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Stochastic dominance for claim spacings from heterogeneous insurance portfolios

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ABSTRACT

This paper investigates the stochastic dominance of claim spacings from heterogeneous insurance portfolios with (in)dependent claim occurrence probabilities and claim severities. We derive explicit distribution functions for the claim range and the second claim spacing. We then establish sufficient conditions for comparing claim ranges in the increasing convex order for two sets of heterogeneous insurance portfolios. Our results show that higher occurrence probabilities, characterized by stochastic arrangement increasing, together with larger claim severities under the conditional increasing convex order, lead to a larger claim range in the increasing convex order. We also show that, when claim indicators are independent, greater heterogeneity in the negative log-transforms of occurrence probabilities leads to a larger claim range in the increasing convex order. In addition, if claim severities are independent and follow proportional hazard rate models, greater heterogeneity in the associated parameters yields a larger claim range in the usual stochastic order. For the second claim spacing, we establish stochastic dominance results under the increasing concave order and the usual stochastic order for two sets of heterogeneous and (in)dependent portfolios. Several numerical examples are provided for illustration.

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1. Introduction

Sample ranges and spacings are fundamental in probability and statistics, attracting interest across fields such as goodness-of-fit testing, actuarial science, reliability theory, and auction theory. The stochastic comparisons between ranges and spacings have been widely studied over the past decades. Early work by Barlow and Proschan (1966) established the usual stochastic order between successive normalized spacings from decreasing failure rate distributions. Since then, many authors have investigated stochastic orders for ranges and spacings from homogeneous or heterogeneous samples, especially under exponential and proportional hazard rate models; see, for example, Pledger and Proschan (1971), Khaledi

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and Kochar (2000), Zhao and Li (2009), Genest et al. (2009), Xu and Balakrishnan (2012), Kochar and Rojo (1996), Ding et al. (2013), Zhao and Zhang (2012), Torrado and Lillo (2015), Zhao et al. (2016), and Balakrishnan et al. (2019). Comprehensive reviews can be found in Kochar (2012) and Kochar (2022).

In insurance applications, ranges and spacings provide simple yet informative summaries of claim dispersion and local loss patterns. Given a set of claims, the claim range, defined as the difference between the largest and smallest claim amounts, measures the overall extreme spread of losses. By contrast, the second claim spacing, defined as the difference between the second smallest and the smallest claim amounts, captures a local gap between neighbouring ordered claims. Actuaries may compare claim ranges across policy years or lines of business to assess variation in risk exposure and detect potential anomalies. In life insurance, range analysis of age-specific mortality rates can help evaluate the smoothness of life tables and identify data outliers. More broadly, the claim range captures the within-scenario amplitude of losses with minimal distributional assumptions, serving as a dispersion measure and a useful statistic for inference. The claim spacing, on the other hand, isolates local marginal deterioration immediately after the first claim trigger, thereby revealing threshold-induced clustering and dependence effects that may be obscured by global metrics. For further background on ranges, spacings, and order statistics, readers are referred to Hartley (1942), Pyke (1965), and David and Nagaraja (2004).

In actuarial science, a predominant approach to modelling claims from insurance portfolios is the *individual risk model* (IRM). The IRM considers an insurance portfolio with n heterogeneous policies. For policy $i \in \{1, \ldots, n\}$, let $I_i \in \{0, 1\}$ denote the claim occurrence indicator and $X_i \geq 0$ the claim severity. The policy-level claim is then given by the mixed discrete-continuous random variable $Y_i := I_i X_i$, which has an atom at zero. A substantial literature has studied stochastic comparisons for aggregate claims and extreme claim amounts in IRMs; see, for example, Frostig (2001), Denuit and Frostig (2006), Khaledi and Ahmadi (2008), Zhang et al. (2018), Zhang and Zhao (2015), Zhang and Cheung (2020), Ariyafar et al. (2020), and Torrado and Navarro (2021). Table 1 summarizes some recent significant developments regarding the stochastic properties of aggregate claims, as well as the smallest and largest claim amounts in IRMs. However, much less is known about claim ranges and claim spacings in this setting. To the best of our knowledge, except for (Balakrishnan et al., 2018), which considered claim ranges for portfolios with only two claims, no general stochastic comparison results are available for portfolios with more than two claims. Moreover, classical results for sample ranges and spacings cannot be directly applied to IRMs, because each policy-level claim amount has a positive mass at zero. This feature fundamentally changes the distributional structure and makes it necessary to derive new distributional formulas and stochastic comparison results for claim ranges and claim spacings.

In this work, we will first present the explicit expressions for the distribution functions of claim ranges and second claim spacings derived from a group of heterogeneous claims, a topic that has not been addressed in existing literature. These useful expressions enable us to establish sufficient conditions for the increasing convex order between claim ranges and the increasing concave order for second claim spacings. We apply conditional stochastic orders to model the dependence among claim severities, noting that occurrence probabilities may also depend on the concept of stochastic arrangement increasing. Additionally, we provide strengthened results based on the usual stochastic order for cases where claim severities follow proportional hazard rates or exponential distributions.

Table 1. Literature review on recent development on stochastic dominance of the aggregation, smallest, and largest claim amounts.

Classification	Literature
Independent claim amount	Khaledi and Ahmadi (2008) (aggregation)
	Barmalzan et al. (2017) (smallest, largest)
	Balakrishnan et al. (2018) (largest)
	Barmalzan et al. (2020) (smallest, largest)
	Zhang (2021) (largest)
Dependent occurrence probability and/or dependent claim severity	Zhang and Zhao (2015) (aggregation)
	Zhang et al. (2018) (aggregation)
	Zhang and Cheung (2020) (aggregation)
	Ariyafar et al. (2020) (aggregation, minimum, maximum)
	Torrado and Navarro (2021) (smallest, largest)

The rest of the paper is organized as follows. Section 2 introduces the basic definitions, concepts, and lemmas used throughout the paper. Section 3 derives exact distribution formulas for the claim range and second claim spacing of heterogeneous claims. Sections 4 and 5 establish stochastic ordering results for claim ranges and second claim spacings under heterogeneous insurance portfolios, considering dependent or independent claim occurrences and severities. Section 6 provides numerical examples, and Section 7 concludes the paper. All proofs are given in the Appendix.

2. Preliminaries

Throughout the paper, we use $\overset{\text{sign}}{=}$ to express that both sides of the equality have the same sign. Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. All random variables are defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and expectations and integrals are assumed to exist whenever they appear. For any two real-valued vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, let $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ and $y_{1:n} \le y_{2:n} \le \cdots \le y_{n:n}$ be their increasing arrangements, respectively. For any (i, j) with $1 \le i < j \le n$, let $\tau_{ij}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$. We define the sets $\mathcal{I}_n = \{\boldsymbol{x} : x_1 \le x_2 \le \cdots \le x_n\}$ and $\mathcal{D}_n = \{\boldsymbol{x} : x_1 \ge x_2 \ge \cdots \ge x_n\}$, which represent the collections of all non-decreasing and non-increasing n-dimensional real vectors, respectively.

2.1. Stochastic orders

Stochastic orders have a central place in current research activity given their applicability in important research areas of probability and statistics, in particular, in reliability theory and actuarial mathematics. Let $F[\overline{F}]$ and $G[\overline{G}]$ be the distribution[survival] functions of two random variables X and Y, respectively.

Definition 2.1: A random variable X is said to be smaller than Y in the

(i) usual stochastic order (denoted by $X \le_{\text{st}} Y$) if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for any increasing function $\phi : \mathbb{R} \to \mathbb{R}$, or equivalently, $\overline{F}(t) \le \overline{G}(t)$, for $t \in \mathbb{R}$;

(ii) increasing convex[concave] order (denoted by $X \le_{\text{icx[icv]}} Y$) if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for any increasing convex[concave] function $\phi : \mathbb{R} \to \mathbb{R}$.

Without loss of generality, the orders in Definition 2.1 can be verified using respectively increasing, increasing convex and increasing concave functions $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(0) = 0$. If $\phi(0) = c \neq 0$, one can define a shifted function $\tilde{\phi}(x) := \phi(x) - c$ such that $\tilde{\phi}(0) = 0$. Since ϕ is increasing and real-valued, the constant $c = \phi(0)$ is finite and can be subtracted from both sides of the inequality $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$.

The usual stochastic order implies both the increasing convex and increasing concave orders, but not conversely in general. By Theorem (4.A.5) in Shaked and Shanthikumar (2007),

$$X \leq_{\text{icx}} Y \iff \int_x^\infty \bar{F}(u)\,\mathrm{d}u \leq \int_x^\infty \bar{G}(u)\,\mathrm{d}u, \quad \forall\, x \in \mathbb{R}; \tag{1}$$

equivalently, $\pi_X(t) := \mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+] =: \pi_Y(t), \forall t \in \mathbb{R}$. Likewise, by Theorem (4.A.7) in Shaked and Shanthikumar (2007),

$$X \leq_{\text{icv}} Y \iff \int_{-\infty}^x F(u)\,\mathrm{d}u \geq \int_{-\infty}^x G(u)\,\mathrm{d}u, \quad \forall\, x \in \mathbb{R}; \tag{2}$$

equivalently, $\tilde{\pi}_X(t) := \mathbb{E}[(t-X)_+] \geq \mathbb{E}[(t-Y)_+] =: \tilde{\pi}_Y(t), \forall t \in \mathbb{R}$.

The following lemma provides a sufficient condition for the increasing convex and increasing concave orders.

Lemma 2.2 (Shaked & Shanthikumar, 2007): *Let X and Y be two random variables with distribution functions F and G, respectively, and with finite expectations such that $\mathbb{E}[X] \leq \mathbb{E}[Y]$. If there exists an x_0 such that $F(x) \leq G(x)$ $[F(x) \geq G(x)]$ for $x < x_0$ and $F(x) \geq G(x)$ $[F(x) \leq G(x)]$ for $x \geq x_0$, then $X \leq_{\text{icx}} Y$ $[X \leq_{\text{icv}} Y]$.*

Christofides and Hadjikyriakou (2015) defined a class of conditional stochastic orders with respect to a sub-σ-algebra; see also Leskelä and Vihola (2017). In this context, we recall the definitions of conditional convex and concave orders, which will be utilized in our main results regarding the dependence structure among claim severities. To the best of the authors' knowledge, there has been limited research on the applications in actuarial science of conditional stochastic orders.[1] Therefore, we hope that our results will contribute to expanding the application scope of these orders.

Definition 2.3: Suppose that X and Y are defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$. A random variable X is said to be smaller than Y in the conditional convex[concave] order (denoted by $X \leq_{\mathcal{F}-\text{icx[icv]}} Y$) if $\mathbb{E}[\phi(X)|\mathcal{F}] \leq \mathbb{E}[\phi(Y)|\mathcal{F}]$ almost surely, for any increasing convex[concave] function ϕ, where $\mathcal{F} \subset \mathcal{G}$ is a sub-σ-algebra.

Unlike unconditional stochastic orders that compare only marginal distributions, conditional stochastic orders incorporate external information $\mathcal{F}$ into the comparison. This allows one to use available information rather than averaging it out. Such orders are particularly suitable and necessary for insurance risk modelling because insurance operations are inherently conditional and mixed: payments are activated by random trigger events, while portfolios exhibit heterogeneity, random exposure, and background risk. Unconditional orders compare overall distributions and may fail to preserve comparisons once information is revealed.

[1] Some recent works have applied the conditional stochastic orders in comparing two different McKean-Vlasov processes or stochastic Volterra equations; see for example (Bernou et al., 2025; Jourdain & Pagès, 2025; Liu & Pagès, 2022, 2023).

In contrast, conditional stochastic orders maintain the required monotonicity under the given information, preventing rank reversals induced by layering, truncation, selection, or reinsurance structures. This stability supports consistent decisions in pricing, reserving, limit setting, and reinsurance design when models are applied in the informational context in which insurance actually operates.

Now, we consider a special case of Definition 2.3 where the σ-algebra $\mathcal{F}$ is generated by a random element Z taking values in a general measurable space S, i.e. $\mathcal{F} = \sigma(Z)$. Then, for any random variable X defined on the same probability space as Z, there exists a probability kernel $\mathbb{P}$ from S to $\mathbb{R}$ such that $\omega \mapsto \mathbb{P}_{Z(\omega)}(B)$ is a version of $\mathbb{E}[\mathbf{1}_{\{X\in B\}} \mid Z]$ for every Borel set $B \subset \mathscr{R}$. Such $\mathbb{P}$ is called a *regular conditional distribution* of X given Z; see Leskelä and Vihola (2017) for formal discussions.

Lemma 2.4 (Leskelä & Vihola, 2017): *Let X and Y be integrable random variables and Z a random element in a measurable space S, all defined on a common probability space. If $\mathbb{P}$ and $\mathbb{Q}$ are regular conditional distributions of X and Y given Z, then*

$$X \leq_{Z-\mathrm{icx}} Y \Longleftrightarrow \mathbb{P}_z \leq_{\mathrm{icx}} \mathbb{Q}_z, \quad \text{for } \mu - a.s.\, z \in S,$$

where μ stands for the distribution of Z.

The following theorem is very helpful to construct the conditional increasing convex and conditional increasing concave orders in our numerical examples.

Theorem 2.5: *Let X, Y, and Z be continuous random variables. Denote the distribution functions of X and Y by F and G, respectively, and let S denote the support of Z. Suppose that X and Y have finite expectations such that $\mathbb{E}[X] \leq \mathbb{E}[Y]$, (X, Z) and (Y, Z) admit the same differentiable copula C.*[2] *Then $X \leq_{Z-\mathrm{icx}} [\leq_{Z-\mathrm{icv}}] Y$ if both of following conditions are satisfied:*

(i) *There exists an x_0 such that $F(x) \leq [\geq] G(x)$ for $x < x_0$ and $F(x) \geq [\leq] G(x)$ for $x \geq x_0$.*
(ii) *$\mathbb{E}[X|Z = z] \leq \mathbb{E}[Y|Z = z]$ holds for all $z \in S$.*

Proof: Here we only establish the result for '$\leq_{Z-\mathrm{icx}}$' and the proof for '$\leq_{Z-\mathrm{icv}}$' can be derived similarly. Suppose Z has distribution function H. Then it can be obtained that (see Property 4.2.13 in Denuit et al. (2005))

$$\mathbb{P}\{X \leq x|Z = z\} = \partial_2 C(F(x), H(z)) \quad \text{and} \quad \mathbb{P}\{Y \leq x|Z = z\} = \partial_2 C(G(x), H(z)),$$

where the notation $\partial_2 C$ refers to the partial derivative of the copula C with respect to its second component. For each fixed $z \in S$, the mapping $u \mapsto \partial_2 C(u, H(z))$ is nondecreasing on $[0, 1]$ and hence acts as a distortion function. Therefore, since F and G cross only once, the conditional distribution functions $x \mapsto \mathbb{P}\{X \leq x \mid Z = z\}$ and $x \mapsto \mathbb{P}\{Y \leq x \mid Z = z\}$ also

[2] For a bivariate random vector (X_1, X_2) with respective marginal distribution functions F_1 and F_2 and joint distribution function F_{X_1,X_2}, it is well known that F_{X_1,X_2} admits the decomposition

$$F_{X_1,X_2}(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \quad x_1, x_2 \in \mathbb{R},$$

where C, referred to as the copula of F_{X_1,X_2}, is a bivariate distribution function on $(0, 1)^2$ with uniform marginals *(cf. Nelsen, 2006)*. If both F_1 and F_2 are continuous, then C is uniquely determined by $C(u, v) = F_{X_1,X_2}(F_1^{-1}(u), F_2^{-1}(v))$.

cross only once, with the same direction of crossing. Together with the assumed conditional mean inequality $\mathbb{E}[X \mid Z = z] \leq \mathbb{E}[Y \mid Z = z]$, Lemma 2.2 implies that, for every $z \in S$, $[X \mid Z = z] \leq_{\text{icx}} [Y \mid Z = z]$. Finally, applying Lemma 2.4, we obtain $X \leq_{Z-\text{icx}} Y$. This completes the proof. ■

By imposing constraints on the crossing pattern of the distribution functions of the random variables, Theorem 2.5 provides sufficient conditions for establishing the conditional stochastic order when the two sets of random variable have a common copula structure. The following example shows that the result may still hold when the copula structures are different.

Example 2.6: Consider three random variables $X \sim U(0, 1/2)$, $Y \sim U(0, 1)$ and $Z \sim U(0, 2)$. Suppose that (X, Y, Z) admits the copula

$$C(u, v, w) = uvw\{1 + \theta_{12}(1 - u)(1 - v) + \theta_{13}(1 - u)(1 - w) + \theta_{23}(1 - v)(1 - w)\}.$$

Let $\theta_{12} = 0.4$, $\theta_{13} = 0$ and $\theta_{23} = 0.3$. Then the dependence structures of (X, Z) and (Y, Z) are not identical: (X, Z) has the independence copula, whereas (Y, Z) has an FGM copula with parameter $\theta_{23} = 0.3$. Nevertheless, the conditional stochastic order still holds. Indeed, since $\theta_{13} = 0$, X and Z are independent, and hence $F_{X|Z=z}(t) = 2t$, for $0 < t < 1/2$. On the other hand, the conditional distribution function of Y given $Z = z$ is

$$F_{Y|Z=z}(t) = t\{1 + \theta_{23}(1 - t)(1 - 2F_Z(z))\}, \qquad 0 < t < 1.$$

Therefore, for $0 < t < 1/2$,

$$F_{Y|Z=z}(t) \leq t(1 + \theta_{23}) = 1.3t \leq 2t = F_{X|Z=z}(t).$$

For $1/2 \leq t < 1$, the same inequality is immediate since $F_{X|Z=z}(t) = 1$. Thus,

$$[X \mid Z = z] \leq_{\text{st}} [Y \mid Z = z], \quad z \in [0, 2],$$

which implies $X \leq_{Z-\text{icx}} Y$. Similarly, by conditioning on $X = x$, one obtains

$$[Y \mid X = x] \leq_{\text{st}} [Z \mid X = x], \quad x \in [0, 1/2],$$

and hence $Y \leq_{X-\text{icx}} Z$. This example illustrates that the conditional increasing convex order may still hold despite differences in the dependence structures.

2.2. Majorization order

The majorization order is a useful tool in establishing various inequalities arising from many research areas, and it describes the degree of heterogeneity among the elements of the concerned vectors.

Definition 2.7: A vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to majorize $\boldsymbol{y} \in \mathbb{R}^n$, denoted by $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}$, for all $j = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$.

Lemma 2.8 (Marshall et al., 2011): *Let ϕ be a real-valued function defined on $\mathcal{D}_n$ and continuously differentiable on the interior of $\mathcal{D}_n$. Denote the partial derivative of ϕ with respect to*

its kth argument by $\phi_{(k)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial z_k$, $k = 1, \ldots, n$. *Then,* $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ *whenever* $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$ *on* $\mathcal{D}_n$ *if and only if* $\phi_{(1)}(\boldsymbol{z}) \ge \phi_{(2)}(\boldsymbol{z}) \ge \cdots \ge \phi_{(n)}(\boldsymbol{z})$, *i.e., the gradient* $\nabla\phi(\boldsymbol{z}) \in \mathcal{D}_n$, *for all* $\boldsymbol{z}$ *in the interior of* $\mathcal{D}_n$.

The majorization order will be used to characterize the degree of heterogeneity in claim severities or occurrence probabilities, facilitating the establishment of stochastic ordering results between claim ranges or second claim spacings. For a more detailed discussion on the theory of majorization and its applications, please refer to Marshall et al. (2011).

2.3. Stochastically arrangement increasing

To present the main results, we give some notations for modelling the dependence of occurrence probabilities. For a given Bernoulli random vector $\boldsymbol{I} = (I_1, I_2, \ldots, I_n)$, we define, whenever no ambiguity arises, $p(\boldsymbol{\lambda}) = \mathbb{P}(\boldsymbol{I} = \boldsymbol{\lambda})$ for vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, that is, $p(\boldsymbol{\lambda}) = \mathbb{P}(I_1 = \lambda_1, I_2 = \lambda_2, \ldots, I_n = \lambda_n)$. Specially, set $p(\boldsymbol{0}) = p(0, 0, \ldots, 0)$ and $p(\boldsymbol{1}) = p(1, 1, \ldots, 1)$. In addition, define the set

$$\Lambda_k = \{\boldsymbol{\lambda} | \lambda_i = 0 \text{ or } 1, i = 1, \ldots, n, \lambda_1 + \lambda_2 + \cdots + \lambda_n = k\}, \quad k = 0, 1, \ldots, n.$$

For all $1 \le i \ne j \le n$ and $k = 1, 2, \ldots, n-1$, denote

$$\Lambda_k^{i,j}(0,1) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 0, \lambda_j = 1\}, \quad \Lambda_k^{i,j}(0,0) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 0, \lambda_j = 0\},$$
$$\Lambda_k^{i,j}(1,0) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 1, \lambda_j = 0\}, \quad \Lambda_k^{i,j}(1,1) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 1, \lambda_j = 1\}.$$

Obviously, $\Lambda_1^{i,j}(1,1) = \Lambda_{n-1}^{i,j}(0,0) = \emptyset$ and

$$\Lambda_k = \Lambda_k^{i,j}(0,1) \cup \Lambda_k^{i,j}(0,0) \cup \Lambda_k^{i,j}(1,0) \cup \Lambda_k^{i,j}(1,1).$$

The notion of stochastic arrangement increasing (SAI) has been widely applied in actuarial science and financial engineering to model the dependence among ordered random risks. This concept not only characterizes the positive dependence structure among random variables but also provides a stochastic ordering of these variables.

Definition 2.9: A random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to be stochastic arrangement increasing (SAI), if $\mathbb{E}[h(\boldsymbol{X})] \ge \mathbb{E}[h(\tau_{ij}(\boldsymbol{X}))]$, for any pair (i, j) such that $1 \le i < j \le n$ and any $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(\boldsymbol{x}) \ge h(\tau_{ij}(\boldsymbol{x}))$, for any $\boldsymbol{x} \in \mathcal{I}_n$.

Utilizing the aforementioned notations, Cai and Wei (2015) presented characterizations of the SAI Bernoulli random vector, which has been employed for modelling occurrence probabilities by Zhang et al. (2018) and Zhang and Cheung (2020).

3. Distribution functions of claim spacings

Let $X_1, X_2, \ldots, X_n$ be independent and continuous nonnegative random variables with probability density function f_i and cumulative distribution function F_i, $i = 1, 2, \ldots, n$. According

to the 2.3.2(b) on Page 26 of David and Nagaraja (2004), the sample range $W_n := X_{n:n} - X_{1:n}$ has the cumulative distribution function

$$F_{W_n}(w) = \sum_{i=1}^{n} \int_0^{\infty} f_i(x) \prod_{j=1, j\neq i}^{n} \left[F_j(x+w) - F_j(x)\right] \mathrm{d}x. \tag{3}$$

However, the expression (3) cannot be applied directly to derive the distribution function of claim ranges from insurance portfolios, as each claim typically has a non-zero mass probability at the point 0. In the remainder of this section, we present the expressions for the distribution functions of the claim range and the second claim spacing.

Consider a set of insurance portfolios with claims $(I_1X_1, \ldots, I_nX_n)$, where $\boldsymbol{I}$ is generally dependent while $\boldsymbol{X}$ can be assumed to be independent and heterogeneous. Let $Y_{n:n}$, $Y_{2:n}$ and $Y_{1:n}$ be the largest, second smallest and smallest claim amounts of the insurance portfolios. Then the claim range and the second claim spacing of this portfolio can be denoted as $R_n := Y_{n:n} - Y_{1:n}$ and $S_n := Y_{2:n} - Y_{1:n}$, respectively.

If both of the claim severities and occurrence probabilities are dependent, one can note that, for any $y \geq 0$,

$$\begin{aligned}
\mathbb{P}(R_n \leq y) &= \mathbb{P}(Y_{n:n} - Y_{1:n} \leq y) \\
&= p(\mathbf{0})\mathbb{P}(0 \leq y) + p(\mathbf{1})\mathbb{P}(X_{n:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})\mathbb{P}\left(\max\{\lambda_1X_1, \lambda_2X_2, \ldots, \lambda_nX_n\}\right. \\
&\quad \left. - \min\{\lambda_1X_1, \lambda_2X_2, \ldots, \lambda_nX_n\} \leq y\right) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(X_{n:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})\mathbb{P}\left(\max\{\lambda_1X_1, \lambda_2X_2, \ldots, \lambda_nX_n\} \leq y\right),
\end{aligned} \tag{4}$$

where the last equality is due to the fact that $\min\{\lambda_1X_1, \lambda_2X_2, \ldots, \lambda_nX_n\} = 0$ a.s. for $\boldsymbol{\lambda} \in \Lambda_k$ and $k = 1, \ldots, n-1$.

Furthermore, if the claim severities X_i's are independent and heterogeneous with distribution functions F_i's, then, by applying (3), the distribution function of R_n can be written as

$$\begin{aligned}
\mathbb{P}(R_n \leq y) &= p(\mathbf{0}) + p(\mathbf{1}) \sum_{i=1}^{n} \int_0^{\infty} f_i(u) \prod_{j=1, j\neq i}^{n} \left[F_j(u+y) - F_j(u)\right] \mathrm{d}u \\
&\quad + \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) \prod_{i=1}^{n} F_i^{\lambda_i}(y) \right\}.
\end{aligned} \tag{5}$$

In particular, if the claim severities X_i's are i.i.d. and have a common distribution function F, then the distribution function of R_n is given by

$$\mathbb{P}(R_n \leq y) = p(\mathbf{0}) + np(\mathbf{1}) \int_0^{\infty} f(u)[F(u+y) - F(u)]^{n-1} \mathrm{d}u$$

$$+\sum_{k=1}^{n-1}\left\{F^k(y)\sum_{\boldsymbol{\lambda}\in\Lambda_k}p(\boldsymbol{\lambda})\right\}. \tag{6}$$

Next, we calculate the distribution of the second claim spacing. If both of the claim severities and occurrence probabilities are dependent, it can be noted that, for any $y \geq 0$,

$$\begin{aligned}
\mathbb{P}(S_n \leq y) &= p(\mathbf{0})\mathbb{P}(0 \leq y) + p(\mathbf{1})\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})\mathbb{P}\left(\{\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n\}_{2:n}\right. \\
&\quad \left. - \min\{\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n\} \leq y\right) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})\mathbb{P}\left(\{\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n\}_{2:n} \leq y\right) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-2}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) + \sum_{i=1}^{n} p(\mathbf{1}_i)\mathbb{P}\left(\min_{j\neq i}\{X_j\} \leq y\right),
\end{aligned} \tag{7}$$

where $\mathbf{1}_i = (1, \ldots, 1, 0, 1, \ldots, 1)$, meaning that the ith component of $\mathbf{1}$ equals to 0.

Moreover, when the claim severities X_i's are independent, (7) can be written as

$$\begin{aligned}
\mathbb{P}(S_n \leq y) &= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-2}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) + \sum_{i=1}^{n}\left\{p(\mathbf{1}_i)\left[1 - \prod_{j\neq i}^{n}\bar{F}_j(y)\right]\right\} \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \\
&\quad + \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) - \prod_{j=1}^{n}\bar{F}_j(y)\sum_{i=1}^{n}\left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i(y)}\right\}.
\end{aligned} \tag{8}$$

In particular, if X_i's are independent and have heterogeneous exponential survival distributions $\exp\{-\lambda_i y\}$, $i = 1, 2, \ldots, n$, (8) can be simplified as

$$\begin{aligned}
\mathbb{P}(S_n \leq y) &= p(\mathbf{0}) + p(\mathbf{1})\left\{1 - \frac{\exp\{-\sum_{i=1}^{n}\lambda_i y\}}{\sum_{i=1}^{n}\lambda_i}\sum_{i=1}^{n}\lambda_i \exp\{\lambda_i y\}\right\} \\
&\quad + \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) - \exp\left\{-\sum_{i=1}^{n}\lambda_i y\right\}\sum_{i=1}^{n}\left\{\frac{p(\mathbf{1}_i)}{\exp\{-\lambda_i y\}}\right\},
\end{aligned} \tag{9}$$

where the expression of the distribution $\mathbb{P}(X_{2:n} - X_{1:n} \leq y)$ is due to Theorem 3.2 of Kochar and Korwar (1996).

As observed from (4) and (7), the computational burden for the distribution functions of the claim range and the second claim spacing increases with sample size, making it challenging to compare their values directly. Therefore, it is meaningful to investigate their stochastic behaviours using various stochastic orders to establish lower or upper bounds for the distribution functions or (dual) stop-loss transform functions. In the next two sections, we will explore the stochastic dominance of the claim range and the second claim spacing in terms of the increasing convex and concave orders, as well as the usual stochastic order.

4. Results on stochastic dominance for claim ranges

Both occurrence levels and claim severities in insurance portfolios often exhibit a positive dependency structure in actuarial practices, particularly in scenarios where insureds in an area are affected by severe natural disasters, such as droughts, floods, and earthquakes. Therefore, we will first establish ordering results for claim ranges derived from two sets of heterogeneous insurance portfolios with dependent occurrence probabilities and claim severities. Subsequently, we will present comparison results for the independent setting.

4.1. Dependent claim severities

In this part, we present sufficient conditions on the increasing convex ordering of claim ranges from two sets of dependent insurance portfolios when exchanging the claim severities of two claims I_iX_i and I_jX_j with occurrence levels unchanged. To proceed, we define a σ-algebra generated by the remaining $n-2$ claim severities:

$$\mathcal{F}_{i,j} = \sigma(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n) = \sigma\left(\underset{r\neq i,j}{\cup} X_r\right).$$

Theorem 4.1: *Let $Y_{n:n}$ and $Y_{1:n}$ be the largest and smallest claims of the insurance portfolio $(I_1X_1, \ldots, I_iX_i, \ldots, I_jX_j, \ldots, I_nX_n)$, and $Y^*_{n:n}$ and $Y^*_{1:n}$ be the largest and smallest claims of the insurance portfolio $(I_1X_1, \ldots, I_iX_j, \ldots, I_jX_i, \ldots, I_nX_n)$, respectively, for $1 \leq i < j \leq n$, where I_i's are independent of X_i's, for $i = 1, 2, \ldots, n$. If $\boldsymbol{I}$ is SAI and $X_i \leq_{\mathcal{F}_{i,j}-\text{icx}} X_j$, then*

$$Y_{n:n} - Y_{1:n} \geq_{\text{icx}} Y^*_{n:n} - Y^*_{1:n}.$$

Theorem 4.1 indicates that a larger occurrence probability, represented by the SAI, when accompanied by a higher claim severity characterized by the conditional increasing convex order, results in a larger claim range in the sense of the increasing convex order.

Next, we investigate the effect of the heterogeneity among the negative log-transform of independent occurrence probabilities on the claim ranges in the sense of the increasing convex order under appropriate conditions.

Theorem 4.2: *Let $Y_{n:n}$ and $Y_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y^*_{n:n}$ and $Y^*_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I^*_1X_1, I^*_2X_2, \ldots, I^*_nX_n)$, respectively, where $I_1, \ldots, I_n$ and $I^*_1, \ldots, I^*_n$ are independent and such that $\mathbb{E}[I_i] = p_i$ and $\mathbb{E}[I^*_i] = p^*_i$, $i = 1, 2, \ldots, n$. Besides,*

suppose both I_i's and I_i^*'s are independent of X_i's. For any $1 \leq i < n$, if $X_i \leq_{\mathcal{F}_{i,i+1}-\text{icx}} X_{i+1}$ and $\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \in \mathcal{I}_n$, then

$$(-\log p_1, -\log p_2, \ldots, -\log p_n) \overset{\text{m}}{\succeq} (-\log p_1^*, -\log p_2^*, \ldots, -\log p_n^*)$$

implies that

$$Y_{n:n} - Y_{1:n} \geq_{\text{icx}} Y_{n:n}^* - Y_{1:n}^*.$$

Note that Theorem 4.2 considers only the majorization order based on the negative log-transformation of the occurrence probabilities to establish the ordering result. It would also be meaningful to investigate whether this result holds under other types of decreasing convex transformations, such as $h(p) = \frac{1-p}{p}$. However, this issue will not be explored in detail, as the primary focus of this section is to examine the effect of the heterogeneity of occurrence probabilities rather than the specific form of the transformation. Therefore, we leave the study of ordering results with general decreasing and convex transformations on the set of occurrence probabilities as an open question.

4.2. Independent claim severities

In this subsection, we present sufficient conditions for the usual stochastic ordering of claim ranges from two sets of insurance portfolios with independent claim severities and (in)dependent occurrence probabilities.

Recall that a random variable is said to follow the PHR model with parameter (failure rate) $\mu > 0$, denoted by $X \sim \text{PHR}(\bar{F}, \mu)$, if its survival function is written as $\bar{F}_X(x) = \bar{F}^{\mu}(x)$, $x \geq 0$, where $\bar{F}$ is an interested baseline survival function. For two sets of independent and heterogeneous PHR random variables such that $X_i \sim \text{PHR}(\bar{F}, \mu_i)$ and $X_i^* \sim \text{PHR}(\bar{F}, \mu_i^*)$, $i = 1, 2, \ldots, n$, it is shown in Theorem 4.2 of Ding et al. (2013) that the condition

$$(\mu_1, \mu_2, \ldots, \mu_n) \overset{\text{m}}{\succeq} (\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$$

implies that

$$X_{n:n} - X_{1:n} \geq_{\text{st}} X_{n:n}^* - X_{1:n}^*.$$

This indicates that greater heterogeneity among the distributions results in a larger sample range in the context of the usual stochastic order. In the following theorem, we will establish the usual stochastic ordering between the claim ranges derived from two sets of heterogeneous insurance portfolios, characterized by dependent occurrence probabilities and independent claim severities that follow PHR distributions.

Theorem 4.3: *Let $Y_{n:n}$ and $Y_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y_{n:n}^*$ and $Y_{1:n}^*$ be the largest and smallest claim amounts of the insurance portfolio $(I_1X_1^*, I_2X_2^*, \ldots, I_nX_n^*)$, where $X_i \sim \text{PHR}(\bar{F}, \mu_i)$, $X_i^* \sim \text{PHR}(\bar{F}, \mu_i^*)$, all of X_i's and X_i^*'s are independent, and both of X_i's and X_i^*'s are independent of I_i's, $i = 1, 2, \ldots, n$. If $\boldsymbol{I}$ is SAI and $\boldsymbol{\mu}, \boldsymbol{\mu}^* \in \mathcal{D}_n$, then we have*

$$(\mu_1, \mu_2, \ldots, \mu_n) \overset{\text{m}}{\succeq} (\mu_1^*, \mu_2^*, \ldots, \mu_n^*) \Longrightarrow Y_{n:n} - Y_{1:n} \geq_{\text{st}} Y_{n:n}^* - Y_{1:n}^*.$$

Theorem 4.3 indicates that more heterogeneity among claim severities results in a larger claim range in the usual stochastic order when the vector of occurrence probabilities is SAI and the claim severities follow PHR distributions.

Remark 4.1: Under the same setting of Theorem 4.3, Balakrishnan et al. (2018) proved the result when $n = 2$. Therefore, our results established in Theorem 4.3 generalize Theorem 4.1 of Balakrishnan et al. (2018) to the case of $n \geq 3$.

Combining Theorems 4.2 and 4.3, we immediately get the following Theorem 4.4.

Theorem 4.4: *Let $Y_{n:n}$ and $Y_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y^*_{n:n}$ and $Y^*_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I^*_1X^*_1, I^*_2X^*_2, \ldots, I^*_nX^*_n)$, where $X_i \sim \mathrm{PHR}(\bar{F}, \mu_i)$, $X^*_i \sim \mathrm{PHR}(\bar{F}, \mu^*_i)$, $\mathbb{E}[I_i] = p_i$, $\mathbb{E}[I^*_i] = p^*_i$, for $i = 1, 2, \ldots, n$, and all of I_i's, I^*_i's, X_i's and X^*_i's are independent of each other. Suppose that $\boldsymbol{\mu}, \boldsymbol{\mu}^* \in \mathcal{D}_n$ and $\boldsymbol{p}, \boldsymbol{p}^* \in \mathcal{I}_n$. Then*

$$(\mu_1, \ldots, \mu_n) \overset{\mathrm{m}}{\succeq} (\mu^*_1, \ldots, \mu^*_n)$$

together with

$$(-\log p_1, -\log p_2, \ldots, -\log p_n) \overset{\mathrm{m}}{\succeq} (-\log p^*_1, -\log p^*_2, \ldots, -\log p^*_n)$$

implies that

$$Y_{n:n} - Y_{1:n} \geq_{\mathrm{icx}} Y^*_{n:n} - Y^*_{1:n}.$$

The following corollary can be derived from Theorem 4.4, providing a lower bound on the integral of the survival function (i.e., the stop-loss transformation function) of the claim range originating from a heterogeneous insurance portfolio comprised of independent occurrence probabilities and independent exponentially distributed claim severities.

Corollary 4.5: *Let $Y_{n:n}$ and $Y_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y^*_{n:n}$ and $Y^*_{1:n}$ be the largest and smallest claim amounts of the insurance portfolio $(I^*_1X^*_1, I^*_2X^*_2, \ldots, I^*_nX^*_n)$, where $X_i \sim \mathrm{Exp}(\mu_i)$, $X^*_i \sim \mathrm{Exp}(\mu)$, $\mathbb{E}[I_i] = p_i$, $\mathbb{E}[I^*_i] = p$, for $i = 1, 2, \ldots, n$, and all of I_i's, I^*_i's, X_i's and X^*_i's are independent of each other. Suppose that $\boldsymbol{\mu} \in \mathcal{D}_n$ and $\boldsymbol{p} \in \mathcal{I}_n$. If $\mu = \frac{1}{n}\sum_{i=1}^n \mu_i$ and $p = (\prod_{i=1}^n p_i)^{\frac{1}{n}}$, then*

$$Y_{n:n} - Y_{1:n} \geq_{\mathrm{icx}} Y^*_{n:n} - Y^*_{1:n},$$

which implies that, for any $y \geq 0$

$$\begin{aligned}\pi_{Y_{n:n}-Y_{1:n}}(y) &= \int_y^\infty \mathbb{P}(Y_{n:n} - Y_{1:n} > s)\,\mathrm{d}s \\ &\geq -\frac{\mathrm{e}^{-j\mu y}}{j\mu}\left\{p^n\sum_{j=1}^{k-1}(-1)^j\binom{k}{j} + \sum_{k=1}^{n-1}\left[\binom{n}{k}p^k(1-p)^{n-k}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\right]\right\}.\end{aligned}$$

5. Results on stochastic dominance for second claim spacings

In this section, we establish ordering results for the second claim spacings derived from two groups of heterogeneous and (in)dependent insurance portfolios.

5.1. Dependent claim severities

This subsection presents sufficient conditions for the increasing concave order of the second claim spacings derived from two sets of dependent occurrence levels and claim severities.

Theorem 5.1: *Let $Y_{2:n}$ and $Y_{1:n}$ be the second smallest and smallest claim amounts from $I_1X_1, \ldots, I_iX_i, \ldots, I_jX_j, \ldots, I_nX_n$, and $Y^*_{2:n}$ and $Y^*_{1:n}$ be the second smallest and smallest claim amounts from $I_1X_1, \ldots, I_iX_j, \ldots, I_jX_i, \ldots, I_nX_n$, for $1 \leq i < j \leq n$, where I_r's are independent of X_r's, $r = 1, 2, \ldots, n$. If $\boldsymbol{I}$ is SAI and $X_i \leq_{\mathcal{F}_{i,j}-\mathrm{icv}} X_j$, then*

$$Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} Y^*_{2:n} - Y^*_{1:n}.$$

Theorem 5.1 indicates that a larger occurrence probability, represented by the SAI, when accompanied by a higher claim severity characterized by the conditional increasing concave order, results in a larger second claim spacing in the sense of the increasing concave order.

Theorem 5.2: *Let $Y_{2:n}$ and $Y_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y^*_{2:n}$ and $Y^*_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I^*_1X_1, I^*_2X_2, \ldots, I^*_nX_n)$, where I_i's and I^*_i's are independent of each other such that $\mathbb{E}[I_i] = p_i$ and $\mathbb{E}[I^*_i] = p^*_i$, $i = 1, 2, \ldots, n$, and both of them are independent of X_i's. For any $1 \leq i < n$, if $X_i \leq_{\mathcal{F}_{i,i+1}-\mathrm{icv}} X_{i+1}$ and $\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \in \mathcal{I}_n$, then*

$$(-\log p_1, -\log p_2, \ldots, -\log p_n) \overset{\mathrm{m}}{\succeq} (-\log p^*_1, -\log p^*_2, \ldots, -\log p^*_n)$$

implies that

$$Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} Y^*_{2:n} - Y^*_{1:n}.$$

Under the assumption that claim severities are ordered by the conditional increasing concave order and that larger claim severities are associated with higher occurrence levels, Theorem 5.2 demonstrates that greater heterogeneity among the negative log-transforms of the occurrence probabilities results in a larger second claim spacing in the context of the increasing concave order.

Remark 5.1: Consider a special case of Theorems 4.2 and 5.2 where $n = 2$ and X_1, X_2 are identically distributed. In this case, $Y_{2:2} - Y_{1:2}$ and $Y_{n:n} - Y_{1:n}$ coincide. Noting that $\mathcal{F}_{1,2} = \{\Omega, \emptyset\}$ when $n = 2$, the conditional stochastic order assumptions reduce to their unconditional forms and are thus satisfied. Furthermore, suppose all other conditions in Theorems 4.2 and 5.2 hold. Then for any increasing function ϕ with $\phi(0) = 0$, we have

$$\begin{aligned}\mathbb{E}[\phi(Y_{2:2} - Y_{1:2})] = {} & (1-p_1)(1-p_2)\mathbb{E}[0] + (1-p_1)p_2\mathbb{E}[\phi(X_2)] \\ & + p_1(1-p_2)\mathbb{E}[\phi(X_1)] + p_1p_2\mathbb{E}[\phi(|X_2 - X_1|)],\end{aligned}$$

and (A1) leads to

$$\mathbb{E}[\phi(Y_{2:2}-Y_{1:2})-\phi(Y^*_{2:2}-Y^*_{1:2})]=(p_2-p_2^*)\mathbb{E}[\phi(X_2)]+(p_1-p_1^*)\mathbb{E}[\phi(X_1)]\geq 0,$$

which implies $Y_{2:2}-Y_{1:2}\geq_{\mathrm{st}} Y^*_{2:2}-Y^*_{1:2}$. Indeed, this holds whenever $X_1\leq_{\mathrm{st}} X_2$. However, when $n\geq 3$, this result does not generally hold, as $X_1\overset{d}{=}X_2$ does not necessarily imply $X_1\leq_{\mathcal{F}_{1,2}-\mathrm{icx}} X_2$ or $X_1\leq_{\mathcal{F}_{1,2}-\mathrm{icv}} X_2$.

5.2. Independent claim severities

The previous subsection examined the effect of heterogeneity among occurrence probabilities on the stochastic behaviours of the second claim spacing. It is natural to investigate the stochastic dominance of the second claim spacing as the vector of claim severities varies, which is the primary focus of this subsection.

For two sets of independent and heterogeneous exponential random variables, Kochar and Korwar (1996) showed that the survival function of the second sample spacing is Schur-convex in the vector of the hazard rates, which means that if $X_i\sim\mathrm{Exp}(\mu_i)$ and $X_i^*\sim\mathrm{Exp}(\mu_i^*)$, $i=1,\ldots,n$, then $(\mu_1,\ldots,\mu_n)\overset{\mathrm{m}}{\succeq}(\mu_1^*,\ldots,\mu_n^*)$ implies that $X_{2:n}-X_{1:n}\geq_{\mathrm{st}} X^*_{2:n}-X^*_{1:n}$. In what follows, we assume that the claim severities are independent and have exponential distributions. The following theorem can be reached based on the result of Kochar and Korwar (1996).

Theorem 5.3: *Let $Y_{2:n}$ and $Y_{1:n}$ be the second smallest and the smallest claim amounts of the insurance portfolio $(I_1X_1,I_2X_2,\ldots,I_nX_n)$, and $Y^*_{2:n}$ and $Y^*_{1:n}$ be the second smallest and the smallest claim amounts of the insurance portfolio $(I_1X_1^*,I_2X_2^*,\ldots,I_nX_n^*)$, where $X_i\sim\mathrm{Exp}(\mu_i)$, $X_i^*\sim\mathrm{Exp}(\mu_i^*)$, for $i=1,\ldots,n$, X_i's are independent, X_i^*'s are independent, and I_i's are independent of X_i's and X_i^*'s, respectively. Suppose that $\boldsymbol{\mu},\boldsymbol{\mu}^*\in\mathcal{D}_n$ and the random vector $\boldsymbol{I}$ is SAI. Then $(\mu_1,\ldots,\mu_n)\overset{\mathrm{m}}{\succeq}(\mu_1^*,\ldots,\mu_n^*)$ implies $Y_{2:n}-Y_{1:n}\geq_{\mathrm{st}} Y^*_{2:n}-Y^*_{1:n}$.*

Under the setting that claim severities are independent, and exponentially distributed and the occurrence levels are SAI and independent of the claim severities, Theorem 5.3 states that more heterogeneity among the hazard rates of claim severities results in larger second claim spacing with respect to the usual stochastic order.

Theorem 5.4: *Let $Y_{2:n}$ and $Y_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I_1X_1,I_2X_2,\ldots,I_nX_n)$, and $Y^*_{2:n}$ and $Y^*_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I_1^*X_1^*,I_2^*X_2^*,\ldots,I_n^*X_n^*)$, where $X_i\sim\mathrm{Exp}(\mu_i)$, $X_i^*\sim\mathrm{Exp}(\mu_i^*)$, all of I_i's, I_i^*'s, X_i's and X_i^*'s are independent and such that $\mathbb{E}[I_i]=p_i$ and $\mathbb{E}[I_i^*]=p_i^*$, for $i=1,2,\ldots,n$. Suppose that $\boldsymbol{\mu},\boldsymbol{\mu}^*\in\mathcal{D}_n$ and $\boldsymbol{p},\boldsymbol{p}^*\in\mathcal{I}_n$. Then*

$$(\mu_1,\ldots,\mu_n)\overset{\mathrm{m}}{\succeq}(\mu_1^*,\ldots,\mu_n^*)$$

together with

$$(-\log p_1,-\log p_2,\ldots,-\log p_n)\overset{\mathrm{m}}{\succeq}(-\log p_1^*,-\log p_2^*,\ldots,-\log p_n^*)$$

implies that

$$Y_{2:n}-Y_{1:n}\geq_{\mathrm{icv}} Y^*_{2:n}-Y^*_{1:n}.$$

Suppose that $X_1, \ldots, X_n$ are independent and exponentially distributed with hazard rates $\mu_1, \ldots, \mu_n$, respectively. Then, according to Theorem 3.2 of Kochar and Rojo (1996), the survival function of $D^X_{2:n} = (n-1)S^X_{2:n} = (n-1)(X_{2:n} - X_{1:n})$ is given by

$$\bar{F}_{D^X_{2:n}}(y) = \frac{1}{\sum_{i=1}^n \mu_i} \exp\left(\frac{-y\sum_{i=1}^n \mu_i}{n-1}\right) \sum_{i=1}^n \left\{\mu_i \exp\left(\frac{y\mu_i}{n-1}\right)\right\}, \quad y \geq 0.$$

Note that, for all $y \geq 0$,

$$\bar{F}_{D^X_{2:n}}(y) = \mathbb{P}\left(X_{2:n} - X_{1:n} \geq \frac{y}{n-1}\right) = \bar{F}_{S^X_n}\left(\frac{y}{n-1}\right).$$

Hence, the survival function of S^X_n can be derived as

$$\bar{F}_{S^X_n}(y) = \bar{F}_{D_{2:n}}((n-1)y) = \frac{1}{\sum_{i=1}^n \mu_i} \exp\left(-y\sum_{i=1}^n \mu_i\right) \sum_{i=1}^n \left\{\mu_i \exp\left(y\mu_i\right)\right\}. \tag{10}$$

Combining (9) with (10), the distribution function of the second claim spacing $S_n = Y_{2:n} - Y_{1:n}$ can be expressed by

$$\begin{aligned} F_{S_n}(y) = p(\mathbf{0}) + p(\mathbf{1}) &\left\{1 - \frac{1}{\sum_{i=1}^n \mu_i} \sum_{i=1}^n \left[\mu_i \exp\left(-y\sum_{j\neq i}^n \mu_j\right)\right]\right\} \\ &+ \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda}) - \sum_{i=1}^n \left[p(\mathbf{1}_i) \exp\left(-y\sum_{j\neq i}^n \mu_j\right)\right], \quad y \geq 0. \end{aligned} \tag{11}$$

As a direct consequence of Theorem 5.4, the following corollary can be obtained, which provides an upper bound of the integral of the cumulative distribution of the second claim spacing stemming from an insurance portfolio consisting of independent and heterogeneous occurrence probabilities as well as independent and heterogeneous exponential claim severities.

Corollary 5.5: *Let $Y_{2:n}$ and $Y_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I_1X_1, I_2X_2, \ldots, I_nX_n)$, and $Y^*_{2:n}$ and $Y^*_{1:n}$ be the second smallest and smallest claim amounts of the insurance portfolio $(I^*_1X^*_1, I^*_2X^*_2, \ldots, I^*_nX^*_n)$, where $X_i \sim \mathrm{Exp}(\mu_i)$, $X^*_i \sim \mathrm{Exp}(\mu)$, all of I_i's, I^*_i's, X_i's and X^*_i's are independent and such that $\mathbb{E}[I_i] = p_i$ and $\mathbb{E}[I^*_i] = p$, for $i = 1, 2, \ldots, n$. Suppose that $\boldsymbol{\mu} \in \mathcal{D}_n$ and $\boldsymbol{p} \in \mathcal{I}_n$. If $\mu = \frac{1}{n}\sum_{i=1}^n \mu_i$ and $p = (\prod_{i=1}^n p_i)^{\frac{1}{n}}$, then*

$$Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} Y^*_{2:n} - Y^*_{1:n},$$

which implies that, for any $y \geq 0$,

$$\int_0^y \mathbb{P}(Y_{2:n} - Y_{1:n} \leq s)\,\mathrm{d}s \leq y - \frac{p^{n-1}}{\mu}\left(\frac{n}{n-1} - p\right)\left(1 - \mathrm{e}^{-(n-1)\mu y}\right).$$

Table 2. Probability mass function for $\boldsymbol{I} = (I_1, I_2, I_3)$.

λ_1	0	0	0	0	1	1	1	1
λ_2	0	0	1	1	0	0	1	1
λ_3	0	1	0	1	0	1	0	1
$p(\boldsymbol{\lambda})$	0.194	0.162	0.15	0.19	0.008	0.022	0.01	0.264

6. Numerical examples

In this section, we implement some simulation examples to illustrate our main findings. Recall that a random variable X is said to follow a Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ if its survival function is given by

$$\bar{F}(x) = \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}, \quad \text{for all } x \in (0, +\infty).$$

For two Weibull random variables $X_1 \sim \text{Weibull}(\alpha_1, \beta_1)$ and $X_2 \sim \text{Weibull}(\alpha_2, \beta_2)$, if $\dfrac{\Gamma(1+1/\beta_1)}{\Gamma(1+1/\beta_2)} \leq \dfrac{\alpha_2}{\alpha_1}$, then $\beta_1 > \beta_2$ implies $X_1 \leq_{\text{icx}} X_2$ while $\beta_1 < \beta_2$ implies $X_1 \leq_{\text{icv}} X_2$; see Table 2.2 in Belzunce et al. (2015) or apply Lemma 2.2.

Further, we adopt the Farlie-Gumbel-Morgenstern (FGM) copula for modelling the dependence structure among claim severities. Consider the 3-dimensional FGM copula C given by

$$C(u, v, w) = uvw(1 + \theta(1-u)(1-v) + \theta(1-u)(1-w) + \theta(1-v)(1-w)), \tag{12}$$

where θ is a parameter belonging to $[-1, 1]$. For the case of $\theta = 0$, it reduces to the independence copula.

We are now ready to present several numerical examples to illustrate the main results developed in Sections 4 and 5. Given that the distribution functions of the claim range and the second claim spacing involve multiple integrations, calculating these functions directly becomes challenging. Therefore, we will employ Monte Carlo simulations. The sample size for the Monte Carlo simulations is set to 10^7 throughout this section.

Example 6.1: Suppose X_1, X_2, X_3 have Weibull distributions with $X_1 \sim \text{Weibull}(2, 1.5)$, $X_2 \sim \text{Weibull}(10, 1)$ and $X_3 \sim \text{Weibull}(40, 0.5)$, and they admit a FGM copula C given by (12) with $\theta = 0.3$. The probability mass function for $\boldsymbol{I} = (I_1, I_2, I_3)$ is displayed in Table 2.

Besides, by Theorem 2.5, we have $X_1 \leq_{\mathcal{F}_{1,2}-\text{icx}} X_2$ and $X_2 \leq_{\mathcal{F}_{2,3}-\text{icx}} X_3$. The verification of the ordering relations among the conditional expectations in Theorem 2.5 is provided in Appendix 2, which also contains the verification of the corresponding ordering relations for the other examples. Denote by

$$Y_{3:3} = \max\{I_1X_1, I_2X_2, I_3X_3\}, \quad Y_{1:3} = \min\{I_1X_1, I_2X_2, I_3X_3\},$$
$$Y^*_{3:3} = \max\{I_1X_2, I_2X_1, I_3X_3\}, \quad Y^*_{1:3} = \min\{I_1X_2, I_2X_1, I_3X_3\},$$

and

$$Y^{**}_{3:3} = \max\{I_1X_1, I_2X_3, I_3X_2\}, \quad Y^{**}_{1:3} = \min\{I_1X_1, I_2X_3, I_3X_2\}.$$

Let $R_3 = Y_{3:3} - Y_{1:3}$, $R^*_3 = Y^*_{3:3} - Y^*_{1:3}$ and $R^{**}_3 = Y^{**}_{3:3} - Y^{**}_{1:3}$. The stop-loss transformation functions for R_3, R^*_3 and R^{**}_3 are denoted by $\pi_{R_3}(x)$, $\pi_{R^*_3}(x)$ and $\pi_{R^{**}_3}(x)$, respectively. The

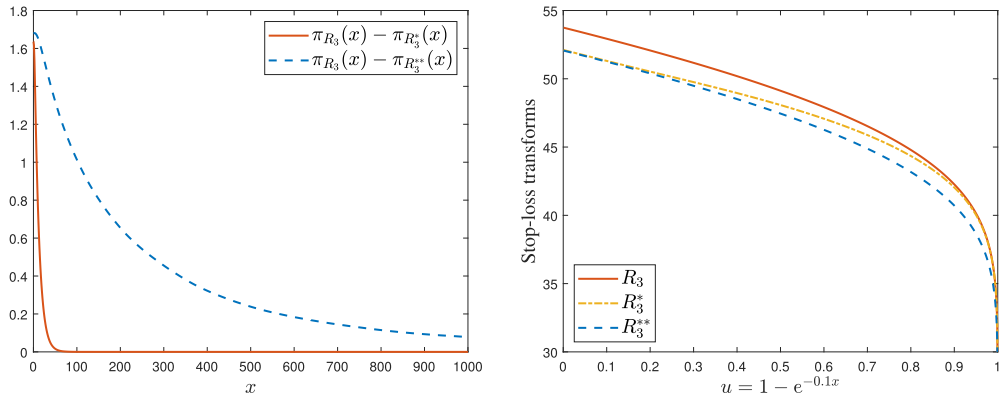


Figure 1. Plots of the difference functions, stop-loss transforms, and empirical stop-loss transforms under different settings.

difference curves for $x \in [0, 1000]$ are plotted in Figure 1(a). The transformation functions are displayed in Figure 1(b) where a transform is imposed on the x-axis. One can directly witness $R_3^* \leq_{\text{icx}} R_3$ and $R_3^{**} \leq_{\text{icx}} R_3$, which agrees with Theorem 4.1.

We introduce probability metrics to quantify the effect of swapping claim occurrence indicators on claim ranges and spacings. Specifically, we compute the probability metric between the claim ranges before and after the swap. A larger value of the probability metric indicates a stronger impact of the swap on the claim range. A standard example of the probability metrics is the *integrated stop-loss distance*; see Kaas (1993). Given two random variables X and Y, the integrated stop-loss distance[3] d_{ISL} is defined by

$$d_{\text{ISL}}(X, Y) = \int_{-\infty}^{+\infty} |\pi_X(t) - \pi_Y(t)| \, \mathrm{d}t. \tag{13}$$

If X and Y are non-negative and $X \leq_{\text{icx}} Y$, then Proposition 9.8.2 of Denuit et al. (2005) yields

$$d_{\text{ISL}}(X, Y) = \int_0^{+\infty} (\pi_Y(t) - \pi_X(t)) \, \mathrm{d}t = \frac{1}{2} \left(\mathbb{E}[Y^2] - \mathbb{E}[X^2] \right). \tag{14}$$

This identity does not directly carry over to the icv order, because $X \leq_{\text{icv}} Y$ is equivalent to $-X \geq_{\text{icx}} -Y$, and the latter variables are non-positive.

It can be calculated that $d_{\text{ISL}}(R_3, R_3^*) = 2.08$ and $d_{\text{ISL}}(R_3, R_3^{**}) = 38.28$. This suggests that the swap between indicators I_1 and I_2 has more influence on the claim range compared with the swap between indicators I_2 and I_3.

Example 6.2: Suppose that $X_1 \sim$ Weibull $(10, 1.5)$, $X_2 \sim$ Weibull $(10, 1)$ and $X_3 \sim$ Weibull $(20, 0.5)$ and they admit a FGM copula with $\theta = 0.5$. Let $\boldsymbol{p}^1 = (\mathrm{e}^{-0.5}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.5})$, $\boldsymbol{p}^2 = (\mathrm{e}^{-0.6}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.4})$, $\boldsymbol{p}^3 = (\mathrm{e}^{-0.7}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.3})$, and $\boldsymbol{p}^4 = (\mathrm{e}^{-0.9}, \mathrm{e}^{-0.4}, \mathrm{e}^{-0.2})$. For any paired

[3] To the best of our knowledge, there is no analogous integrated distance based on dual stop-loss transforms, since $|\tilde{\pi}_X(t) - \tilde{\pi}_Y(t)|$ becomes constant on $\{X < t\} \cap \{Y < t\}$, leading to a divergent integral.

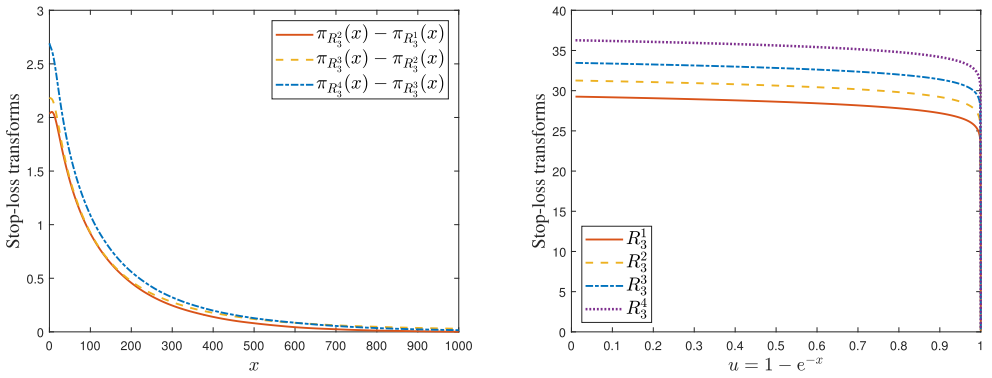


Figure 2. Plots of the difference functions, stop-loss transforms, and empirical stop-loss transforms under different settings.

Table 3. The ISL distance matrix among R_3^1, R_3^2, R_3^3 and R_3^4.

	R_3^1	R_3^2	R_3^3	R_3^4
R_3^1	0	31.14	63.38	102.31
R_3^2	31.14	0	32.24	71.17
R_3^3	63.38	32.24	0	38.93
R_3^4	102.31	71.17	38.93	0

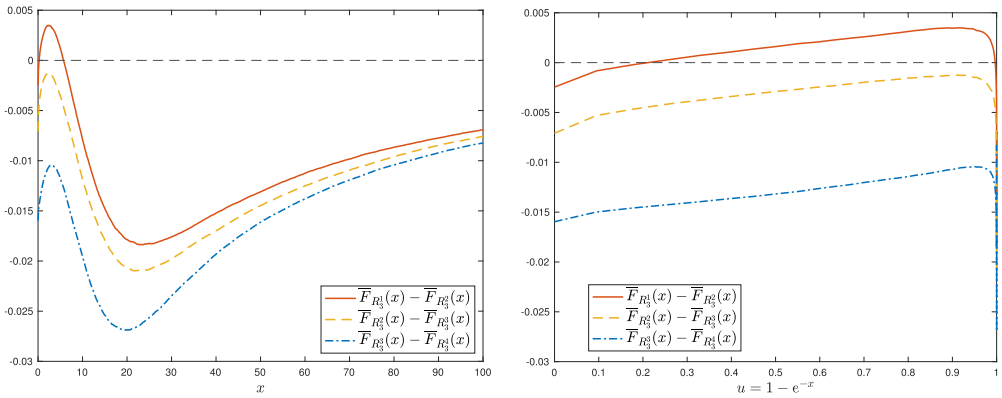
integers $1 \le i < j \le 4$, we have

$$(-\log p_1^j, -\log p_2^j, \ldots, -\log p_n^j) \overset{\mathrm{m}}{\succeq} (-\log p_1^i, -\log p_2^i, \ldots, -\log p_n^i).$$

Besides, it can be checked that $X_1 \le_{\mathrm{icx}} X_2 \le_{\mathrm{icx}} X_3$ and $\mathbb{E}[X_1] < \mathbb{E}[X_2] < \mathbb{E}[X_3]$. Hence, by Theorem 2.5, $X_1 \le_{\mathcal{F}_{1,2}-\mathrm{icx}} X_2$ and $X_2 \le_{\mathcal{F}_{2,3}-\mathrm{icx}} X_3$. Let $Y_{3:3}^i = \max\{I_1^i X_1, I_2^i X_2, I_3^i X_3\}$, $Y_{1:3}^i = \min\{I_1^i X_1, I_2^i X_2, I_3^i X_3\}$ and $R_3^i = Y_{3:3}^i - Y_{1:3}^i$, for $i = 1, 2, 3, 4$. The difference functions among $\pi_{R_3^1}(x), \pi_{R_3^2}(x), \pi_{R_3^3}(x)$ and $\pi_{R_3^4}(x)$ for $x \in [0, 1000]$ are presented in Figure 2(a). The curves of stop-loss transforms are presented in Figure 2(b) where a transform is imposed on the x-axis. Figure 2(a,b) illustrate the icx-dominance chain $R_3^1 \le_{\mathrm{icx}} R_3^2 \le_{\mathrm{icx}} R_3^3 \le_{\mathrm{icx}} R_3^4$, consistent with the result established in Theorem 4.2. The values of integrated stop-loss distance among R_3^1, R_3^2, R_3^3 and R_3^4 are summarized in Table 3. It can be noticed that $d_{\mathrm{ISL}}(R_3^1, R_3^4)$ is the greatest one, which is in line with our intuition since $\boldsymbol{p}^4$ exhibits the greatest degree of dispersion among $\boldsymbol{p}^i$.

We continue to use the parameters in Example 6.2 to present a counterexample, which demonstrates that Theorem 4.2 does not, in general, hold for the usual stochastic order when $n \ge 3$. The differences of survival functions for $x \in [0, 100]$ are plotted in Figure 3(a,b), where a transform is imposed on the x-axis for the latter. One can immediately notice that though $R_3^2 \le_{\mathrm{st}} R_3^3$ and $R_3^3 \le_{\mathrm{st}} R_3^4$ under the same parametric settings, $R_3^1 \not\le_{\mathrm{st}} R_3^2$ and $R_3^1 \not\ge_{\mathrm{st}} R_3^2$.

Example 6.3: Suppose X_1, X_2, X_3 have Weibull distributions with $X_1 \sim \mathrm{Weibull}(2, 0.5)$, $X_2 \sim \mathrm{Weibull}(5, 1)$ and $X_3 \sim \mathrm{Weibull}(40, 5)$ and admit the FGM copula C given in (12) with $\theta = 0.5$. We adopt another set of probability mass functions for (I_1, I_2, I_3) given in Table 4.



(a) (b)

Figure 3. Plots of the difference of survival functions under the setting of Example 6.2.

Table 4. Probability mass function for $\boldsymbol{I} = (I_1, I_2, I_3)$.

λ_1	0	0	0	0	1	1	1	1
λ_2	0	0	1	1	0	0	1	1
λ_3	0	1	0	1	0	1	0	1
$p(\boldsymbol{\lambda})$	0.012	0.198	0.178	0.198	0.03	0.144	0.004	0.236

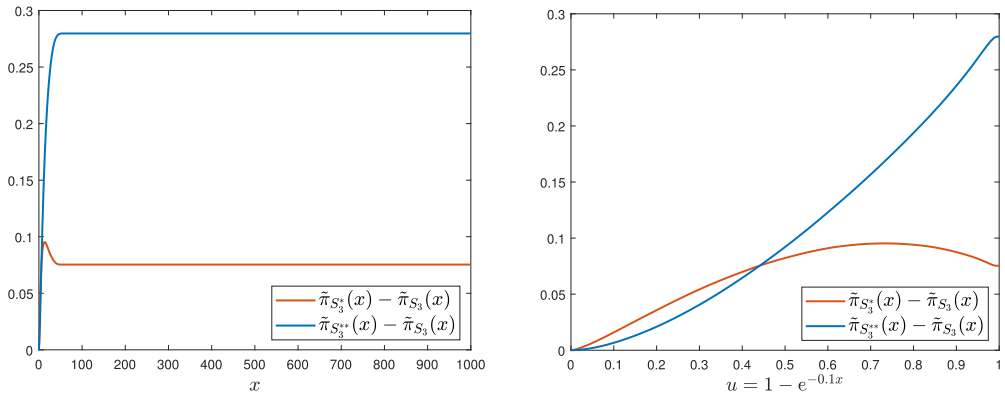
It is easy to check that $\boldsymbol{I}$ is SAI. Besides, by Theorem 2.5, we have $X_1 \leq_{\mathcal{F}_{1,2}-\mathrm{icv}} X_2$ and $X_2 \leq_{\mathcal{F}_{2,3}-\mathrm{icv}} X_3$. Let $Y_{2:3}$ and $Y_{1:3}$ be the second smallest and smallest claim amounts of (I_1X_1, I_2X_2, I_3X_3), respectively; let $Y^*_{2:3}$ and $Y^*_{1:3}$ be those of (I_1X_2, I_2X_1, I_3X_3); and let $Y^{**}_{2:3}$ and $Y^{**}_{1:3}$ be those of (I_1X_1, I_2X_3, I_3X_2). Define $S_3 = Y_{2:3} - Y_{1:3}$, $S^*_3 = Y^*_{2:3} - Y^*_{1:3}$, and $S^{**}_3 = Y^{**}_{2:3} - Y^{**}_{1:3}$. The difference functions $\tilde{\pi}_{S^*_3}(x) - \tilde{\pi}_{S_3}(x)$ and $\tilde{\pi}_{S^{**}_3}(x) - \tilde{\pi}_{S_3}(x)$ for $x \in [0, 1000]$ are plotted in Figure 4(a,b), where a transform is imposed on the x-axis. One can infer that $S^*_3 \leq_{\mathrm{icv}} S_3$ and $S^{**}_3 \leq_{\mathrm{icv}} S_3$, which agrees with the result in Theorem 5.1.

Note that the numerical plots only validate the increasing concave order rather than the increasing convex order, which means that Equation (14) cannot be applied here for the calculation. Even so, we can still resort to the expression of d_{ISL} in (13) to calculate the corresponding values for different second spacings to check the influence induced by the swap between occurrence indicators. It can be calculated that $d_{\mathrm{ISL}}(S_3, S^*_3) = 0.065$ and $d_{\mathrm{ISL}}(S_3, S^{**}_3) = 0.361$. This suggests that the swap between indicators I_1 and I_2 has more influence on the second claim spacings compared with the swap between indicators I_2 and I_3.

Example 6.4: Suppose X_1, X_2, X_3 have Weibull distributions with $X_1 \sim \mathrm{Weibull}(2, 0.5)$, $X_2 \sim \mathrm{Weibull}(6, 1)$ and $X_3 \sim \mathrm{Weibull}(40, 1.5)$ and admit the FGM copula C given in (12) with $\theta = 0.5$. Let $\boldsymbol{p}^1 = (\mathrm{e}^{-0.5}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.5})$, $\boldsymbol{p}^2 = (\mathrm{e}^{-0.6}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.4})$, $\boldsymbol{p}^3 = (\mathrm{e}^{-0.7}, \mathrm{e}^{-0.5}, \mathrm{e}^{-0.3})$, and $\boldsymbol{p}^4 = (\mathrm{e}^{-0.9}, \mathrm{e}^{-0.4}, \mathrm{e}^{-0.2})$. For any paired integers $1 \leq i < j \leq 4$,

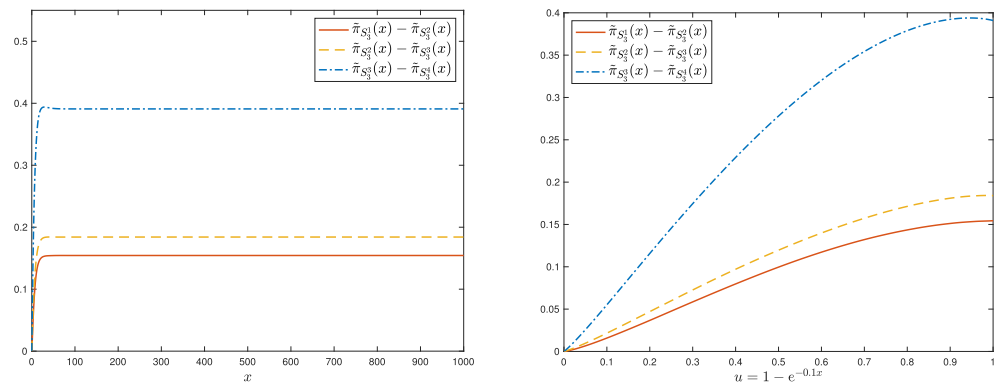
$$(-\log p^j_1, -\log p^j_2, \ldots, -\log p^j_n) \overset{\mathrm{m}}{\succeq} (-\log p^i_1, -\log p^i_2, \ldots, -\log p^i_n).$$

Besides, it follows that $X_1 \leq_{\mathrm{icv}} X_2 \leq_{\mathrm{icv}} X_3$ and $\mathbb{E}[X_1] < \mathbb{E}[X_2] < \mathbb{E}[X_3]$. By Theorem 2.5, $X_1 \leq_{\mathcal{F}_{1,2}-\mathrm{icv}} X_2$ and $X_2 \leq_{\mathcal{F}_{2,3}-\mathrm{icv}} X_3$. Let $Y^i_{2:3}$ and $Y^i_{1:3}$ be the second smallest and smallest claim amounts of $(I^i_1X_1, I^i_2X_2, I^i_3X_3)$, respectively, and let $S^i_3 = Y^i_{2:3} - Y^i_{1:3}$, for $i = 1, 2, 3, 4$.



(a) (b)

Figure 4. Plots of the difference functions.



(a) (b)

Figure 5. Plots of the difference functions under different settings.

Table 5. The ISL distance matrix among S_3^1, S_3^2, S_3^3 and S_3^4.

	S_3^1	S_3^2	S_3^3	S_3^4
S_3^1	0	2.54	4.14	7.43
S_3^2	2.54	0	1.60	4.89
S_3^3	4.14	1.60	0	3.29
S_3^4	7.43	4.89	3.29	0

The difference functions $\tilde{\pi}_{S_3^1}(x) - \tilde{\pi}_{S_3^2}(x)$, $\tilde{\pi}_{S_3^2}(x) - \tilde{\pi}_{S_3^3}(x)$ and $\tilde{\pi}_{S_3^3}(x) - \tilde{\pi}_{S_3^4}(x)$ for $x \in [0, 1000]$ are plotted in Figure 5(a) and a scaled version is displayed in Figure 5(b). One can observe that $S_3^1 \leq_{\text{icv}} S_3^2 \leq_{\text{icv}} S_3^3 \leq_{\text{icv}} S_3^4$, which agrees with the theoretical finding in Theorem 5.2. The values of integrated stop-loss distance among S_3^1, S_3^2, S_3^3 and S_3^4 are summarized in Table 5. It can be noticed that $d_{\text{ISL}}(S_3^1, S_3^4)$ is the greatest one, which is in line with our intuition since $\boldsymbol{p}^4$ exhibits the greatest degree of dispersion among $\boldsymbol{p}^i$.

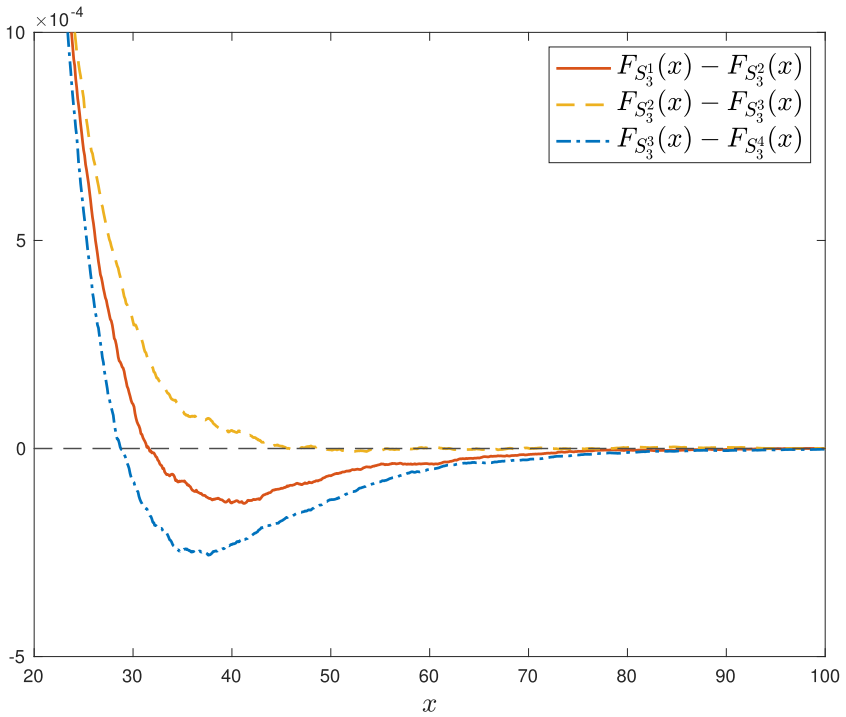


Figure 6. Plot of the difference of cumulative distribution function under the setting of Example 6.4.

Aligning with the parameters in Example 6.4, Figure 6 exhibits another counterexample, where Theorem 5.2 does not hold for the usual stochastic order in a general sense when $n \geq 3$. The differences of cumulative distribution functions for $x \in [0, 100]$ are plotted in Figure 6. One can immediately notice that, in either direction, the usual stochastic order does not hold for any pair among (S_3^1, S_3^2), (S_3^2, S_3^3), and (S_3^3, S_3^4).

7. Conclusions

The claim range and claim spacings are crucial for insurance companies to effectively manage various portfolios, as they provide valuable information for measuring the heterogeneity and dispersiveness of claims within the insurance portfolio. We have established sufficient conditions for the increasing convex [concave] order between the claim ranges [second claim spacings] derived from two sets of insurance portfolios, which may include dependent occurrence probabilities and/or dependent claim severities. These results address a gap in the study of the stochastic properties of claim ranges and claim spacings. Additionally, we presented numerical examples to illustrate the main findings.

Due to the complexity of the distribution function of the claim spacing, it remains an open question whether one can establish the usual stochastic ordering between the second claim spacings arising from two sets of insurance portfolios with claim severities that follow PHR distributions, where the vector of hazard rates of one group is majorized by that of another. Additionally, it would be worthwhile to investigate whether the results in Theorems 4.3 and 5.2 still hold when the $-\log(x)$ function is replaced by a general decreasing

and convex function. Another interesting avenue for future research would be to explore the stochastic dominance for other types of claim spacings, such as $Y_{i:n} - Y_{i-1:n}$, $i = 3, \ldots, n$.

Author contributions

CRediT: **Tong Pu:** Formal analysis, Investigation, Methodology, Validation, Writing – original draft; **Yaodi Yong:** Formal analysis, Software, Validation, Writing – original draft, Writing – review & editing; **Jiandong Zhang:** Funding acquisition, Investigation, Software, Writing – original draft, Writing – review & editing; **Yiying Zhang:** Conceptualization, Formal analysis, Methodology, Project administration, Supervision, Writing – original draft, Writing – review & editing

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Appendices

Appendix 1. Proofs of all main results

We begin this section by firstly introducing the following notations and then presenting the proofs of main results. Let $\mathbf{1}$ denote the n-dimensional vector whose components are all equal to 1. Given n-dimensional (random) vectors $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n\}$, we define

$$\{\boldsymbol{\lambda} \circ \boldsymbol{X}\} := \{\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n\}.$$

In the sequel, we adopt $\max\{\boldsymbol{\lambda} \circ \boldsymbol{X}\}$ and $\min\{\boldsymbol{\lambda} \circ \boldsymbol{X}\}$ to denote the maxima and minima in the vector $\{\boldsymbol{\lambda} \circ \boldsymbol{X}\}$, and use $\{\boldsymbol{\lambda} \circ \boldsymbol{X}\}_{i:n}$ to denote the ith smallest value of $\{\boldsymbol{\lambda} \circ \boldsymbol{X}\}$. Further, let $\boldsymbol{\lambda}_{(i,j)}^{(a,b)}$ denote the vector obtained by replacing the ith and jth components of $\boldsymbol{\lambda}$ with their exact values a and b, respectively, which means that

$$\boldsymbol{\lambda}_{(i,j)}^{(a,b)} = \{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, a, \lambda_{i+1}, \ldots, \lambda_{j-1}, b, \lambda_{j+1}, \ldots, \lambda_n\}.$$

A.1 Proof of Theorem 4.1

Proof: The idea of the proof is based on Definition 2.1(i). For any increasing convex function ϕ, and the claim range $R_n = Y_{n:n} - Y_{1:n}$, by considering the values of the ith and jth components of $\boldsymbol{\lambda}$, we have

$$\begin{aligned}
\mathbb{E}[\phi(R_n)] = {} & p(\mathbf{0})\mathbb{E}[\phi(0)] + p(\mathbf{1})\mathbb{E}\left[\phi(X_{n:n} - X_{1:n})\right] \\
& + \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_1^{i,j}(0,0)} p(\boldsymbol{\lambda}) \mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)} \circ \boldsymbol{X}\} - \min\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)} \circ \boldsymbol{X}\}\right)\right] \right. \\
& \left. + \sum_{\boldsymbol{\lambda} \in \Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda}) \mathbb{E}\left[\phi\left(X_j\right)\right] + \sum_{\boldsymbol{\lambda} \in \Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda}) \mathbb{E}\left[\phi\left(X_i\right)\right] \right\} \\
& + \sum_{k=2}^{n-2} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,0)} p(\boldsymbol{\lambda}) \mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)} \circ \boldsymbol{X}\} - \min\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)} \circ \boldsymbol{X}\}\right)\right] \right. \\
& \left. + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda}) \mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)} \circ \boldsymbol{X}\} - \min\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)} \circ \boldsymbol{X}\}\right)\right] \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}-\min\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}\right)\right] \\
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}-\min\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}\right)\right]\Bigg\} \\
&+ \Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}-\min\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}\right)\right] \\
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}-\min\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}\right)\right] \\
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}-\min\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}\right)\right]\Bigg\} \\
&= p(\mathbf{0})\mathbb{E}[\phi(0)] + p(\mathbf{1})\mathbb{E}\left[\phi(X_{n:n}-X_{1:n})\right] \\
&+ \Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}\right)\right] \\
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_j\right)\right] + \sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_i\right)\right]\Bigg\} \\
&+ \sum_{k=2}^{n-2}\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}\right)\right] + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})A_j \\
&+ \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})A_i + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,1)} p(\boldsymbol{\lambda})A_{ij}\Bigg\} \\
&+ \Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})A_j + \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})A_i + \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})A_{ij}\Bigg\},
\end{aligned}
$$

where

$$
A_j := \mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}\right)\right], \quad A_i := \mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}\right)\right]
$$

and

$$
A_{ij} := \mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}\right)\right].
$$

Similarly, for the claim range $R_n^* = Y_{n:n}^* - Y_{1:n}^*$, a case analysis on $\boldsymbol{\lambda}$ for the expectation $\mathbb{E}[\phi(R_n)]$ gives rise to

$$
\begin{aligned}
\mathbb{E}[\phi(R_n^*)] &= \mathbb{E}[\phi(Y_{n:n}^* - Y_{1:n}^*)] \\
&= p(\mathbf{0})\mathbb{E}[\phi(0)] + p(\mathbf{1})\mathbb{E}\left[\phi(X_{n:n}-X_{1:n})\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}\right)\right]\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_i\right)\right]+\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_j\right)\right]\right\}\\
&+\sum_{k=2}^{n-2}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}\right)\right]+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})A_i\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})A_j+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,1)} p(\boldsymbol{\lambda})A_{ij}\right\}\\
&+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})A_i+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})A_j+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})A_{ij}\right\}.
\end{aligned}
$$

Note that, for any constant $a\in\mathbb{R}$, the function

$$
f_1(x):=\max\{x,a\}=\begin{cases} a, & x\le a,\\ x, & x\ge a,\end{cases}
$$

is increasing and convex in $x\in\mathbb{R}$. Then, $X_i\le_{\mathcal{F}_{i,j}-\mathrm{icx}} X_j$ implies that

$$
A_j-A_i=\mathbb{E}\left[\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}\right)|\mathcal{F}_{i,j}\right]-\mathbb{E}\left[\phi\left(\max\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}\right)\Big|\mathcal{F}_{i,j}\right]\right]\ge 0.
$$

Hence,

$$
\begin{aligned}
&\mathbb{E}[\phi(R_n)]-\mathbb{E}[\phi(R_n^*)]\\
&=\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_j\right)\right]+\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_i\right)\right]\right\}\\
&\quad+\sum_{k=2}^{n-2}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})A_j+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})A_i\right\}\\
&\quad+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})A_j+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})A_i\right\}\\
&\quad-\left[\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_i\right)\right]+\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(X_j\right)\right]\right\}\right.\\
&\quad\left.+\sum_{k=2}^{n-2}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})A_i+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})A_j\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})A_i+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})A_j\right\}\Bigg]\\
&=\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{1}^{i,j}(0,1)} p(\boldsymbol{\lambda})\left\{\mathbb{E}\left[\phi\left(X_j\right)\right]-\mathbb{E}\left[\phi\left(X_i\right)\right]\right\}\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_{1}^{i,j}(1,0)} p(\boldsymbol{\lambda})\left\{\mathbb{E}\left[\phi\left(X_i\right)\right]-\mathbb{E}\left[\phi\left(X_j\right)\right]\right\}\right\}\\
&\quad+\sum_{k=2}^{n-2}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{k}^{i,j}(0,1)} p(\boldsymbol{\lambda})(A_j-A_i)+\sum_{\boldsymbol{\lambda}\in\Lambda_{k}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_i-A_j)\right\}\\
&\quad+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})(A_j-A_i)+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_i-A_j)\right\}\\
&\geq\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{1}^{i,j}(1,0)} p(\boldsymbol{\lambda})\left\{\mathbb{E}\left[\phi\left(X_j\right)\right]-\mathbb{E}\left[\phi\left(X_i\right)\right]\right\}\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_{1}^{i,j}(1,0)} p(\boldsymbol{\lambda})\left\{\mathbb{E}\left[\phi\left(X_i\right)\right]-\mathbb{E}\left[\phi\left(X_j\right)\right]\right\}\right\}\\
&\quad+\sum_{k=2}^{n-2}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{k}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_j-A_i)+\sum_{\boldsymbol{\lambda}\in\Lambda_{k}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_i-A_j)\right\}\\
&\quad+\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_j-A_i)+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(A_i-A_j)\right\}=0,
\end{aligned}
$$

where the last inequality is from the SAI property of $\boldsymbol{I}$ and the fact that $X_i \leq_{\mathcal{F}_{i,j}-\mathrm{icx}} X_j$ implies $X_i \leq_{\mathrm{icx}} X_j$. As a result, we have $\mathbb{E}[\phi(R_n)] \geq \mathbb{E}[\phi(R_n^*)]$. By the arbitrariness of ϕ, we can reach that $Y_{n:n} - Y_{1:n} \geq_{\mathrm{icx}} Y_{n:n}^* - Y_{1:n}^*$. ■

A.2 Proof of Theorem 4.2

Proof: By applying the nature of majorization order, we only need to prove the desired result when

$$(-\log p_1, -\log p_2, \ldots, -\log p_n) \overset{\mathrm{m}}{\succeq} (-\log p_1^*, -\log p_2^*, \ldots, -\log p_n^*)$$

such that $p_1 \leq p_2$, $p_i = p_i^*$, for $i = 3, 4, \ldots, n$, and $(-\log p_1, -\log p_2) \overset{\mathrm{m}}{\succeq} (-\log p_1^*, -\log p_2^*)$. This implies that $p_1 p_2 = p_1^* p_2^*$, and

$$p_2 - p_2^* \geq p_1^* - p_1 \geq 0. \tag{A1}$$

Considering the possible values of the first and second components of the random vector $\boldsymbol{I}$ leads to

$$\mathbb{E}[\phi(R_n)] = (1-p_1)(1-p_2)\mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}_{(1,2)}^{(0,0)} \circ \boldsymbol{X}\} - \min\{\boldsymbol{I}_{(1,2)}^{(0,0)} \circ \boldsymbol{X}\}\right]\right\}$$

$$
\begin{aligned}
&+ (1-p_1)p_2\mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(0,1)}_{(1,2)}\circ\boldsymbol{X}\} - \min\{\boldsymbol{I}^{(0,1)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}\\
&+ p_1(1-p_2)\mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(1,0)}_{(1,2)}\circ\boldsymbol{X}\} - \min\{\boldsymbol{I}^{(1,0)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}\\
&+ p_1p_2\mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(1,1)}_{(1,2)}\circ\boldsymbol{X}\} - \min\{\boldsymbol{I}^{(1,1)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}\\
&=: (1-p_1)(1-p_2)A_1 + (1-p_1)p_2A_2 + p_1(1-p_2)A_3 + p_1p_2A_4,
\end{aligned}
$$

where

$$
\begin{aligned}
&A_1 := \mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(0,0)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}, \quad A_2 := \mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(0,1)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\},\\
&A_3 := \mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(1,0)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}, \quad \text{and}\\
&A_4 := \mathbb{E}\left\{\phi\left[\max\{\boldsymbol{I}^{(1,1)}_{(1,2)}\circ\boldsymbol{X}\} - \min\{\boldsymbol{I}^{(1,1)}_{(1,2)}\circ\boldsymbol{X}\}\right]\right\}.
\end{aligned}
$$

Since $X_1 \le_{\mathcal{F}_{1,2}-\text{icx}} X_2$, we have

$$
A_2 = \mathbb{E}\left[\mathbb{E}\left\{\phi\left(\max\{\boldsymbol{I}^{(0,1)}_{(1,2)}\circ\boldsymbol{X}\}\right)\big|\mathcal{F}_{1,2}\right\}\right] \ge \mathbb{E}\left[\mathbb{E}\left\{\phi\left(\max\{\boldsymbol{I}^{(1,0)}_{(1,2)}\circ\boldsymbol{X}\}\right)\Big|\mathcal{F}_{1,2}\right\}\right] = A_3. \tag{A2}
$$

Combining (A1) with (A2), we have

$$
\begin{aligned}
&\mathbb{E}[\phi(R_n)] - \mathbb{E}[\phi(R^*_n)]\\
&\quad= (1-p_1)(1-p_2)A_1 + (1-p_1)p_2A_2 + p_1(1-p_2)A_3 + p_1p_2A_4\\
&\qquad- \left[(1-p^*_1)(1-p^*_2)A_1 + (1-p^*_1)p^*_2A_2 + p^*_1(1-p^*_2)A_3 + p^*_1p^*_2A_4\right]\\
&\quad= \left[p_2 - p^*_2 - (p^*_1 - p_1)\right][A_2 - A_1] + (p^*_1 - p_1)[A_2 - A_3] \ge 0.
\end{aligned}
$$

Hence, the desired result is obtained. ■

A.3 Proof of Theorem 4.3

Before proving Theorem 4.3, we first introduce a lemma that will be used in the proof. The following lemma characterizes an SAI multivariate Bernoulli random vector.

Lemma A.1: *A multivariate Bernoulli random vector $\boldsymbol{I}$ is SAI if and only if $p(\tau_{ij}(\boldsymbol{\lambda})) \le p(\boldsymbol{\lambda})$, for any $\boldsymbol{\lambda} \in \Lambda^{i,j}_k(0,1)$, $1 \le i < j \le n$ and $k = 1, 2, \ldots, n-1$.*

Proof: The proof is equivalent to checking the sign of the difference between $F_{R_n}(y)$ and $F_{R^*_n}(y)$, for all $y \in \mathbb{R}_+$. By applying Theorem 4.2 of Ding et al. (2013) and the assumption $(\mu_1, \mu_2, \ldots, \mu_n) \overset{\text{m}}{\succeq} (\mu^*_1, \mu^*_2, \ldots, \mu^*_n)$, one immediately has

$$
\mathbb{P}(X_{n:n} - X_{1:n} \le y) \le \mathbb{P}(X^*_{n:n} - X^*_{1:n} \le y), \quad \text{for all } y \in \mathbb{R}_+.
$$

Hence, we have

$$
\begin{aligned}
F_{R_n}(y) - F_{R^*_n}(y) &= p(\mathbf{1})[\mathbb{P}(X_{n:n} - X_{1:n} \le y) - \mathbb{P}(X^*_{n:n} - X^*_{1:n} \le y)]\\
&\quad+ \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k}\left(p(\boldsymbol{\lambda})\prod_{i=1}^{n}\left[1-\bar{F}^{\mu_i}(y)\right]^{\lambda_i}\right) - \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k}\left(p(\boldsymbol{\lambda})\prod_{i=1}^{n}\left[1-\bar{F}^{\mu^*_i}(y)\right]^{\lambda_i}\right)\\
&\le \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k}\left(p(\boldsymbol{\lambda})\prod_{i=1}^{n}\left[1-\bar{F}^{\mu_i}(y)\right]^{\lambda_i}\right) - \sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k}\left(p(\boldsymbol{\lambda})\prod_{i=1}^{n}\left[1-\bar{F}^{\mu^*_i}(y)\right]^{\lambda_i}\right)\\
&=: \psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\mu}^*),
\end{aligned}
$$

where

$$\psi(\boldsymbol{\mu}) = \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right)$$

and

$$\psi(\boldsymbol{\mu}^*) = \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i^*}(y)\right]^{\lambda_i} \right).$$

Next, we show that $\psi(\boldsymbol{\mu}) \leq \psi(\boldsymbol{\mu}^*)$, for all $y \in \mathbb{R}_+$. By applying the nature of majorization order, we only need to show the inequality when $(\mu_1, \mu_2, \ldots, \mu_n) \overset{\text{m}}{\succeq} (\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$ such that $(\mu_1, \mu_2) \overset{\text{m}}{\succeq} (\mu_1^*, \mu_2^*)$, $\mu_1 \geq \mu_2$, $\mu_1^* \geq \mu_2^*$, and $\mu_i = \mu_i^*$, for $i = 3, 4, \ldots, n$. Note that

$$\begin{aligned}
\psi(\boldsymbol{\mu}) &= \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \\
&= \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,0)} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right. \\
&\quad \left. + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,0)} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,1)} \left(p(\boldsymbol{\lambda}) \prod_{i=1}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right\} \\
&= \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,0)} \left(p(\boldsymbol{\lambda}) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right. \\
&\quad + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} \left(p(\boldsymbol{\lambda}) (1 - \bar{F}^{\mu_2}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \\
&\quad + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,0)} \left(p(\boldsymbol{\lambda}) (1 - \bar{F}^{\mu_1}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \\
&\quad \left. + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,1)} \left(p(\boldsymbol{\lambda}) (1 - \bar{F}^{\mu_1}(y))(1 - \bar{F}^{\mu_2}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\partial \psi(\boldsymbol{\mu})}{\partial \mu_1} &= \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,0)} \left(p(\boldsymbol{\lambda}) \bar{F}^{\mu_1}(y)(-\ln \bar{F}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right. \\
&\quad \left. + \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1,1)} \left(p(\boldsymbol{\lambda}) \bar{F}^{\mu_1}(y)(-\ln \bar{F}(y))(1 - \bar{F}^{\mu_2}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right\}
\end{aligned}$$

and

$$\frac{\partial \psi(\boldsymbol{\mu})}{\partial \mu_2} = \sum_{k=1}^{n-1} \left\{ \sum_{\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)} \left(p(\boldsymbol{\lambda}) \bar{F}^{\mu_2}(y)(-\ln \bar{F}(y)) \prod_{i=3}^{n} \left[1 - \bar{F}^{\mu_i}(y)\right]^{\lambda_i} \right) \right.$$

$$+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)(-\ln\bar{F}(y))(1-\bar{F}^{\mu_1}(y))\prod_{i=3}^{n}\left[1-\bar{F}^{\mu_i}(y)\right]^{\lambda_i}\right)\Bigg\}.$$

Since $-\ln\bar{F}(y)\prod_{i=3}^{n}[1-\bar{F}^{\mu_i}(y)]^{\lambda_i}\geq 0$ for all $y\geq 0$, we have

$$\begin{aligned}
&\frac{\partial\psi(\boldsymbol{\mu})}{\partial\mu_1}-\frac{\partial\psi(\boldsymbol{\mu})}{\partial\mu_2}\\
&\overset{\text{sign}}{=}\sum_{k=1}^{n-1}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_1}(y)\right)+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_1}(y)(1-\bar{F}^{\mu_2}(y))\right)\right\}\\
&\quad-\sum_{k=1}^{n-1}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)\right)+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)(1-\bar{F}^{\mu_1}(y))\right)\right\}\\
&=\sum_{k=1}^{n-1}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_1}(y)\right)-\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)\right)\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\left[\bar{F}^{\mu_1}(y)-\bar{F}^{\mu_2}(y)\right]\right)\right\}\\
&=\sum_{k=1}^{n-1}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_1}(y)\right)-\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)\right)+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)\right)\right.\\
&\quad\left.-\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)}\left(p(\boldsymbol{\lambda})\bar{F}^{\mu_2}(y)\right)+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\left[\bar{F}^{\mu_1}(y)-\bar{F}^{\mu_2}(y)\right]\right)\right\}\\
&=\sum_{k=1}^{n-1}\left\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})\left[\bar{F}^{\mu_1}(y)-\bar{F}^{\mu_2}(y)\right]\right)+\bar{F}^{\mu_2}(y)\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}\left(p(\boldsymbol{\lambda})-p(\tau_{1,2}(\boldsymbol{\lambda}))\right)\right.\\
&\quad\left.+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)}\left(p(\boldsymbol{\lambda})\left[\bar{F}^{\mu_1}(y)-\bar{F}^{\mu_2}(y)\right]\right)\right\}\\
&\leq 0,
\end{aligned}$$

where the inequality is due to $\mu_1\geq\mu_2$ and the SAI property of $\boldsymbol{I}$ as stated in Lemma A.1. Thus, $\boldsymbol{\mu}\overset{\text{m}}{\succeq}\boldsymbol{\mu}^*$ implies $\psi(\boldsymbol{\mu})\leq\psi(\boldsymbol{\mu}^*)$ in accordance with Lemma 2.8, which means that $F_{R_n}(y)\leq F_{R_n^*}(y)$, for all $y\in\mathbb{R}_+$. Therefore, the desired result is proved. ■

A.4 Proof of Theorem 4.4

Proof: Consider a constructed insurance portfolio $(I_1^*X_1, I_2^*X_2,\ldots,I_n^*X_n)$ with its largest and smallest claim amounts denoted by $\tilde{Y}_{n:n}$ and $\tilde{Y}_{1:n}$, respectively. Due to $\boldsymbol{\mu}\in\mathcal{D}_n$, we have that $X_1\leq_{\text{st}}X_2\leq_{\text{st}}\cdots\leq_{\text{st}}X_n$ implies $X_1\leq_{\text{icx}}X_2\leq_{\text{icx}}\cdots\leq_{\text{icx}}X_n$. Hence, by applying Theorem 4.2 it follows that $Y_{n:n}-Y_{1:n}\geq_{\text{icx}}\tilde{Y}_{n:n}-\tilde{Y}_{1:n}$. Meanwhile, it holds from Theorem 4.3 that $\tilde{Y}_{n:n}-\tilde{Y}_{1:n}\geq_{\text{icx}}Y_{n:n}^*-Y_{1:n}^*$.

By virtue of the transitivity of the increasing convex order, we immediately have $Y_{n:n} - Y_{1:n} \geq_{\mathrm{icx}} Y^*_{n:n} - Y^*_{1:n}$, which yields the desired result. ■

A.5 Proof of Corollary 4.5

Proof: By applying (6), we first calculate the distribution function of $Y^*_{n:n} - Y^*_{1:n}$. Note that $\mathbb{P}(Y^*_{n:n} - Y^*_{1:n} = 0) = (1-p)^n$. Further, for any $y \geq 0$, the binomial theorem yields that

$$\mathbb{P}(Y^*_{n:n} - Y^*_{1:n} \leq y) = (1-p)^n + np^n \int_0^\infty \mu \mathrm{e}^{-\mu s}[\mathrm{e}^{-\mu s} - \mathrm{e}^{-\mu(s+y)}]^{n-1}\,\mathrm{d}s + \sum_{k=1}^{n-1}\left[(1-\mathrm{e}^{-\mu y})^k \binom{n}{k} p^k (1-p)^{n-k}\right].$$

Hence, the survival function of $Y^*_{n:n} - Y^*_{1:n}$ can be given by

$$\mathbb{P}(Y^*_{n:n} - Y^*_{1:n} > y) = \left[1 - (1-\mathrm{e}^{-\mu y})^{n-1}\right]p^n - \sum_{k=1}^{n-1}\left[\binom{n}{k} p^k (1-p)^{n-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \mathrm{e}^{-ujy}\right].$$

Thus, for any $y \geq 0$, we have

$$\begin{aligned}
&\int_y^\infty \mathbb{P}(Y^*_{n:n} - Y^*_{1:n} > s)\,\mathrm{d}s \\
&= p^n \int_y^\infty \left[1 - (1-\mathrm{e}^{-\mu s})^{n-1}\right]\mathrm{d}s - \int_y^\infty \sum_{k=1}^{n-1}\left[\binom{n}{k} p^k (1-p)^{n-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \mathrm{e}^{-ujs}\right]\mathrm{d}s \\
&= -\sum_{j=1}^{k-1} (-1)^j \binom{k}{j} p^n \int_y^\infty \mathrm{e}^{-\mu s j}\,\mathrm{d}s - \sum_{k=1}^{n-1}\left[\binom{n}{k} p^k (1-p)^{n-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \int_y^\infty \mathrm{e}^{-ujs}\,\mathrm{d}s\right] \\
&= -\frac{\mathrm{e}^{-j\mu y}}{j\mu} p^n \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} - \frac{\mathrm{e}^{-j\mu y}}{j\mu} \sum_{k=1}^{n-1}\left[\binom{n}{k} p^k (1-p)^{n-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j}\right] \\
&= -\frac{\mathrm{e}^{-j\mu y}}{j\mu}\left\{p^n \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} + \sum_{k=1}^{n-1}\left[\binom{n}{k} p^k (1-p)^{n-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j}\right]\right\}.
\end{aligned}$$

Hence, by using (1), the desired result is established. ■

A.6 Proof of Theorem 5.1

Proof: Recall that the second claim spacing is defined by $S_n = Y_{2:n} - Y_{1:n}$. Then, for any increasing concave function ϕ, by considering the possible values of the ith and jth components of $\boldsymbol{\lambda}$, we obtain

$$\begin{aligned}
\mathbb{E}[\phi(S_n)] = p(\mathbf{0})\mathbb{E}[\phi(0)] + p(\mathbf{1})\mathbb{E}\left[\phi(X_{2:n} - X_{1:n})\right] \\
+ \left\{\sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}^{(0,0)}_{(i,j)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{\lambda}^{(0,0)}_{(i,j)} \circ \boldsymbol{X}\}_{1:n}\right)\right]\right. \\
\left. + \sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi(0)\right] + \sum_{\boldsymbol{\lambda}\in\Lambda_1^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi(0)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=2}^{n-2}\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(0,0)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\\
&\quad+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\\
&\quad+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\\
&\quad+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\Bigg\}\\
&+\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\\
&\quad+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\\
&\quad+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{2:n}-\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{1:n}\right)\right]\Bigg\}\\
&=\phi(0)\sum_{k=0}^{n-2}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})+p(\mathbf{1})\mathbb{E}\left[\phi(X_{2:n}-X_{1:n})\right]\\
&\quad+\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})B_j+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})B_i\\
&\qquad+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{2:n}\right)\right]\Bigg\},
\end{aligned}
$$

where

$$
B_i=\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)}\circ\boldsymbol{X}\}_{2:n}\right)\right]\quad\text{and}\quad B_j=\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)}\circ\boldsymbol{X}\}_{2:n}\right)\right].
$$

Similarly, for the claim spacing $S_n^*=Y_{2:n}^*-Y_{1:n}^*$, we have

$$
\begin{aligned}
\mathbb{E}[\phi(S_n^*)]&=\phi(0)\sum_{k=0}^{n-2}\sum_{\boldsymbol{\lambda}\in\Lambda_k} p(\boldsymbol{\lambda})+p(\mathbf{1})\mathbb{E}\left[\phi(X_{2:n}-X_{1:n})\right]\\
&\quad+\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})B_i+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})B_j\\
&\qquad+\sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,1)} p(\boldsymbol{\lambda})\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,1)}\circ\boldsymbol{X}\}_{2:n}\right)\right]\Bigg\}.
\end{aligned}
$$

Note that

$$f_2(x) := \min\{x, a\} = \begin{cases} x, & x \le a, \\ a, & x \ge a, \end{cases}$$

is an increasing concave function in x, for any $a \in \mathbb{R}$. Then, $X_i \le_{\mathcal{F}_{i,j}-\mathrm{icv}} X_j$ implies that

$$B_j = \mathbb{E}\left[\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(0,1)} \circ \boldsymbol{X}\}_{2:n}\right)|\mathcal{F}_{i,j}\right]\right] \ge \mathbb{E}\left[\mathbb{E}\left[\phi\left(\{\boldsymbol{\lambda}_{(i,j)}^{(1,0)} \circ \boldsymbol{X}\}_{2:n}\right)\Big|\mathcal{F}_{i,j}\right]\right] = B_i.$$

Hence,

$$\begin{aligned}
&\mathbb{E}[\phi(S_n)] - \mathbb{E}[\phi(S_n^*)] \\
&= \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})B_j + \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})B_i - \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})B_i - \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})B_j \\
&= \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(0,1)} p(\boldsymbol{\lambda})(B_j - B_i) + \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(B_i - B_j) \\
&\ge \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(B_j - B_i) - \sum_{\boldsymbol{\lambda}\in\Lambda_{n-1}^{i,j}(1,0)} p(\boldsymbol{\lambda})(B_j - B_i) = 0,
\end{aligned}$$

where the last inequality is from the SAI property of $\boldsymbol{I}$ and the fact $B_j \ge B_i$. Hence, we have $\mathbb{E}[\phi(S_n)] \ge \mathbb{E}[\phi(S_n^*)]$. By the arbitrariness of ϕ, the proof is finished. ■

A.7 Proof of Theorem 5.2

Proof: By the nature of majorization order, it is sufficient to prove the desired result under the setting $(-\log p_1, -\log p_2) \overset{\mathrm{m}}{\succeq} (-\log p_1^*, -\log p_2^*)$, $(p_1, p_2) \in \mathcal{I}_2$ and $p_i = p_i^*$, for $i = 3, \ldots, n$. Under this assumption, we have $p_1 p_2 = p_1^* p_2^*$ and $p_2 - p_1^* \ge 0$. Then, for any increasing concave function ϕ, the desired result is equivalent to showing that

$$\begin{aligned}
\mathbb{E}[\phi(S_n)] &= \mathbb{E}\left\{\phi\left[\{\boldsymbol{I} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&\ge \mathbb{E}\left\{\phi\left[\{\boldsymbol{I}^* \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}^* \circ \boldsymbol{X}\}_{1:n}\right]\right\} = \mathbb{E}[\phi(S_n^*)].
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{E}[\phi(S_n)] &= \mathbb{E}\left\{\phi\left[\{\boldsymbol{I} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&= (1-p_1)(1-p_2)\,\mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(0,0)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}_{(1,2)}^{(0,0)} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&\quad + (1-p_1)p_2\,\mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(0,1)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}_{(1,2)}^{(0,1)} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&\quad + p_1(1-p_2)\,\mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(1,0)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}_{(1,2)}^{(1,0)} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&\quad + p_1 p_2\,\mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(1,1)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}_{(1,2)}^{(1,1)} \circ \boldsymbol{X}\}_{1:n}\right]\right\} \\
&= (1-p_1)p_2 C_1 + p_1(1-p_2)C_2 + p_1 p_2 C_3.
\end{aligned}$$

where

$$C_1 = \mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(0,1)} \circ \boldsymbol{X}\}_{2:n}\right]\right\}, \quad C_2 = \mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(1,0)} \circ \boldsymbol{X}\}_{2:n}\right]\right\},$$

and

$$C_3 = \mathbb{E}\left\{\phi\left[\{\boldsymbol{I}_{(1,2)}^{(1,1)} \circ \boldsymbol{X}\}_{2:n} - \{\boldsymbol{I}_{(1,2)}^{(1,1)} \circ \boldsymbol{X}\}_{1:n}\right]\right\}.$$

With the condition $X_1 \le_{\mathcal{F}_{1,2}-\mathrm{icv}} X_2$, it follows that

$$C_1 = \mathbb{E}\left[\mathbb{E}\left\{\phi\left(\{\boldsymbol{I}_{(1,2)}^{(0,1)} \circ \boldsymbol{X}\}_{2:n}\right)|\mathcal{F}_{1,2}\right\}\right] \ge \mathbb{E}\left[\mathbb{E}\left\{\phi\left(\{\boldsymbol{I}_{(1,2)}^{(1,0)} \circ \boldsymbol{X}\}_{2:n}\right)\Big|\mathcal{F}_{1,2}\right\}\right] = C_2. \tag{A3}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}[\phi(S_n)] - \mathbb{E}[\phi(S_n^*)] &= (1-p_1)p_2C_1 + p_1(1-p_2)C_2 + p_1p_2C_3 \\ &\quad - \left[(1-p_1)p_2C_1 + p_1(1-p_2)C_2 + p_1p_2C_3\right] \\ &= \left[p_2 - p_1\right][C_1 - C_2] + (p_2 - p_1^*)C_2 \\ &\geq 0.\end{aligned}$$

Hence, the proof is completed. ■

A.8 Proof of Theorem 5.3

Proof: First, in light of Theorem 3.2 of Kochar and Korwar (1996), it holds that

$$\mathbb{P}(X_{2:n} - X_{1:n} \leq y) \leq \mathbb{P}(X_{2:n}^* - X_{1:n}^* \leq y), \quad \text{for all } y \in \mathbb{R}_+. \tag{A4}$$

By applying (8) and (9), it can be calculated that

$$\begin{aligned}F_{S_n}(y) - F_{S_n^*}(y) &= p(\mathbf{1})\left[\mathbb{P}(X_{2:n} - X_{1:n} \leq y) - \mathbb{P}(X_{2:n}^* - X_{1:n}^* \leq y)\right] \\ &\quad + \prod_{j=1}^{n} \bar{F}_j^*(y) \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i^*(y)}\right\} - \prod_{j=1}^{n} \bar{F}_j(y) \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i(y)}\right\} \\ &\leq \prod_{j=1}^{n} \bar{F}_j^*(y) \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i^*(y)}\right\} - \prod_{j=1}^{n} \bar{F}_j(y) \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i(y)}\right\} \\ &=: \mathrm{e}^{-\sum_{j=1}^{n} \mu_j y}[\phi(\boldsymbol{\mu}^*) - \phi(\boldsymbol{\mu})],\end{aligned}$$

where

$$\phi(\boldsymbol{\mu}^*) := \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i^*(y)}\right\} = \sum_{i=1}^{n} \left\{p(\mathbf{1}_i)\mathrm{e}^{\mu_i^* y}\right\}, \quad \phi(\boldsymbol{\mu}) := \sum_{i=1}^{n} \left\{\frac{p(\mathbf{1}_i)}{\bar{F}_i(y)}\right\} = \sum_{i=1}^{n} \left\{p(\mathbf{1}_i)\mathrm{e}^{\mu_i y}\right\},$$

and the inequality is based on (A4) and the assumption that $(\mu_1, \ldots, \mu_n) \overset{\mathrm{m}}{\succeq} (\mu_1^*, \ldots, \mu_n^*)$. Since $\boldsymbol{I}$ is SAI, we have $p(\mathbf{1}_i) = p(\tau_{i,j}(\mathbf{1}_j)) \geq p(\mathbf{1}_j)$, for any $1 \leq i < j \leq n$, by applying Lemma A.1. Hence, it can be noted that

$$\frac{\partial \phi(\boldsymbol{\mu})}{\partial \mu_1} - \frac{\partial \phi(\boldsymbol{\mu})}{\partial \mu_2} = y[p(\mathbf{1}_1)\mathrm{e}^{\mu_1 y} - p(\mathbf{1}_2)\mathrm{e}^{\mu_2 y}] \geq 0,$$

where the inequalities are due to the conditions that $\mu_1 \geq \mu_2$ and $p(\mathbf{1}_2) \leq p(\mathbf{1}_1)$. Thus, we have $\phi(\boldsymbol{\mu}) \geq \phi(\boldsymbol{\mu}^*)$ by applying Lemma 2.8, which further implies that $F_{S_n}(y) - F_{S_n^*}(y) \leq 0$, for all $y \in \mathbb{R}_+$. Hence, the proof is finished. ■

A.9 Proof of Theorem 5.4

Proof: Consider a fictive set of insurance portfolio $(I_1^* X_1, I_2^* X_2, \ldots, I_n^* X_n)$ with its second smallest claim amount and the smallest claim amount denoted by $\tilde{Y}_{2:n}$ and $\tilde{Y}_{1:n}$, respectively. Since $\boldsymbol{\mu} \in \mathcal{D}_n$, we have $X_1 \leq_{\mathrm{st}} X_2 \leq_{\mathrm{st}} \ldots \leq_{\mathrm{st}} X_n$ and thus $X_1 \leq_{\mathrm{icv}} X_2 \leq_{\mathrm{icv}} \ldots \leq_{\mathrm{icv}} X_n$. Hence, by applying Theorem 5.2 it follows that $Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} \tilde{Y}_{2:n} - \tilde{Y}_{1:n}$. On the other hand, it holds from Theorem 5.3 that $\tilde{Y}_{2:n} - \tilde{Y}_{1:n} \geq_{\mathrm{icv}} Y_{2:n}^* - Y_{1:n}^*$. By virtue of the transitivity of the increasing concave ordering, we immediately have $Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} Y_{2:n}^* - Y_{1:n}^*$, which proves the desired result. ■

A.10 Proof of Corollary 5.5

Proof: First, it can be deduced from Theorem 5.4 that $Y_{2:n} - Y_{1:n} \geq_{\mathrm{icv}} Y_{2:n}^* - Y_{1:n}^*$. Further, by applying (11), we first calculate the distribution function of $Y_{2:n}^* - Y_{1:n}^*$. Note that $\mathbb{P}(Y_{2:n}^* - Y_{1:n}^* = 0) =$

$(1-p)^n$. Further, for any $s \geq 0$, it can be derived that

$$\mathbb{P}(Y^*_{2:n} - Y^*_{1:n} \leq s) = (1-p)^n + p^n(1-\mathrm{e}^{-(n-1)\mu s}) + \sum_{k=1}^{n-1} \binom{n}{k} p^k (1-p)^{n-k} - np^{n-1}(1-p)\mathrm{e}^{-(n-1)\mu s}.$$

Thus, for any $y \geq 0$, we have

$$\begin{aligned}
&\int_0^y \mathbb{P}(Y^*_{2:n} - Y^*_{1:n} \leq s)\,\mathrm{d}s \\
&= (1-p)^n y + p^n y + y \sum_{k=1}^{n-1} \binom{n}{k} p^k (1-p)^{n-k} \\
&\quad - \int_0^y \left(p^n \mathrm{e}^{-(n-1)\mu s} + np^{n-1}(1-p)\mathrm{e}^{-(n-1)\mu s} \right) \mathrm{d}s \\
&= y\left[(1-p)^n + p^n + \sum_{k=1}^{n-1} \binom{n}{k} p^k (1-p)^{n-k} \right] - \frac{p^n + np^{n-1}(1-p)}{(n-1)\mu} \left(1 - \mathrm{e}^{-(n-1)\mu y} \right) \\
&= y \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} - \frac{p^{n-1}}{\mu} \left(\frac{n}{n-1} - p \right) \left(1 - \mathrm{e}^{-(n-1)\mu y} \right) \\
&= y - \frac{p^{n-1}}{\mu} \left(\frac{n}{n-1} - p \right) \left(1 - \mathrm{e}^{-(n-1)\mu y} \right).
\end{aligned}$$

Hence, the desired result is established by using (2). ■

Appendix 2. Verification of conditional expectations

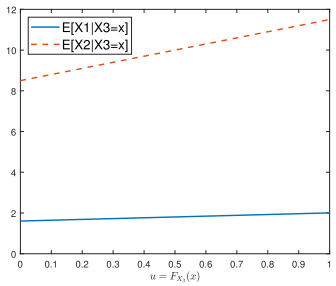
In Examples 6.1 and 6.2, we obtained the conditional increasing convex ordering relations

$$X_1 \leq_{\mathcal{F}_{1,2}-\mathrm{icx}} X_2 \quad \text{and} \quad X_2 \leq_{\mathcal{F}_{2,3}-\mathrm{icx}} X_3,$$

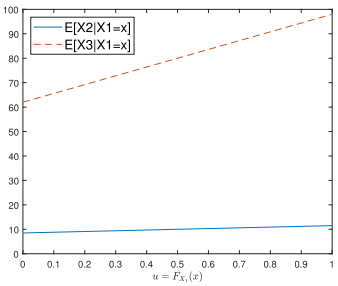
whereas in Examples 6.3 and 6.4, we obtained the conditional increasing concave ordering relations

$$X_1 \leq_{\mathcal{F}_{1,2}-\mathrm{icv}} X_2 \quad \text{and} \quad X_2 \leq_{\mathcal{F}_{2,3}-\mathrm{icv}} X_3.$$

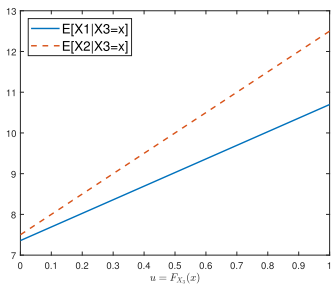
These relations were obtained by applying Theorem 2.5. The application of Theorem 2.5 relies on the required ordering relations among the relevant conditional expectations. In this section, we verify these ordering relations graphically. As shown in Figure A1, the required ordering relations among the conditional expectations are satisfied.



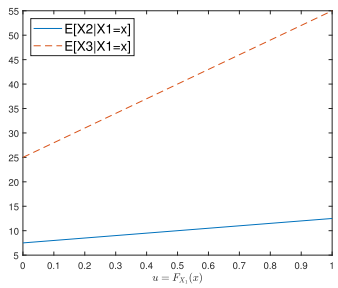
(a) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.1.



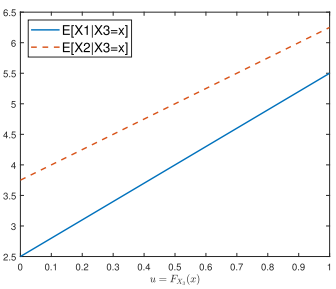
(b) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.1.



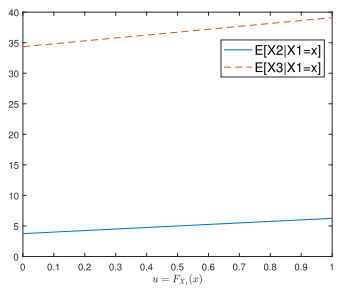
(c) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.2.



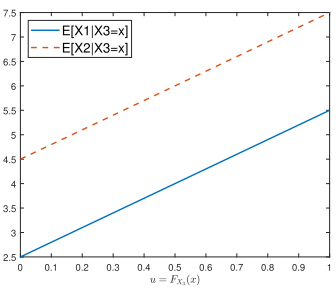
(d) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.2.



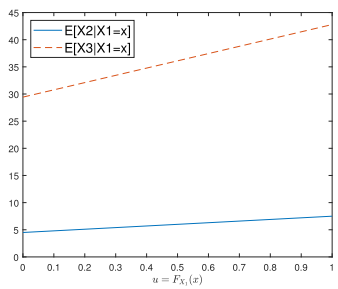
(e) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.3.



(f) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.3.



(g) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.4.



(h) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.4.

Figure A1. Plots of the conditional expectations. (a) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.1. (b) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.1. (c) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.2. (d) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.2. (e) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.3. (f) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.3. (g) Conditional expectations of X_1 and X_2 given X_3 in the setting of Example 6.4 and (h) Conditional expectations of X_2 and X_3 given X_1 in the setting of Example 6.4.