC | | | −186.249 | | | −211.737 | |

and TED process model from the perspective of AB, MSE and CP. Specially, for the dispersion parameter $\lambda$, two wrong models give outrageous estimation results.

### 5.2. Reliability analysis

According to Theorems 3.1 and 3.2, we derive the failure-time distribution, MTTFs, RUL distribution, and MRLs based on the BS distribution approximation. The PDFs and CDFs of failure-time are shown in Figures 1 and A1 in Appendix C. The PDFs and CDFs of RUL when the current degradation values are 0, 4, 8, respectively, are shown in Figures 2 and A2. The three-dimension PDFs and CDFs of RUL based on the true model and TED process can be found in Figures 3 and A3. The MTTFs and MRLS can be seen in Tables 3 and 4. In view of any aspects of reliability analysis, the proposed TED process has close performance to the right model, while the two wrong models have worse reliability estimation results.

## 6. Illustrative examples

### 6.1. GaAs laser data

The GaAs laser data is used to illustrate the proposed TED process with random drifts. The GaAs laser data can be found from Table C.17. in Meeker et al. (1998). The quality characteristic of the GaAs laser device is its operating current. The device is considered a failure, when the operating current reaches a predefined threshold $D_f = 10$. There are 15 GaAs laser

**Table 6.** Absolute relative changes ($\breve{G}$) in MTTF for each OLED under sensitivity analyses with respect to the prior distribution and pseudo likelihood function.

| | | Prior sensitivity analysis | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(\gamma, \eta_\mu)$ | precision | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 | #9 | #10 | #11 | #12 | #13 | #14 | #15 |
| (3, 10) | 0.0001 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| (3, 10) | 0.0002 | 0.77 | 0.44 | 0.00 | 0.00 | 0.00 | 0.39 | 0.00 | 0.61 | 0.50 | 0.01 | 0.53 | 0.50 | 0.48 | 0.00 | 0.00 |
| (3, 10) | 0.0003 | 0.39 | 0.43 | 1.12 | 0.00 | 0.51 | 0.38 | 0.00 | 0.61 | 0.50 | 0.70 | 0.53 | 0.00 | 0.00 | 0.00 | 1.16 |
| (3, 4) | 0.0001 | 0.77 | 0.44 | 0.56 | 0.62 | 0.52 | 0.38 | 0.00 | 0.00 | 0.50 | 0.00 | 0.00 | 0.50 | 0.49 | 0.00 | 0.58 |
| (3, 4) | 0.0002 | 0.38 | 0.00 | 0.56 | 0.62 | 0.00 | 0.00 | 0.00 | 0.00 | 0.50 | 0.36 | 0.53 | 0.50 | 0.49 | 0.56 | 0.58 |
| (3, 4) | 0.0003 | 0.77 | 0.00 | 0.56 | 0.62 | 0.51 | 0.38 | 0.00 | 0.61 | 0.50 | 0.71 | 0.53 | 0.50 | 0.00 | 0.00 | 0.58 |
| (3, 13) | 0.0001 | 0.38 | 0.44 | 0.56 | 0.62 | 0.51 | 0.38 | 0.54 | 0.00 | 0.50 | 0.35 | 0.52 | 0.00 | 0.00 | 0.56 | 0.00 |
| (3, 13) | 0.0002 | 0.38 | 0.00 | 0.56 | 0.00 | 0.00 | 0.38 | 0.54 | 0.61 | 0.50 | 0.35 | 0.52 | 0.00 | 0.00 | 0.00 | 0.58 |
| (3, 13) | 0.0003 | 0.38 | 0.00 | 0.00 | 0.00 | 0.00 | 0.38 | 0.54 | 0.00 | 0.00 | 0.35 | 0.52 | 0.50 | 0.49 | 0.00 | 0.58 |
| (1, 10) | 0.0001 | 0.38 | 0.00 | 0.56 | 0.62 | 0.00 | 0.38 | 0.00 | 0.61 | 0.50 | 0.35 | 0.52 | 0.50 | 0.00 | 0.00 | 0.58 |
| (1, 10) | 0.0002 | 0.01 | 0.44 | 0.00 | 0.00 | 0.00 | 0.39 | 0.00 | 0.00 | 0.50 | 0.35 | 0.00 | 0.00 | 0.97 | 0.00 | 0.58 |
| (1, 10) | 0.0003 | 0.77 | 0.43 | 0.00 | 1.24 | 0.00 | 0.00 | 0.54 | 0.61 | 0.00 | 0.01 | 0.53 | 0.99 | 0.00 | 0.00 | 1.16 |
| (1, 4) | 0.0001 | 0.76 | 0.00 | 0.00 | 0.00 | 0.00 | 0.39 | 0.00 | 0.61 | 0.50 | 0.01 | 0.52 | 0.50 | 0.00 | 0.00 | 0.58 |
| (1, 4) | 0.0002 | 0.39 | 0.87 | 0.56 | 0.62 | 0.51 | 0.00 | 0.00 | 0.00 | 0.00 | 0.70 | 0.53 | 0.99 | 0.00 | 0.00 | 1.16 |
| (1, 4) | 0.0003 | 1.15 | 0.00 | 0.56 | 0.00 | 0.00 | 0.00 | 0.54 | 0.61 | 0.00 | 0.35 | 0.52 | 0.00 | 0.49 | 0.00 | 0.58 |
| (1, 13) | 0.0001 | 0.00 | 0.00 | 0.56 | 0.62 | 0.00 | 0.00 | 0.00 | 0.61 | 0.50 | 0.00 | 0.52 | 0.50 | 0.00 | 0.00 | 0.00 |
| (1, 13) | 0.0002 | 0.38 | 0.44 | 0.56 | 0.00 | 0.00 | 0.38 | 0.54 | 0.00 | 0.00 | 0.00 | 1.05 | 0.50 | 0.49 | 0.00 | 0.00 |
| (1, 13) | 0.0003 | 0.01 | 0.44 | 0.56 | 0.00 | 0.52 | 0.77 | 0.54 | 0.61 | 0.00 | 0.71 | 0.53 | 1.00 | 0.48 | 0.00 | 0.58 |
| (6, 10) | 0.0001 | 0.01 | 0.86 | 0.56 | 0.62 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.34 | 0.00 | 0.00 | 0.48 | 0.00 | 0.58 |
| (6, 10) | 0.0002 | 0.77 | 0.01 | 0.56 | 0.62 | 0.51 | 0.39 | 0.54 | 0.00 | 0.00 | 0.72 | 0.00 | 0.00 | 0.00 | 0.56 | 0.58 |
| (6, 10) | 0.0003 | 0.00 | 0.00 | 0.00 | 0.62 | 0.52 | 0.38 | 0.54 | 0.61 | 0.00 | 0.71 | 0.00 | 0.00 | 0.00 | 0.00 | 0.58 |
| (6, 4) | 0.0001 | 0.77 | 0.00 | 0.00 | 0.62 | 0.00 | 0.00 | 0.00 | 0.61 | 0.00 | 0.00 | 0.00 | 0.00 | 0.49 | 0.56 | 0.58 |
| (6, 4) | 0.0002 | 0.39 | 0.00 | 0.00 | 0.00 | 0.00 | 0.38 | 0.00 | 0.00 | 0.00 | 0.00 | 0.53 | 0.50 | 0.49 | 0.00 | 0.58 |
| (6, 4) | 0.0003 | 0.39 | 0.44 | 0.00 | 0.00 | 0.52 | 0.01 | 0.54 | 0.00 | 0.50 | 0.01 | 1.05 | 0.49 | 0.49 | 0.56 | 0.58 |
| (6, 13) | 0.0001 | 0.39 | 0.43 | 0.56 | 0.63 | 0.01 | 0.37 | 0.55 | 0.00 | 0.50 | 0.37 | 0.53 | 0.99 | 0.01 | 0.00 | 0.00 |
| (6, 13) | 0.0002 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.76 | 0.00 | 0.61 | 0.50 | 0.35 | 1.05 | 0.50 | 0.49 | 0.56 | 0.58 |
| (6, 13) | 0.0003 | 0.78 | 0.01 | 0.56 | 0.62 | 0.01 | 0.01 | 0.54 | 0.00 | 0.49 | 0.34 | 0.00 | 0.49 | 0.48 | 0.56 | 0.00 |

devices tested at 80°C. The measured frequency of its operating current is every 150 hours, and the degradation test is terminated at 4000 hours. Figure 4 shows the degradation paths along with the averaged degradation path for 15 tested units by transforming the time unit to 1000 hours.

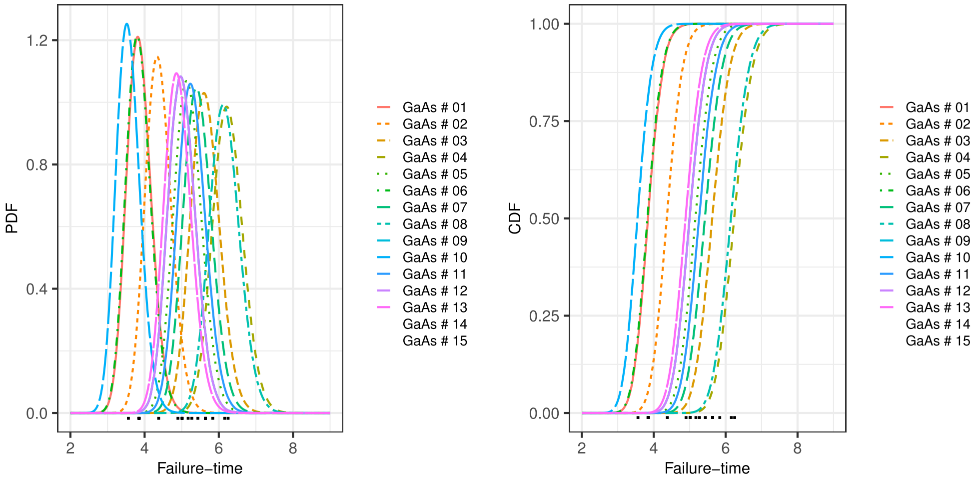


**Figure 5.** The PDF and CDF curves of the lifetime for each GaAs laser based on the proposed TED process.

**Table 7.** The MTTFs for each GaAs laser based on four models.

| GaAs | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 | #9 | #10 | #11 | #12 | #13 | #14 | #15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wiener | 3.80 | 4.39 | 5.69 | 6.26 | 5.23 | 3.77 | 5.49 | 6.18 | 5.07 | 3.44 | 5.34 | 5.07 | 4.95 | 5.69 | 5.88 |
| gamma | 3.86 | 4.38 | 5.65 | 6.26 | 5.18 | 3.84 | 5.45 | 6.16 | 5.02 | 3.56 | 5.29 | 5.02 | 4.91 | 5.65 | 5.85 |
| IG | 3.98 | 4.42 | 5.56 | 6.14 | 5.12 | 3.97 | 5.37 | 6.06 | 4.97 | 3.75 | 5.22 | 4.97 | 4.88 | 5.56 | 5.74 |
| TED | 3.86 | 4.38 | 5.64 | 6.25 | 5.17 | 3.84 | 5.44 | 6.16 | 5.01 | 3.56 | 5.27 | 5.01 | 4.90 | 5.64 | 5.84 |

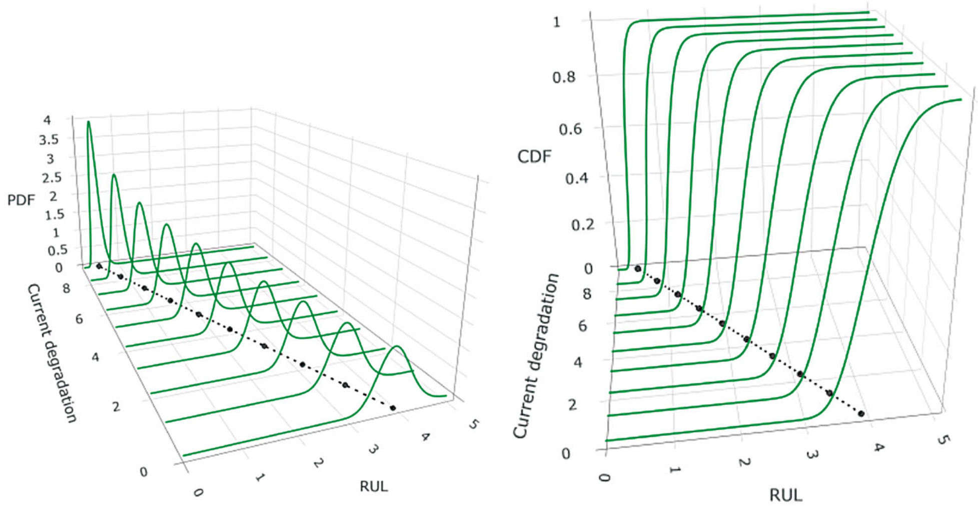


**Figure 6.** The PDF and CDF of the RUL for the first GaAs laser based on TED process.

### 6.1.1. Parameter estimation

The TED process with random drifts is adopted to fit the GaAs laser degradation paths. The posterior estimation of the parameters is shown in Table 5. The estimated $p$ in TED process is 1.838, which is close to 2. Note that $p = 2$ corresponds to the gamma process. We also find that the estimated drifts in TED process are more close to gamma process than Wiener process and IG process. The AIC and DIC to evaluate the model are calculated as

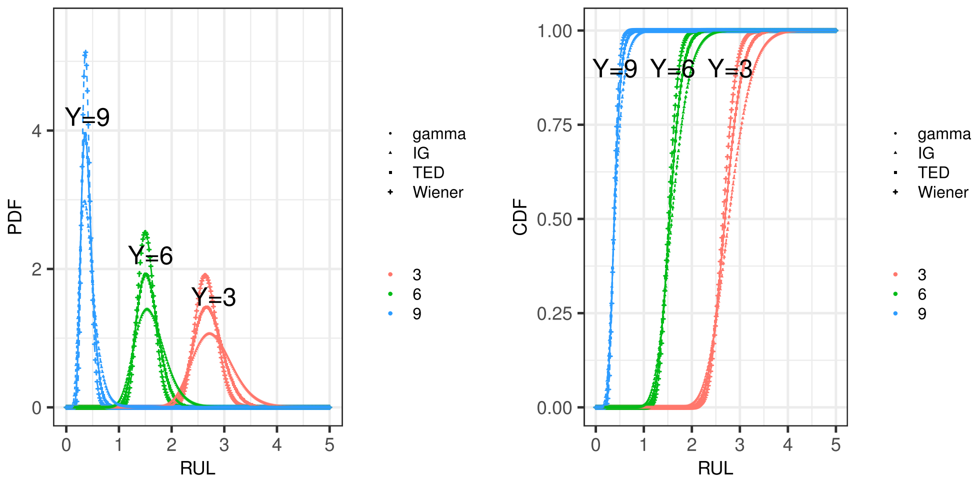


**Figure 7.** The PDF and CDF of the RUL for the first GaAs laser based on Wiener process, gamma process, inverse Gaussian process and TED process when the current degradation values equal 3, 6, 9, respectively.

**Table 8.** The MRLs for each GaAs laser based on four models.

| | Current degradation | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.00 | 1.00 | 2.00 | 3.00 | 4.00 | 5.00 | 6.00 | 7.00 | 8.00 | 9.00 |
| Wiener | 3.80 | 3.42 | 3.04 | 2.66 | 2.28 | 1.90 | 1.52 | 1.14 | 0.77 | 0.39 |
| gamma | 3.86 | 3.47 | 3.09 | 2.70 | 2.32 | 1.94 | 1.55 | 1.17 | 0.78 | 0.40 |
| IG | 3.98 | 3.59 | 3.19 | 2.80 | 2.40 | 2.00 | 1.61 | 1.21 | 0.82 | 0.42 |
| TED | 3.86 | 3.47 | 3.09 | 2.70 | 2.32 | 1.94 | 1.55 | 1.17 | 0.78 | 0.40 |

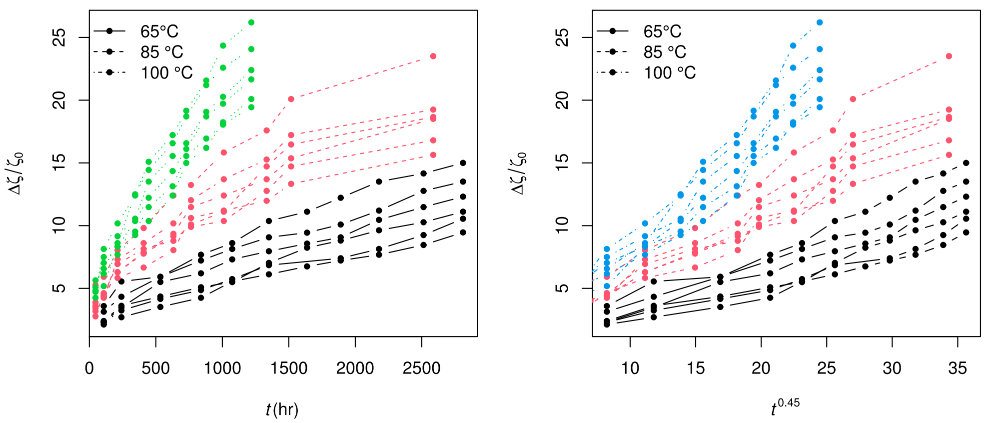


**Figure 8.** The degradation paths for the stress relaxation data along the the averaged mean trend.

$\text{AIC} = -2\log(L) + 2N$ and $\text{DIC} = \overline{D} + p_D$, where $L$ is the likelihood, $N$ is the number of parameters, $D(\theta) = -2L + C$ is the deviance, $\overline{D}$ and $p_D$ are the average and variance of $D(\theta)$. Both the log-likelihood and AIC for each model are also shown in Table 5. The proposed TED

**Table 9.** The posterior estimation of the drift parameters $\mu_i$, dispersion $\lambda$ and $p$ along with the log-likelihood and AIC for TED process along with Wiener process, gamma process and IG process for the stress relaxation data.

| | | | Wiener | | | gamma | | |
|---|---|---|---|---|---|---|---|---|
| Parameter | | Unit | mean | 2.5% | 97.5% | mean | 2.5% | 97.5% |
| $\mu$ | 65°C | # 1 | 0.291 | 0.143 | 0.432 | 0.286 | 0.463 | 0.879 |
| | | # 2 | 0.319 | 0.175 | 0.462 | 0.326 | 0.476 | 0.881 |
| | | # 3 | 0.344 | 0.204 | 0.490 | 0.360 | 0.507 | 0.910 |
| | | # 4 | 0.351 | 0.210 | 0.498 | 0.369 | 0.406 | 0.930 |
| | | # 5 | 0.372 | 0.230 | 0.523 | 0.394 | 0.646 | 0.948 |
| | | # 6 | 0.309 | 0.164 | 0.450 | 0.313 | 0.648 | 1.002 |
| | 85°C | # 7 | 0.500 | 0.348 | 0.639 | 0.505 | 0.567 | 0.666 |
| | | # 8 | 0.501 | 0.350 | 0.640 | 0.506 | 0.553 | 0.668 |
| | | # 9 | 0.553 | 0.409 | 0.694 | 0.214 | 0.575 | 0.704 |
| | | # 10 | 0.541 | 0.397 | 0.681 | 0.248 | 0.624 | 0.728 |
| | | # 11 | 0.559 | 0.416 | 0.701 | 0.277 | 0.441 | 0.744 |
| | | # 12 | 0.608 | 0.469 | 0.757 | 0.286 | 0.429 | 0.798 |
| | 100°C | # 13 | 0.837 | 0.647 | 1.013 | 0.306 | 0.449 | 1.113 |
| | | # 14 | 0.840 | 0.651 | 1.016 | 0.238 | 0.495 | 1.113 |
| | | # 15 | 0.880 | 0.699 | 1.058 | 0.381 | 0.715 | 1.146 |
| | | # 16 | 0.911 | 0.733 | 1.093 | 0.382 | 0.698 | 1.175 |
| | | # 17 | 0.938 | 0.763 | 1.120 | 0.375 | 0.724 | 1.200 |
| | | # 18 | 1.033 | 0.847 | 1.242 | 0.422 | 0.786 | 1.281 |
| $\mu_0$ | 40°C | | 0.139 | 0.065 | 0.246 | 0.139 | 0.065 | 0.246 |
| $p$ | | | 0 | | | 2 | | |
| $\lambda$ | | | 4.208 | 1.555 | 0.596 | 2.170 | 4.208 | 1.555 |
| $\beta$ | | | 1.887 | 1.296 | 2.560 | 1.983 | 1.796 | 2.170 |
| log likelihood | | | | −187.609 | | | −174.154 | |
| AIC | | | | 413.218 | | | 386.308 | |
| DIC | | | | 392.964 | | | 363.232 | |

| | | | IG | | | TED | | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Unit | | mean | 2.5% | 97.5% | mean | 2.5% | 97.5% |
| $\mu$ | 65°C | # 1 | 0.288 | 0.475 | 0.922 | 0.287 | 0.467 | 0.869 |
| | | # 2 | 0.331 | 0.488 | 0.924 | 0.323 | 0.477 | 0.871 |
| | | # 3 | 0.366 | 0.521 | 0.945 | 0.355 | 0.508 | 0.903 |
| | | # 4 | 0.376 | 0.412 | 0.958 | 0.364 | 0.415 | 0.925 |
| | | # 5 | 0.400 | 0.709 | 0.970 | 0.388 | 0.640 | 0.944 |
| | | # 6 | 0.318 | 0.710 | 1.005 | 0.312 | 0.642 | 1.008 |
| | 85°C | # 7 | 0.528 | 0.587 | 0.648 | 0.503 | 0.563 | 0.668 |
| | | # 8 | 0.529 | 0.575 | 0.650 | 0.504 | 0.550 | 0.670 |
| | | # 9 | 0.225 | 0.594 | 0.677 | 0.202 | 0.570 | 0.708 |
| | | # 10 | 0.259 | 0.636 | 0.696 | 0.235 | 0.620 | 0.733 |
| | | # 11 | 0.288 | 0.440 | 0.708 | 0.263 | 0.435 | 0.753 |
| | | # 12 | 0.294 | 0.429 | 0.749 | 0.272 | 0.424 | 0.814 |
| | 100°C | # 13 | 0.314 | 0.447 | 1.261 | 0.292 | 0.444 | 1.077 |
| | | # 14 | 0.248 | 0.487 | 1.262 | 0.224 | 0.490 | 1.078 |
| | | # 15 | 0.385 | 0.780 | 1.294 | 0.375 | 0.706 | 1.118 |
| | | # 16 | 0.386 | 0.763 | 1.309 | 0.375 | 0.689 | 1.143 |
| | | # 17 | 0.375 | 0.790 | 1.333 | 0.388 | 0.714 | 1.168 |
| | | # 18 | 0.429 | 0.845 | 1.386 | 0.428 | 0.774 | 1.258 |
| $\mu_0$ | 40°C | | 0.129 | 0.103 | 0.158 | 0.139 | 0.065 | 0.246 |
| $p$ | | | 3 | | | 1.438 | 1.002 | 1.866 |
| $\lambda$ | | | 0.596 | 2.170 | 4.208 | 1.555 | 0.596 | 2.170 |
| $\beta$ | | | 1.983 | 1.795 | 2.173 | 1.985 | 1.800 | 2.170 |
| log-likelihood | | | | −194.929 | | | −167.167 | |
| AIC | | | | 427.858 | | | 374.334 | |
| DIC | | | | 402.791 | | | 350.862 | |

process achieves the maximum loglikelihood and minimum AIC value. Although the results for TED process are close to the gamma process, the TED process outperforms the gamma process from the perspective of AIC and DIC.

### 6.2. Sensitivity analysis

To investigate whether the changes of the prior distribution will affect the reliability inference, we performed sensitivity analyses with respect to prior distributions. To reflect the effects of the priors, we set three levels for the mean hyper-parameters $(\gamma, \eta_\mu)$, which are $(3, 10)$, $(1, 4)$, and $(6, 13)$, and three levels for precision hyper-parameters $(1/\sigma^2, \lambda_\mu, \nu_\mu, \xi_\mu, \alpha, \beta)$ which are 0.0001, 0.0002, and 0.0003. For different priors, we can observe in Table 6 that absolute relative changes of MTTF compared with the first scenario are all less than 1.24Ğ, showing that the changes of prior distributions do not significantly affect the posterior inference.

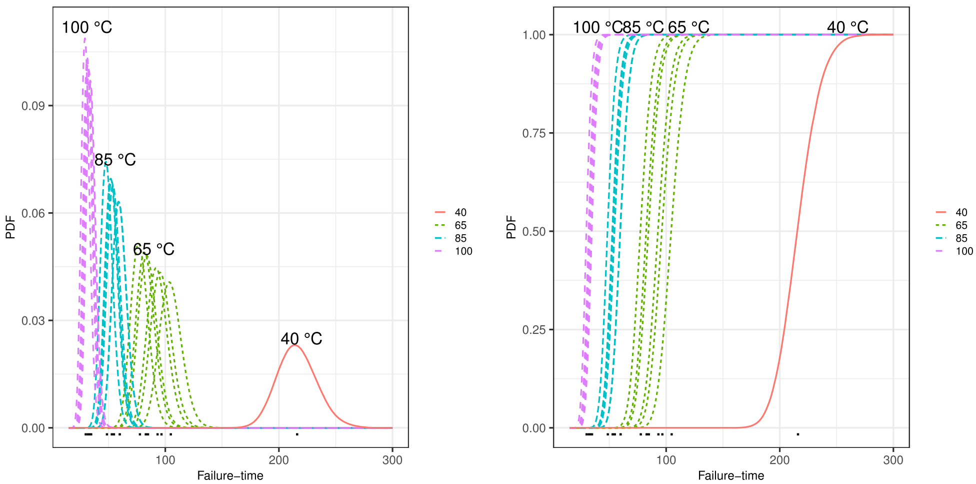


**Figure 9.** The PDF and CDF curves of the lifetime for each stress loss device based on the proposed TED process.

**Table 10.** The MTTFs for each stress loss device under for accelerated temperature stress and usage stress based on four models.

| | 65°C | | | | | | 85°C | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| GaAs | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 | #9 | #10 | #11 | #12 |
| Wiener | 104.46 | 95.33 | 88.14 | 86.36 | 81.50 | 98.33 | 60.46 | 60.32 | 54.68 | 55.87 | 54.02 | 49.66 |
| gamma | 105.25 | 92.37 | 83.70 | 81.58 | 76.50 | 96.14 | 59.74 | 59.57 | 53.25 | 54.53 | 52.52 | 48.38 |
| IG | 104.44 | 90.89 | 82.16 | 80.19 | 75.32 | 94.67 | 57.27 | 57.19 | 51.63 | 52.67 | 51.01 | 47.67 |
| TED | 104.88 | 93.20 | 84.80 | 82.85 | 77.69 | 96.66 | 59.98 | 59.89 | 53.59 | 54.85 | 52.92 | 48.67 |

| | 100°C | | | | | | 40°C |
|---|---|---|---|---|---|---|---|
| GaAs | #13 | #14 | #15 | #16 | #17 | #18 | Usage stress |
| Wiener | 36.03 | 35.87 | 34.23 | 33.07 | 32.12 | 29.15 | 221.49 |
| gamma | 34.46 | 34.37 | 33.30 | 32.56 | 31.98 | 30.26 | 239.37 |
| IG | 33.30 | 33.23 | 32.54 | 32.13 | 31.75 | 30.69 | 232.85 |
| TED | 34.79 | 34.69 | 33.47 | 32.66 | 32.01 | 29.99 | 216.06 |

### 6.2.1. Reliability inference

According to Theorem 3.1, we can obtain the PDF and CDF of the failure time for each unit, which are shown in Figure 5. The MTTFs for each unit are illustrated in Table 7. Based on Theorem 3.2, the PDF and CDF of the RUL are shown in Figure 6. The PDF and CDF of the RUL for the first GaAs laser based on Wiener process, gamma process, inverse Gaussian process and TED process when the current degradation values equal 3, 6, 9/,m, respectively, are shown in Figure 7, which shows that PDFs and CDFs of the RUL based on gamma process are very close to that of the TED process. The MRLs for each GaAs laser based on four models are shown in Table 8.

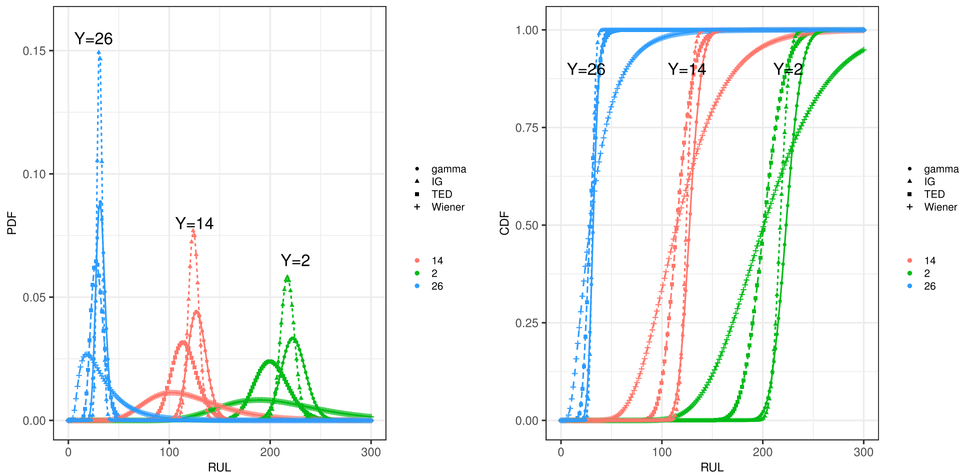


**Figure 10.** The PDF and CDF of the RUL for the first stress relaxation unit based on Wiener process, gamma process, inverse Gaussian process and TED process when the current degradation values equal 2, 14, 26, respectively.

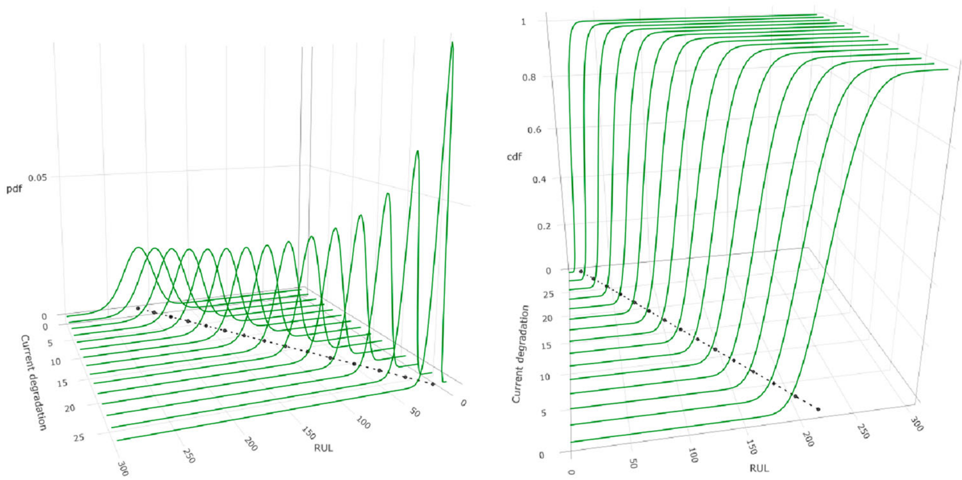


**Figure 11.** The estimated PDFs and CDFs of the RULs based on the TED process with random drifts and covariates along with the MRLs (dotted line) under usage stress at different time points.

### 6.3. Stress relaxation data

The Stress relaxation data in Yang (2007) (Example 8.7, pp. 351) are used in this section to illustrate the proposed TED process with random drifts and covariates. This data including stress loss and measurement time epochs are available from Ye et al. (2014). The stress relaxation is the loss of stress in a component subject to a constant strain over time. The device fails due to excessive stress relaxation, for example, the electrical connector is considered to be a failure if the stress relaxation exceeds $D_f = 30\%$ of its initial stress relaxation. The accelerated degradation data are collected under three temperature stress level: 65°C, 85°C and 100°C. This data set is modelled by many models, such as the regression method (Yang, 2007) and inverse Gaussian process (Ye et al., 2014). To be consistent with Yang (2007), we also assume the usage temperature is $S_0 = 40°\text{C}$ and the highest allowable temperature $S_H = 100°\text{C}$, respectively. The transformed stress is $\phi(S_k) = \frac{\frac{1}{S_k} - \frac{1}{S_0}}{\frac{1}{S_H} - \frac{1}{S_0}}$. The time transformation is $t^{0.45}$ according to Hong and Ye (2017) and Tseng and Lee (2016). Let $\zeta_0$ be the initial stress and $\Delta\zeta$ be the stress loss, and then the degradation characteristic is defined as $\Delta\zeta/\zeta_0$. The degradation paths versus the original time scale for each unit and the degradation paths versus the power transformed time scale can be found in Figure 8.

**Table 11.** The MRLs for each stress loss device based on four models.

| | Wiener process | | | | gamma process | | | |
|---|---|---|---|---|---|---|---|---|
| Cur. Deg. | #1 | #7 | #13 | Usage stress | #1 | #7 | #13 | Usage stress |
| 0.00 | 104.46 | 60.46 | 36.03 | 221.49 | 105.25 | 59.74 | 34.46 | 239.37 |
| 2.00 | 97.59 | 56.46 | 33.64 | 207.13 | 98.26 | 55.78 | 32.18 | 223.43 |
| 4.00 | 90.72 | 52.46 | 31.25 | 192.77 | 91.26 | 51.82 | 29.91 | 207.49 |
| 6.00 | 83.85 | 48.47 | 28.86 | 178.41 | 84.27 | 47.86 | 27.63 | 191.56 |
| 8.00 | 76.98 | 44.47 | 26.47 | 164.06 | 77.27 | 43.89 | 25.36 | 175.62 |
| 10.00 | 70.11 | 40.47 | 24.08 | 149.70 | 70.28 | 39.93 | 23.08 | 159.68 |
| 12.00 | 63.24 | 36.47 | 21.69 | 135.34 | 63.28 | 35.97 | 20.80 | 143.75 |
| 14.00 | 56.37 | 32.47 | 19.30 | 120.98 | 56.29 | 32.01 | 18.53 | 127.81 |
| 16.00 | 49.50 | 28.47 | 16.91 | 106.63 | 49.29 | 28.05 | 16.25 | 111.88 |
| 18.00 | 42.63 | 24.47 | 14.52 | 92.27 | 42.29 | 24.09 | 13.98 | 95.94 |
| 20.00 | 35.75 | 20.47 | 12.12 | 77.91 | 35.30 | 20.13 | 11.70 | 80.00 |
| 22.00 | 28.88 | 16.47 | 9.73 | 63.55 | 28.30 | 16.17 | 9.42 | 64.07 |
| 24.00 | 22.01 | 12.47 | 7.34 | 49.20 | 21.31 | 12.21 | 7.15 | 48.13 |
| 26.00 | 15.14 | 8.47 | 4.95 | 34.84 | 14.31 | 8.24 | 4.87 | 32.19 |
| 28.00 | 8.27 | 4.47 | 2.56 | 20.48 | 7.32 | 4.28 | 2.60 | 16.26 |
| | IG process | | | | TED process | | | |
| Cur. Deg. | #1 | #7 | #13 | Usage stress | #1 | #7 | #13 | Usage stress |
| 0.00 | 104.44 | 57.27 | 33.30 | 232.85 | 104.88 | 59.98 | 34.79 | 216.06 |
| 2.00 | 97.50 | 53.48 | 31.13 | 217.33 | 97.92 | 56.01 | 32.49 | 201.70 |
| 4.00 | 90.55 | 49.69 | 28.96 | 201.81 | 90.96 | 52.03 | 30.18 | 187.35 |
| 6.00 | 83.60 | 45.91 | 26.79 | 186.30 | 84.00 | 48.05 | 27.88 | 172.99 |
| 8.00 | 76.66 | 42.12 | 24.63 | 170.78 | 77.04 | 44.08 | 25.58 | 158.63 |
| 10.00 | 69.71 | 38.33 | 22.46 | 155.27 | 70.08 | 40.10 | 23.27 | 144.27 |
| 12.00 | 62.76 | 34.54 | 20.29 | 139.75 | 63.12 | 36.12 | 20.97 | 129.92 |
| 14.00 | 55.82 | 30.75 | 18.12 | 124.24 | 56.16 | 32.15 | 18.67 | 115.56 |
| 16.00 | 48.87 | 26.96 | 15.95 | 108.72 | 49.19 | 28.17 | 16.37 | 101.20 |
| 18.00 | 41.92 | 23.17 | 13.78 | 93.20 | 42.23 | 24.20 | 14.06 | 86.84 |
| 20.00 | 34.98 | 19.39 | 11.62 | 77.69 | 35.27 | 20.22 | 11.76 | 72.49 |
| 22.00 | 28.03 | 15.60 | 9.45 | 62.17 | 28.31 | 16.24 | 9.46 | 58.13 |
| 24.00 | 21.08 | 11.81 | 7.28 | 46.66 | 21.35 | 12.27 | 7.16 | 43.77 |
| 26.00 | 14.14 | 8.02 | 5.11 | 31.14 | 14.39 | 8.29 | 4.85 | 29.41 |
| 28.00 | 7.19 | 4.23 | 2.94 | 15.62 | 7.43 | 4.32 | 2.55 | 15.06 |

#### 6.3.1. Parameter estimation

We fit the TED process with random drifts and covariates defined in Section 4.1 based on the prior specified in Section 4.2 to the transformed stress loss degradation data. We first run 2000 iterations as the burn-in period of the Markov chains, and then another 40,000 iterations are further run to obtain the posterior samples for posterior inference. According to the trace plot for the posterior samples, we can justify the convergency of the Markov chain. The posterior estimation of the drift parameters $\mu_i$, dispersion $\lambda$, $p$, $\mu_0$ and $\beta$ along with the log-likelihood, AIC and DIC for TED process along with Wiener process, gamma process and IG process for the stress loss data are shown in Table 9. The TED process with $p = 1.438$ has the maximal value of the log-likelihood and minimal AIC and DIC.

#### 6.3.2. Reliability inference

According to Theorem 4.1, we derive the failure-time distribution for the stress-loss data shown in Figure 9. The MTTFs are given in Table 10. As the temperature increases, the MTTF decreases. The estimated PDFs of RUL under usage stress when the current degradations are 2, 14, 26, respectively, can be derived based on Theorem 4.2 shown in Figure 10. The estimated PDFs concentrate on the MLR, as the unit #1 degrades towards the end of its life. Figure 11 shows the estimated PDFs and CDFs of the RULs based on the TED process with random drifts and covariates along with the MRLs (dotted line) under usage stress at different time points. The corresponding MRLs for units #1, #7, #13 at different time points are shown in Table 11.

## 7. Conclusion

In this paper, we proposes a TED process with random drifts and a TED process with random drifts and covariates. We demonstrate the applicative effectiveness of the proposed models by the classical GaAs laser degradation data and stress relaxation accelerated degradation data, respectively. The closed-form of the failure-time distribution, MTTFs, RUL distribution and MRLs are derived. The proposed model performs better than the traditional stochastic process. We expected to extend these two model to more complication degradation paths.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

## Funding

Wang's research was partially supported by the National Natural Science Foundation of China [grant number 12001266] and Natural Science Foundation of Jiangsu Province of China [grant number BK20180813]. Tang's research was partially supported by the National Natural Science Foundation of China [grant number 12271168, 12531013] and the 111 Project of China [grant number B14019].

## Ethics approval

The study was approved by the Danish Data Protection Agency (Record number 2013-41-2569).

## Data availability statement

Data are available upon reasonable request. No additional data are available.

## ORCID

*Pingping Wang* http://orcid.org/0000-0003-2174-049X
*Yincai Tang* http://orcid.org/0000-0001-6756-6461

## References

Bae, S. J., Kuo, W., & Kvam, P. H. (2007). Degradation models and implied lifetime distributions. *Reliability Engineering & System Safety*, *92*(5), 601–608. https://doi.org/10.1016/j.ress.2006.02.002

Balakrishnan, N., & Kundu, D. (2019). Birnbaum–Saunders distribution: A review of models, analysis, and applications. *Applied Stochastic Models in Business and Industry*, *35*(1), 4–49. https://doi.org/10.1002/asmb.v35.1

Bernardo, J. M., & Smith, A. (1994). *Bayesian Theory*. JohnWiley & Sons.

Chen, Z., Xia, T., Li, Y., & Pan, E. (2019). Tweedie exponential dispersion processes for degradation modeling, prognostic, and accelerated degradation test planning. *IEEE Transactions on Reliability*, *69*(3), 887–902. https://doi.org/10.1109/TR.24

Chen, Z., Xia, T., Li, Y., & Pan, E. (2021). Random-effect models for degradation analysis based on nonlinear Tweedie exponential-dispersion processes. *IEEE Transactions on Reliability*, *71*(1), 47–62. https://doi.org/10.1109/TR.2021.3107050

Daniels, H. E. (1954). Saddlepoint approximations in statistics. *The Annals of Mathematical Statistics*, *25*(4), 631–650. https://doi.org/10.1214/aoms/1177728652

dos Santos, T. R., & Colosimo, E. A. (2015). A modified approximate method for analysis of degradation data. *Journal of Applied Statistics*, *42*(6), 1322–1331. https://doi.org/10.1080/02664763.2014.999651

Duan, F., & Wang, G. (2018). Exponential-dispersion degradation process models with random effects and covariates. *IEEE Transactions on Reliability*, *67*(3), 1128–1142. https://doi.org/10.1109/TR.24

Dunn, P. K., & Smyth, G. K. (2008). Evaluation of Tweedie exponential dispersion model densities by Fourier inversion. *Statistics and Computing*, *18*(1), 73–86. https://doi.org/10.1007/s11222-007-9039-6

Fang, G., Pan, R., & Wang, Y. (2022). Inverse Gaussian processes with correlated random effects for multivariate degradation modeling. *European Journal of Operational Research*, *300*(3), 1177–1193. https://doi.org/10.1016/j.ejor.2021.10.049

Hong, L., & Ye, Z. (2017). When is acceleration unnecessary in a degradation test?*Statistica Sinica*, *27*(3), 1461–1483.

Jørgensen, B. (1986). Some properties of exponential dispersion models. *Scandinavian Journal of Statistics*, *13*(3), 187–197.

Jørgensen, B. (1997). *The Theory of Dispersion Models*. CRC Press.

Lawless, J., & Crowder, M. (2004). Covariates and random effects in a Gamma process model with application to degradation and failure. *Lifetime Data Analysis*, *10*(3), 213–227. https://doi.org/10.1023/B:LIDA.0000036389.14073.dd

Lim, H., & Yum, B.-J. (2011). Optimal design of accelerated degradation tests based on Wiener process models. *Journal of Applied Statistics*, *38*(2), 309–325. https://doi.org/10.1080/02664760903406488

Limon, S., Yadav, O. P., & Liao, H. (2017). A literature review on planning and analysis of accelerated testing for reliability assessment. *Quality and Reliability Engineering International*, *33*(8), 2361–2383. https://doi.org/10.1002/qre.v33.8

Lu, C. J., & Meeker, W. O. (1993). Using degradation measures to estimate a time-to-failure distribution. *Technometrics*, *35*(2), 161–174. https://doi.org/10.1080/00401706.1993.10485038

Luo, C., Shen, L., & Xu, A. (2022). Modelling and estimation of system reliability under dynamic operating environments and lifetime ordering constraints. *Reliability Engineering & System Safety*, *218*,108136. https://doi.org/10.1016/j.ress.2021.108136

Meeker, W. Q., Escobar, L. A., & Lu, C. J. (1998). Accelerated degradation tests: Modeling and analysis. *Technometrics*, *40*(2), 89–99. https://doi.org/10.1080/00401706.1998.10485191

Peng, C.-Y. (2015). Inverse Gaussian processes with random effects and explanatory variables for degradation data. *Technometrics*, *57*(1), 100–111. https://doi.org/10.1080/00401706.2013.879077

Shat, H. (2024). Optimal design of stress levels in accelerated degradation testing for multivariate linear degradation models. *Journal of Applied Statistics*, *51*(3), 534–554. https://doi.org/10.1080/02664763.2022.2140332
Shi, Y., & Meeker, W. Q. (2011). Bayesian methods for accelerated destructive degradation test planning. *IEEE Transactions on Reliability*, *61*(1), 245–253. https://doi.org/10.1109/TR.2011.2170115
Tseng, S.-T., & Lee, I.-C. (2016). Optimum allocation rule for accelerated degradation tests with a class of exponential-dispersion degradation models. *Technometrics*, *58*(2), 244–254. https://doi.org/10.1080/00401706.2015.1033109
Wang, X. (2008). A pseudo-likelihood estimation method for nonhomogeneous Gamma process model with random effects. *Statistica Sinica*, *18*(3), 1153–1163.
Wang, X. (2010). Wiener processes with random effects for degradation data. *Journal of Multivariate Analysis*, *101*(2), 340–351. https://doi.org/10.1016/j.jmva.2008.12.007
Yang, G. (2007). *Life Cycle Reliability Engineering*. John Wiley & Sons.
Ye, Z.-S., Chen, L.-P., Tang, L. C., & Xie, M. (2014). Accelerated degradation test planning using the inverse Gaussian process. *IEEE Transactions on Reliability*, *63*(3), 750–763. https://doi.org/10.1109/TR.24
Ye, Z.-S., & Xie, M. (2015). Stochastic modelling and analysis of degradation for highly reliable products. *Applied Stochastic Models in Business and Industry*, *31*(1), 16–32. https://doi.org/10.1002/asmb.v31.1
Zhou, S., & Xu, A. (2019). Exponential dispersion process for degradation analysis. *IEEE Transactions on Reliability*, *68*(2), 398–409. https://doi.org/10.1109/TR.24

# Appendices

## Appendix 1. Full conditional posterior distribution for TED process

Before inferencing the full conditional posterior distribution of each parameter, we introduce one notation to make our express more explicit. $\boldsymbol{\Theta}_{\backslash\boldsymbol{\eta}}$ indicates a parameter vector with $\boldsymbol{\eta}$ removing from $\boldsymbol{\Theta}$. The full conditional posterior distributions of each parameter are derived as follows.

(1) With the joint posterior distribution proportional to (6), we derive the full conditional posterior distribution of $(\mu_k, \sigma_k^2)$ as

$$\pi(\eta, \sigma^2 \mid \boldsymbol{\Theta}_{\backslash(\eta,\sigma^2)}, \boldsymbol{\Delta}y) \propto (\sigma^2)^{(1/2+n/2+\nu_\mu+1)} \times \exp\left\{-\frac{2\xi_\mu + \lambda_\mu(\eta-\eta_\mu)^2 + \sum_{i=1}^n(\mu_i-\eta)^2}{2\sigma^2}\right\}, \tag{A1}$$

which is $\mathcal{NIG}(\lambda'_\mu, \eta'_\mu, \nu'_\mu, \xi'_\mu)$, where $\lambda'_\mu = \lambda_\mu + n$, $\eta'_\mu = (\lambda_\mu\eta_\mu + \sum_{i=1}^n \mu_i)/(\lambda_\mu + n)$, $\nu'_\mu = n/2 + \nu_\mu$, and $\xi'_\mu = \xi_\mu + \lambda_\mu\eta_\mu^2/2 + \sum_{i=1}^n \mu_i^2/2 - (\lambda_\mu\eta_\mu + \sum_{i=1}^n \mu_i)^2/(2(\lambda_\mu + n))$.

(2) To obtain the full conditional distribution of $\lambda$, we derive the following display

$$\pi(\lambda \mid \boldsymbol{\Theta}_{\backslash\lambda}, \boldsymbol{\Delta}y) \propto \lambda^{-\left(\alpha+\frac{\sum_{i=1}^n m_i - n}{2}-1\right)} \times \exp\left[-\lambda\left\{\beta + \sum_{i=1}^n\sum_{j=1}^{m_i-1}\frac{\Delta t_{i,j}}{2} d\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}; \mu_i, p\right)\right\}\right], \tag{A2}$$

which is Gamma $\left(\alpha + \frac{\sum_{i=1}^n m_i - n}{2}, \beta + \sum_{i=1}^n \sum_{j=1}^{m_i-1} \frac{\Delta t_{i,j}}{2} d\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}; \mu_i, p\right)\right)$.

(3) With the joint posterior distribution proportional to (6), it is easy to obtain the full conditional distributions of $p$, which are given by

$$\pi(p \mid \boldsymbol{\Theta}_{\backslash p}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{(p-\gamma)^2}{2\delta^2}\right\}$$
$$\times \prod_{i=1}^{n}\prod_{j=1}^{m_i-1} \exp\left[-\lambda\Delta t_{ij}\left\{\frac{\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}\right)^{2-p}}{(1-p)(2-p)} - \frac{\frac{\Delta y_{i,j}}{\Delta t_{i,j}}\mu_i^{1-p}}{1-p} + \frac{\mu_i^{2-p}}{2-p}\right\}\right]. \tag{A3}$$

(4) For $\mu_i, i = 1, \ldots, n$, the full conditional posterior distribution has the following form

$$\pi(\mu_i \mid \boldsymbol{\Theta}_{\backslash \mu_i}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{1}{2\sigma^2}(\mu_i-\eta)^2\right\}$$
$$\times \prod_{j=1}^{m_i-1} \exp\left[-\lambda\Delta t_{ij}\left\{\frac{\left(\frac{\Delta y_{i,j}}{\Delta t_{i,j}}\right)^{2-p}}{(1-p)(2-p)} - \frac{\frac{\Delta y_{i,j}}{\Delta t_{i,j}}\mu_i^{1-p}}{1-p} + \frac{\mu_i^{2-p}}{2-p}\right\}\right]. \tag{A4}$$

## Appendix 2. Full conditional posterior distribution for accelerated TED process

(1) With the joint posterior distribution proportional to (12), we derive the full conditional posterior distribution of $\mu_0$ as

$$\pi(\mu_0 \mid \boldsymbol{\Theta}_{\backslash \mu_0}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{(\mu_0-\vartheta_\mu)}{2\phi_\mu^2}\right\}\prod_{k=1}^{K}\prod_{i=1}^{n_k}\exp\left[-\frac{\{\mu_{ki}-\mu_0\exp(\beta\phi(S_k))\}^2}{2\sigma_k^2}\right]$$
$$\propto \exp\left(-\frac{1}{2}\left[\left\{\frac{1}{\phi_\mu^2}+\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\exp(2\beta\phi(S_k))}{\sigma_k^2}\right\}\mu_0^2\right.\right.$$
$$\left.\left. -2\left\{\frac{\vartheta_\mu}{\phi_\mu^2}+\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\exp(\beta\phi(S_k))\mu_{ki}}{\sigma_k^2}\right\}\mu_0\right]\right), \tag{A5}$$

which is $\mathcal{N}(\frac{B}{A}, \frac{1}{A})$, where $A = \frac{1}{\phi_\mu^2} + \sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\exp(2\beta\phi(S_k))}{\sigma_k^2}$ and $B = \frac{\vartheta_\mu}{\phi_\mu^2} + \sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\exp(\beta\phi(S_k))\mu_{ki}}{\sigma_k^2}$.

(2) The full conditional posterior distribution of $\sigma_k^2, k = 1, \ldots, K$ is

$$\pi(\sigma_k^2 \mid \boldsymbol{\Theta}_{\backslash(\sigma_k^2)}, \boldsymbol{\Delta}y) \propto (\sigma_k^2)^{-(\kappa+\frac{n_k}{2}+1)}\exp\left\{-\frac{\eta+\frac{1}{2}\sum_{i=1}^{n_k}\{\mu_{ki}-\mu_0\exp(\beta\phi(S_k))\}^2}{\sigma_k^2}\right\}, \tag{A6}$$

which is $\mathcal{IG}\left(\kappa+\frac{n_k}{2}, \eta+\frac{1}{2}\sum_{i=1}^{n_k}\{\mu_{ki}-\mu_0\exp(\beta\phi(S_k))\}^2\right)$.

(3) To obtain the full conditional distribution of $\lambda$, we derive the following display

$$\pi(\lambda \mid \boldsymbol{\Theta}_{\backslash\lambda}, \boldsymbol{\Delta}y) \propto \lambda^{-\left(\alpha+\frac{\sum_{k=1}^{K}n_k(m_k-1)}{2}-1\right)}$$
$$\times \exp\left[-\lambda\left\{\xi+\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_k-1}\frac{\Delta t_{k,i,j}}{2}d\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}};\mu_{ki},p\right)\right\}\right], \tag{A7}$$

which is Gamma $\left(\alpha+\frac{\sum_{k=1}^{K}n_k(m_k-1)}{2}, \xi+\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j=1}^{m_k-1}\frac{\Delta t_{k,i,j}}{2}d\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}};\mu_{ki},p\right)\right)$.

(4) For the full conditional posterior distribution of $\beta$, it can be derived as

$$\pi(\beta \mid \boldsymbol{\Theta}_{\setminus\beta}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{(\beta-\vartheta_\beta)^2}{2\phi_\beta^2}\right\}\exp\left[-\sum_{k=1}^{K}\sum_{i=1}^{n_k}\frac{\{\mu_{ki}-\mu_0\exp(\beta\phi(S_k))\}^2}{2\sigma_k^2}\right]. \tag{A8}$$

(5) The full conditional distribution of $p$ is given by

$$\pi(p \mid \boldsymbol{\Theta}_{\setminus p}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{(p-\gamma)^2}{2\delta^2}\right\}\prod_{k=1}^{K}\prod_{i=1}^{n_k}\prod_{j=1}^{n_i-1}\exp\left[-\lambda\Delta t_{k,i,j}\right. \tag{A9}$$

$$\left.\times\left\{\frac{\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\right)^{2-p}}{(1-p)(2-p)}-\frac{\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\mu_i^{1-p}}{1-p}+\frac{\mu_{ki}^{2-p}}{2-p}\right\}\right]. \tag{A10}$$

(5) For $\mu_{ki}, k=1,\dots,K; i=1,\dots,n_k$, the full conditional posterior distribution has the following form

$$\pi(\mu_{ki} \mid \boldsymbol{\Theta}_{\setminus\mu_{ki}}, \boldsymbol{\Delta}y) \propto \exp\left\{-\frac{\{\mu_{ki}-\mu_0\exp(\beta\phi(S_k))\}^2}{2\sigma_k^2}\right\}$$

$$\times\prod_{j=1}^{m_k-1}\exp\left[-\lambda\Delta t_{k,i,j}\left\{\frac{\left(\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\right)^{2-p}}{(1-p)(2-p)}-\frac{\frac{\Delta y_{k,i,j}}{\Delta t_{k,i,j}}\mu_{ki}^{1-p}}{1-p}+\frac{\mu_{ki}^{2-p}}{2-p}\right\}\right]. \tag{A11}$$

## Appendix 3. CDF figures

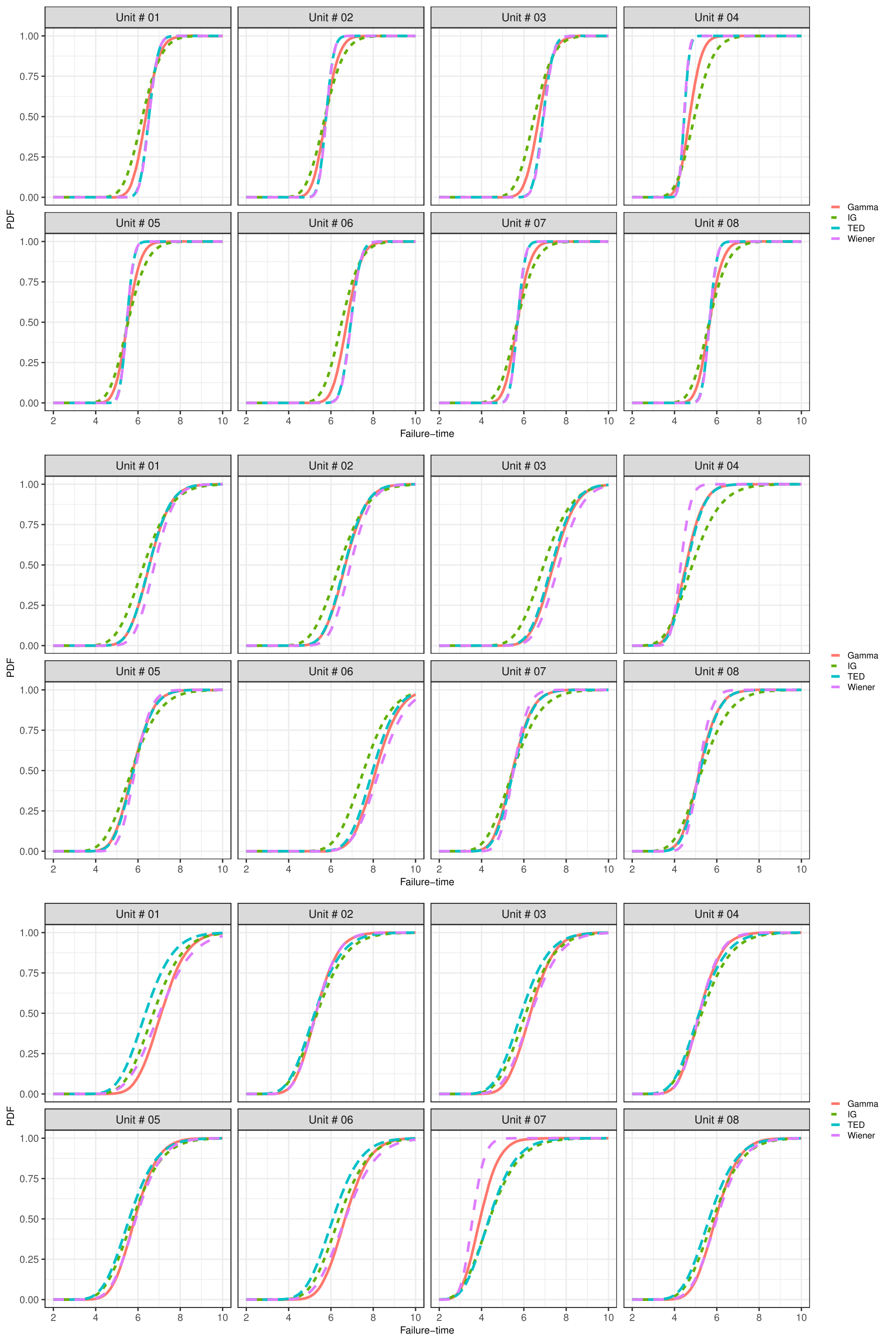


**Figure A1.** CDFs of failure-time based on Wiener process, gamma process, IG process and TED process model and BS distribution: Upper panel is for scenario I; Middle panel is for scenario II; Lower panel is for scenario III in the simulation study.

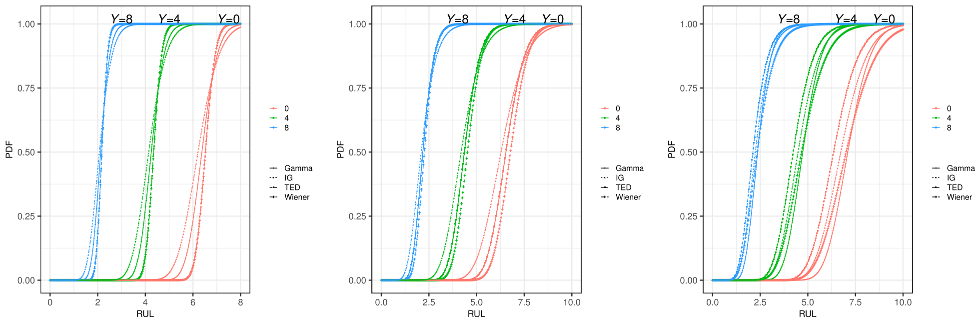


**Figure A2.** Left panel is the CDFs of RUL based on Wiener process, gamma process, IG process and TED process at current degradations 0, 4 and 8 for scenario I; Middle panel is for scenario II; Right panel is for scenario III in the simulation study.

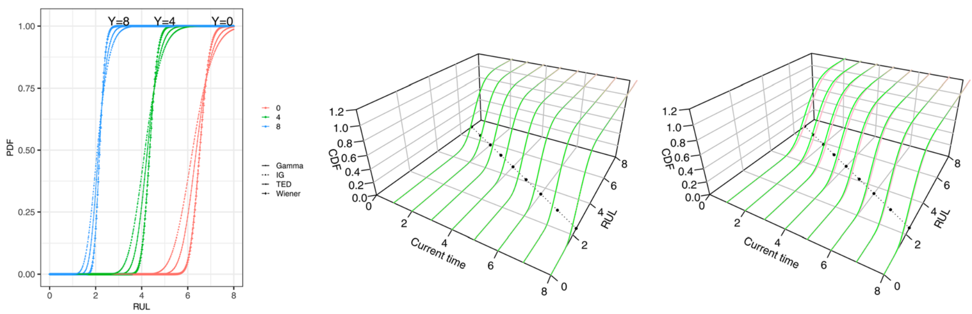


**Figure A3.** The pink solid line and green dashed line are the CDFs of the RUL distribution derived based on the estimated parameters obtained from true model and TED process, respectively. The black dashed line is the MRLs based on the true parameters. Left panel is for scenario I, middle panel is for scenario II and right panel is for scenario III in the simulation study.